A GENERALIZATION OF THE PROCEDURE OF KARP AND MILLER TO WELL STRUCTURED TRANSITION SYSTEMS

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1. INTRODUCTION

Transition systems form a general model for specification and verification of usual properties of parallel system. When the reachability set (i.e. the set of reachable states) is finite we can, at least theoretically, verify the traditional properties such as deadlock freedom, quasi-liveness, liveness, mutual exclusion, existence of infinite sequence...
But if the reachability set is infinite, verification of these properties with the study of the infinite reachability graph becomes impossible.
Let us remark that there exist some transition systems, for example the ones associated with **Petri nets** [Peterson 81] [Hack 75] or with certain **Fifo nets** [Memmi ...85],[Finkel 86],[Rosier... 85] for which an infinite set of states doesn't prevent from the analysis of usual properties.
In these models the analysis is made by associating to the infinite set of reachable states a finite set of states and "limits of states"; this finite set allows the verification of usual properties. In the frame of Petri nets the reduction of the number of states is achieved with the help of the **coverability tree** [Karp...69].
The aim of this work is to draw the fundamental concepts used for the construction of the coverability tree of a Petri net. Then we shall define the general **structure** allowing the analysis of a transition system by reducing the number of states thanks to a coverability tree.
One of the main properties of Petri nets is the existence of an **ordering** $\leq$ on the reachability set. This ordering gives a property of **monotonicity** to the net: it means that if from a marking (state) M we can fire a transition t then from every marking M' larger than M, we can fire t [Brams 83].
A second property of Petri nets is that the reachability tree has a ***finite* degree** (because there is a finite number of transitions in a Petri net). This enables us to apply Koenig's lemma.
The fact that the ordering $\leq$ is a **well ordering** [Dickson 13][Kruskal 72] is the third important property of Petri nets. Let us recall that a well ordering is an ordering such that from every infinite sequence one can extract an infinite increasing subsequence. At last, the ordering $\leq$ and the equality are **decidable** for markings. It means that for two vectors M,M' one can decide M≤M' and M=M'.
These four properties allow to decide, for example, the finiteness of the Petri net language. The algorithm [Karp...69] consists in developing the construction of the reachability tree till we meet two comparable states M,M' on a same branch so that M' is reachable from M and M≤M'. As the reachability tree has a finite degree and the ordering $\leq$ is a decidable well ordering, there are only two possibilities: either the reachability tree is finite then the Petri net is equivalent to a finite automaton and the finiteness of a regular language is decidable; or the reachability tree is infinite, then we eventually reach two states M,M' such that M' is reachable from M and M≤M'. As Petri nets are monotonous, we can conclude that there exists an infinite sequence of transitions and that the language is infinite. In conclusion, in those two cases, we can decide the finiteness of a Petri net language.
The proof that the finiteness of the reachability set is a decidable problem for Petri nets is based on the following property (the strong monotonicity): if we can fire a transition t from a marking M and reach a marking M_1 then from any marking M'>M we can fire the transition t and reach a marking $M_1' > M_1$.
To prove the decidability of the finiteness of the reachability set, we apply the same reasoning as previously; we obtain either a finite reachability tree or an infinite strictly increasing sequence of accessibles states. The second case implies that the reachability set is infinite.
Let us consider now any transition system S. We say that S is **structured** when there exists a quasi-ordering $\leq$ (reflexive and transitive) on the reachability set so that $(S,\leq)$ is monotonous, $\leq$ and = are decidable, $\leq$ is a well quasi-ordering and the reachability tree has a finite degree. We define a few types of structured transition systems according to the monotonicity induced by the quasi-ordering.
We show that for the most general structured transition systems, the finiteness of the language is a decidable problem. Finiteness of the reachability set is also a decidable problem but for less general type of structured transition systems.

Problems like quasi-liveness and deadlock freedom arise naturally in the framework of structured transition systems. The method to solve these problems in the framework of Petri nets, is the use of the coverability tree. This tree (finite) "covers" (for the quasi-ordering) all the states of the reachability set.
Now what allows us to construct a finite coverability tree ?
At first, Petri nets are strongly monotonous (and not only monotonous) for the usual ordