Limit Cycles of Controlled Switched Systems: Existence, Stability, Sensitivity

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Abstract. We present a control method which makes the trajectories of a switched system converge to a stable limit cycle lying in a desired region of equilibrium. The method is illustrated on the boost DC-DC converter example. We also point out in this example the sensitivity of limit cycles to parameter variations by showing how the limit cycle evolves in presence of small perturbations of some system parameters. This suggests that limit cycles are good candidates for reliable estimations of the physical parameters of switched systems using an appropriate inverse approach.

1. Introduction

Sampled switched systems are more and more used in electrical and mechanical industry, e.g. power electronics and automotive industry: this is due to their flexibility and simplicity for controlling accurately industrial mechanismes. These systems are governed by piecewise dynamics that are periodically sampled with a given period τ (see, e.g., [3]). At each sampling time, the "mode" of the system, i.e. the parameters of the dynamics, are switched according to a control rule. A classical example of sampled switched system is the boost DC-DC converter. The control of a switching cell puts the system, according to its position (open or closed), into one mode or another one. When the system stays in the same mode, it evolves towards a unique equilibrium point. In contrast, by adopting switching control rules, one can steer the system to a desired region R that is centered at a different equilibrium point, around which the system oscillates with some variability.

In [5], we have thus designed a state-dependent control method which decomposes R into a set of sub-boxes associated with specific control modes: at each sampling time, depending on the loacation of the system state, one activates a specific mode until the next sampling time. We show here that such a control not only ensures the invariance of the system state inside R, but makes the state trajectories converge to *limit cycles* inside R.

We also point out the sensitivity of limit cycles to parameter variations by showing the evolution of limit cycles in presence of small perturbations of system parameters. As indicated, e.g., in [9], this suggests that limit cycles are are good candidates for reliable estimation of physical parameters of the system

1.1. Related work

The presence of limit cycles in switched systems such as those used in power electronics has been often observed (see, e.g., [10]). The existence and stability of limit cycles is often shown using

Lyapunov techniques (see, e.g., [2, 11]). An other common technique for proving existence of limit cycles is based on Poincaré map technique [7, 9]. Finally, sensibility functions w.r.t. initial conditions are used in order to compute a limit cycle [4], as well as describing functions [12].

We use here an original method showing that the existence of limit cycles follows from a property of piecewise invariance of a state space region, when the dynamics of the modes are contractive.

2. Decomposable Area of Interest

2.1. Model of sampled switched systems

Sampled switched systems, as defined, e.g., in [6], is a subclass of affine hybrid systems [8]. A sampled switched system Σ is defined by a triple $\langle \tau, U, \mathcal{F} \rangle$ where:

- $\tau \in \mathbb{R}_{\geq 0}$ is a "time sampling parameter",
- $U = \{1, \dots, q\}$ is a finite set of modes,
- $\mathcal{F}$ is a set of the form $\{(A_u, B_u)\}_{u \in U}$ with $(A_u, B_u) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1}$.

We say that a piecewise $\mathcal{C}^1$ function $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a *trajectory* of Σ if it is continuous and satisfies, for all $k \in \mathbb{N}$:

$$\exists u \in U \ \forall t \in [k\tau, (k+1)\tau) \qquad \dot{\mathbf{x}}(t) = f_u \mathbf{x}(t),$$

with $(A_u, B_u) \in \mathcal{F}$. The trajectory is fully determined by the values $u_1, u_2, \dots$ of u at sampling times $\tau, 2\tau, \dots$. These values define a control function $\mathbf{u}(t)$, which is constant on each interval $[k\tau, (k+1)\tau)$, for all $k \in \mathbb{N}$. Between two sampling times, the system is governed by a differential equation of the form: $\dot{\mathbf{x}}(t) = A_u \mathbf{x}(t) + B_u$ with $u \in U$. We will use $\mathbf{x}(t, x, u)$ to denote the point reached by Σ at time t under mode u from the initial condition x. This defines a transition relation $\to_\tau^u$ given by: $x \to_\tau^u x'$ iff $\mathbf{x}(\tau, x, u) = x'$ for x and x' in $\mathbb{R}^n$.

NB: The expression $\mathbf{x}(\tau, x, u)$ can be put under the algebraic form: $(C_u \cdot x + D_u)$, with $C_u = e^{A_u \tau}$ and $D_u = e^{(A_u \tau - \mathcal{I}_n) A_u^{-1}} \cdot B_u$, where $\mathcal{I}_n$ denotes the identity matrix.

We define: $Post_u(X) = \{x' \mid x \to_\tau^u x' \text{ for some } x \in X\}$. We say that a zone X is *R-invariant via mode u* if $Post_u(X) \subseteq R$. We will group modes together into sequences called "patterns". Given a pattern π of the form $(u_1 \dots u_m)$, we have:
$Post_\pi(X) = \{x' \mid x \to_\tau^{u_1} \cdots \to_\tau^{u_m} x' \text{ for some } x \in X\}$.
We say that X is *R-invariant via π* if $Post_\pi(X) \subseteq R$.

We say that R is "piecewise k-invariant" if it can be decomposed into a finite set of subsets that are R-invariant. Formally:

Definition 1 *A set $R \subseteq \mathbb{R}^n$ is* piecewise k-invariant *if there exists a finite set I of indices, a set $\{V_i\}_{i \in I}$ of subsets of $\mathbb{R}^n$ and a set $\{\pi_i\}_{i \in I}$ of patterns of length at most k such that:*

- $R = \bigcup_{i \in I} V_i$
- *V_i is R-invariant via π_i (i.e., $Post_{\pi_i}(V_i) \subseteq R$), for all $i \in I$.*

In practice, R and V_i ($i \in I$) are *boxes*, i.e., cartesian products of closed intervals of $\mathbb{R}^n$. The set $\{(V_i, \pi_i)\}_{i \in I}$ is said to be a *k-invariant decomposition* of R.

Given a decomposition Δ of the from $\{V_i, \pi_i)\}_{i \in I}$, we define $Post_\Delta$ as follows:

$$Post_\Delta(X) = \bigcup_{i \in I} Post_{\pi_i}(X \cap V_i), \quad \text{for all } X \subset R.$$

In [5], we have proposed a method in order to show that, given a set R, it is piecewise k-invariant by constructing a k-invariant decomposition of R. This method is recalled in Appendix.

In the rest of this paper, we will suppose that we are given a piecewise k-invariant set R and a k-invariant decomposition $\Delta = \{(V_i, \pi_i)\}_{i\in I}$. The decomposition Δ induces a state-dependent control that makes any trajectory starting fom R go back to R within at most k steps: Given a starting state x_0 in R, we know that $x_0 \in V_i$ for some $i \in I$ (since $R = \bigcup_{i\in I} V_i$); one thus applies π_i to x_0, which gives a new state x_1 that belongs to R (since V_i is R-invariant via π_i); the process is repeated on x_1, and so on iteratively. Formally, for any $x_0 \in R$, the above process leads to the construction of a sequence of points $\{x_i\}_{i\geq 0}$ of R, and a sequence of indices $\{k_i\}_{i\geq 0}$ defined, for all $i \geq 0$, by:

- $x_{i+1} = Post_{\pi_{k_i}}(x_i)$
- $k_i = argmin\{j \in I \mid x_i \in V_j\}$.

In the following, we will suppose that every mode of U is "contractive". We say that a mode $u \in U$ is *contractive* if there exists ρ_u such that $0 < \rho_u < 1$ and, for all $x, y \in \mathbb{R}^n$: $\|Post_u(x) - Post_u(y)\| \leq \rho_u \|x - y\|$.

2.2. Boost DC-DC Converter

Let us illustrate the model of sampled switched system and the result of the decomposition procedure (see Appendix).

Example 1 *This example is taken from [1] (see also, e.g., [6, 13]). This is a boost DC-DC converter with one switching cell (see Fig. 1). There are two operation modes depending on the position of the switching cell. An example of pattern of length* 4 *is illustrated on Fig. 2: it corresponds to the application of mode 2 on* $(0, \tau]$ *and mode 1 on* $(\tau, 4\tau]$*. The controlled system is steered to a zone where the output voltage stabilizes around a desired value. The state of the system is* $x(t) = [i_l(t), v_c(t)]^T$ *where* i_l *is the current intensity in inductor, and* $v_c(t)$ *the voltage of capacitor* C*. The dynamics associated with mode u is of the form* $\dot{x}(t) = A_u x(t) + b_u$ $(u = 1, 2)$ *with*

$$A_1 = \begin{pmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{pmatrix} \quad b_1 = \begin{pmatrix} \frac{v_s}{x_l} \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -\frac{1}{x_l}(r_l + \frac{r_0.r_c}{r_0+r_c}) & -\frac{1}{x_l}\frac{r_0}{r_0+r_c} \\ \frac{1}{x_c}\frac{r_0}{r_0+r_c} & -\frac{1}{x_c}\frac{1}{r_0+r_c} \end{pmatrix} \quad b_2 = \begin{pmatrix} \frac{v_s}{x_l} \\ 0 \end{pmatrix}$$

We will use the numerical values of [1], expressed in the per unit system: $x_c = 70$*,* $x_l = 3$*,* $r_c = 0.005$*,* $r_l = 0.05$*,* $r_0 = 1$*,* $v_s = 1$*. The sampling period is* $\tau = 0.5$*. We can check that modes 1 and 2 are contractive (i.e.,* $\|e^{A_u\tau}\| = \rho_u < 1$ *for* $u = 1, 2$*). For the equilibrium region, we take* $R = [1.55, 2.15] \times [1.0, 1.4]$*, which corresponds to a medium value* 1.85 *for* i_l *with* ± 0.3 *for variability, and medium value* 1.20 *for* v_c *with* ± 0.2 *for variability.*

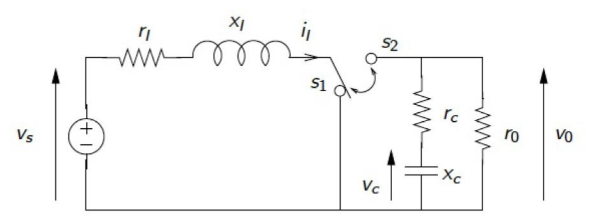

Figure 1. Scheme of the boost DC-DC converter

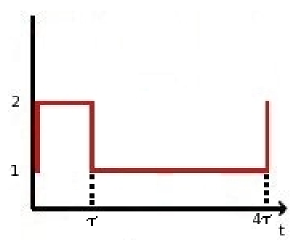

Figure 2. Switching pattern (2.1.1.1)

The application of the algorithm Decomposition (see Appendix) to $R = [1.55, 2.15] \times [1.0, 1.4]$*, with* $k = 5$ *and* $d = 1$*, yields a decomposition of* $R = \bigcup_{j=1}^{4} V_j$ *depicted in Figure 3. The associated patterns are:* $\pi_1 = (11222)$*,* $\pi_2 = (2)$*,* $\pi_3 = (212)$*,* $\pi_4 = (1)$*, with:* $V_1 = [1.55, 1.85] \times [1.00, 1.20]$*,* $V_2 = [1.85, 2.15] \times [1.00, 1.20]$*,* $V_3 = [1.85, 2.15] \times [1.20, 1.40]$*,* $V_4 = [1.55, 1.85] \times [1.20, 1.40]$*.*

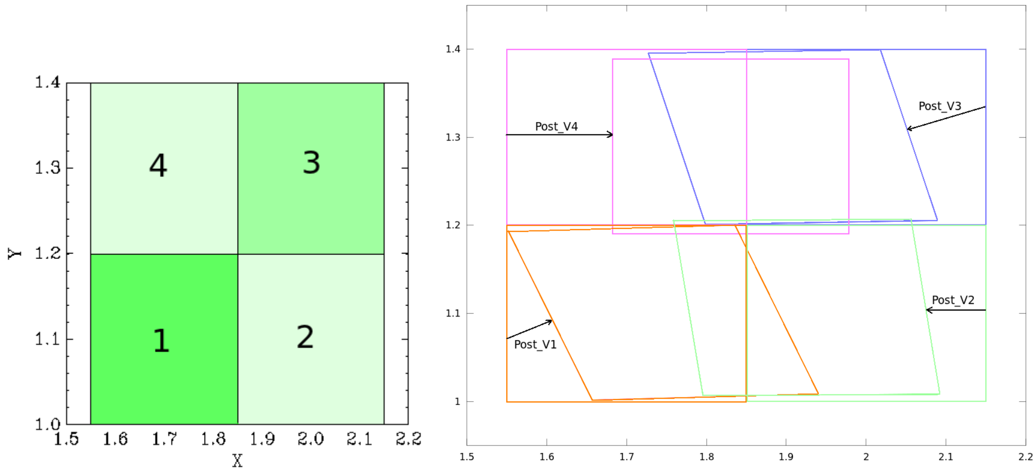

Figure 3. Left: Decomposition of R into 4 subsets. Right: visualization of R-invariance ($Post_{\pi_i}(V_i) \subset R$)

In Figure 4, we depict runs that start from the four corners of R*, and follows the control strategy induced by this decomposition . In all cases, one observes a phenomenon of convergence of the trajectories to a unique limit cycle. This phenomenon follows from the existence of the decomposition of* R*, as explained in next Section.*

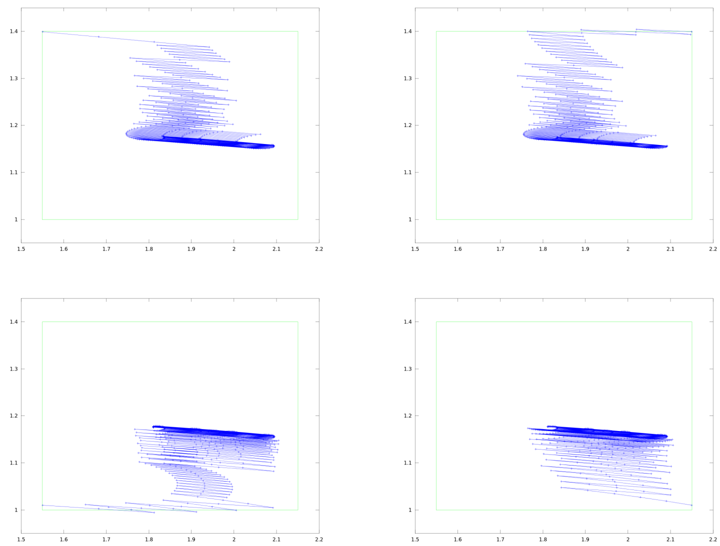

Figure 4. Runs starting from the four corners of R, following the control strategy induced by the decomposition

3. Limit Cycles

3.1. Existence and stability

If all the modes are contractive (i.e., $\|e^{A_u \tau}\| = \rho_u < 1$ for all $u \in U$), the existence of a k-invariant decomposition $\Delta : \{(V_i, \pi_i)\}_{i \in I}$ of R entails the existence of stable limit cycles to which any trajectory $\mathcal{T}$ issued from R converges provided that all the points of $\mathcal{T}$ remain at a minimum distance $e > 0$ (of arbitrary positive value) from any border of a sub-box V_i ($i \in I$). Formally, we have:

Theorem 1 *Consider a trajectory $\mathcal{T} : \{x_0, x_1, x_2, \cdots\}$ of points of R induced by Δ, which all stay at a minimum distance $e > 0$ from the borders of V_i. Then there exists a set of points $\mathcal{C} = \{y_0, y_1, \cdots, y_{m-1}\}$ of R, and an integer N such that, for all $j \in \{0, \cdots, m-1\}$:*

- $Post_\Delta(y_j) = y_{j+1}$ *(where y_m stands for y_0),*
- $Post_\Delta(x_{N+j+i.m}) = x_{N+j+i.m+1}$,
- $\lim_{i \to \infty} x_{N+j+i.m} = y_j$.

The proof of the above statement (see Appendix 5) shows also that $\mathcal{C}$ is a stable limit cycle for the trajectories of R, in the sense that a small perturbation occurring in the trajectory $\mathcal{T}$ does not prevent the subsequent trajectory to converge to $\mathcal{C}$. For the boost converter, the corresponding limit cycle is depicted in Figure 5.

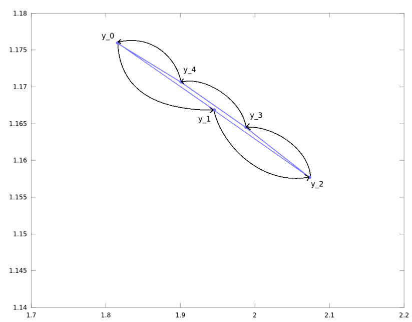

Figure 5. Limit cycle for $r_0 = 1$ (corresponding to pattern π_1 : (11222)): one has $y_0 \to^1 y_1 \to^1 y_2 \to^2 y_3 \to^2 y_4 \to^2 y_0$

3.2. Sensitivity

Given the region R and fixed values for the physical parameters of the system, the decomposition procedure gives a decomposition $\Delta : \{(V_i, \pi_i)\}_{i \in I}$. Actually, even in presence of small variations of parameters of the system, the same decomposition of Δ can ensure the piecewsie invariance of R. For example, the decomposition Δ found for the boost DC-DC converter for $r_0 = 1$ and $R = [1.55, 2.15] \times [1.0, 1.4]$ (see Figure 3), still ensures the piecewise invariance of R when r_0 varies from 0.975 to 1.005. This guaranteed that the control found $r_0 = 1$ makes the system converge to a stable limit cycle of R, for small variations of r_0. However, as shown in Figure 6, the form and position of the limit cycle is very sensitive to the actual value of r_0:

For $r_0 = 0.965$, the limit cycle corresponds to repeated pattern $(\pi_3\pi_1^3\pi_3\pi_1^2\pi_3\pi_1^3\pi_3\pi_1^3\pi_3\pi_1^3)$ (with $\pi_1 = (11222)$, $\pi_3 = (212)$); for $r_0 = 0.975$, the associated pattern is $(\pi_3\pi_1^5)$, while, for $r_0 = 1$ and $r_0 = 1.005$, the pattern is (π_1).

This suggests that limit cycles are good candidates for reliable estimations of the physical parameters of switched systems, using an appropriate inverse approach.

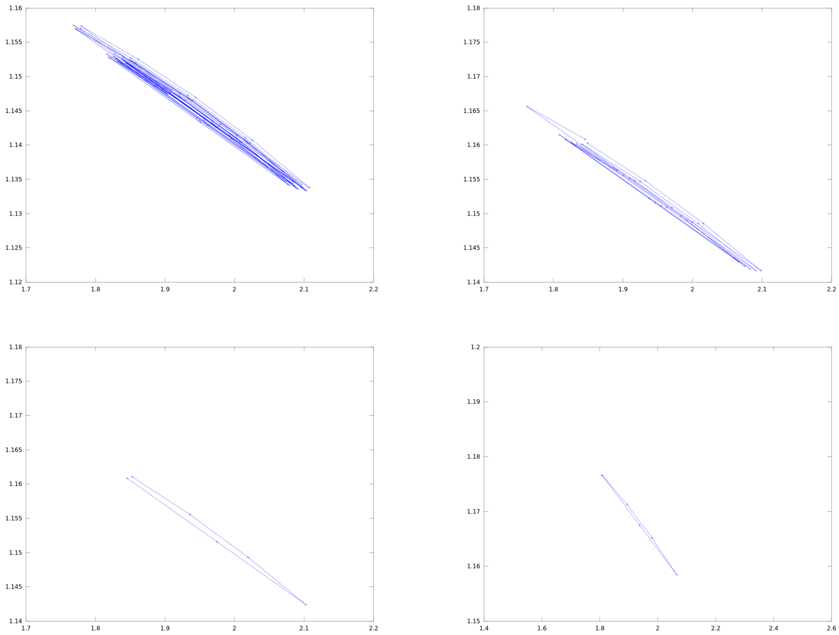

Figure 6. Limit cycles for $r_0 = 0.965$ (pattern $(\pi_3\pi_1^3\pi_3\pi_1^2\pi_3\pi_1^3\pi_3\pi_1^3\pi_3\pi_1^3)$) on the upper left, for $r_0 = 0.975$ (pattern $(\pi_3\pi_1^5)$) on the upper right, for $r_0 = 1$ (pattern π_1) on the lower left, and for $r_0 = 1.005$ (pattern π_1) on the lower right

4. Final Remarks

We have shown that, for controlled switched systems, the existence of an invariance decomposition for a given zone of interest, entails the convergence of the system to a stable limit cycle within the zone.

We have also shown the high sensitivity of the form and position of limit cycles when some physical parameter of the system slighty varies, such as load resistance in DC-DC boost converters. This suggests that limit cycles are good candidates for reliable estimations of the physical parameters of switched systems,

5. References

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Appendix: Decomposition Procedure [5]

Suppose we are given a box $R \subseteq \mathbb{R}^n$. We now give a procedure in order to show that R is piecewise k-invariant. If we find a pattern π of length at most k such that R is invariant, then we are done. Otherwise, we decompose R dichotomously into 2^n sub-boxes $V_1, \cdots, V_{2^n}$ of equal size. If for each box V_i, we find a pattern π_i of length at most k such that V_i is R-invariant via π_i, we are done. If, for some V_j, no such pattern exists, we recursively apply the process to V_j. The process ends with success when a k-invariant decomposition of R is found, or with failure when a maximal degree d of decomposition has been reached. The algorithmic form of this method is given in Algorithms 1 and 2. For the sake of simplicity, we have considered the case of space dimension $n = 2$ (the extension to the general case is straightforward). The procedure Decomposition(W,R,D,K) is called with R as input value for W, d for input value for D, and k as input value for K. The procedure Decomposition(W,R,D,K) calls the subprocedure Find_Pattern(W,R,K) in order to find a pattern for which W is R-invariant. This is simply done by testing all the patterns by increasing length order, until one is found that maps W into R. The correctness of the decomposition procedure is stated formally as follows ([5]): If the procedure Decomposition(R,R,d,k) terminates with success (i.e: $\Delta \neq False$), then it returns a k-invariant decomposition Δ of R.

Algorithm 1: Decomposition(W,R,D,K)

Input: A box W, a box R, a degree D of decomposition, a length K of pattern
Output: Δ which is either a K-invariant decomposition of W or *False*

$\pi := Find\_Pattern(W, R, K)$
if $\pi \neq False$ **then**
  return $\Delta := (W, \pi)$
else
  if $D = 0$ **then**
    return $\Delta := False$
  else
    Dichotomously decompose W into (W_1, W_2, W_3, W_4) /* (case $n = 2$) */
    $\Delta_1 :=$ Decomposition(W_1,R,$D-1$,K)
    $\Delta_2 :=$ Decomposition(W_2,R,$D-1$,K)
    $\Delta_3 :=$ Decomposition(W_3,R,$D-1$,K)
    $\Delta_4 :=$ Decomposition(W_4,R,$D-1$,K)
    if $\Delta_1 = False \vee \Delta_2 = False \vee \Delta_3 = False \vee \Delta_4 = False$ **then**
      return $\Delta := False$
    else
      return $\Delta := \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$

Appendix: Sketch of Proof of Theorem 1

Let us assume that we have a decomposition $\Delta = \{(V_i; \pi_i)\}_{i \in I}$ of R. Let us consider a trajectory $\mathcal{T} = \{x_0, x_1, \cdots\}$, where all x_i stay at minimum distance $e > 0$ from the borders of Δ. For every point x_i of $\mathcal{T}$, we consider the ball B_i centered at x_i of radius e. Clearly, every B_i is included in $\mathring{V}_j$ for some $j \in I$. Furthermore we have $Post_\Delta(B_i) \subset Bi + 1$, since all the modes of U are contractive. We claim that there exists $\varepsilon > 0$ such that: either $\lambda(B_{i+1} \cap \overline{\bigcup_{j \leq i} B_j})$ or $B_{i+1} \subset \bigcup_{j \leq i} B_j$. The first subcase cannot occur indefinitely, since eventually all R is covered in a finite number of steps. Hence, the second subcase occurs eventually ($B_{i+1} \subset \bigcup_{j \leq i} B_j$), this entails the existence of a cycle passing by a point of B_{i+1}.

Algorithm 2: Find_Pattern(W,R,K)

Input: A box W, a box R, a length K of pattern
Output: π wich is either a pattern mapping W into R or $False$

for $i = 1 \dots K$ **do**
    $\Pi :=$ set of patterns of length i
    while Π *is non empty* **do**
        Select π in Π
        $\Pi := \Pi \setminus \{\pi\}$
        if $Post_\pi(W) \subseteq R$ **then**
            return π
return $\pi := False$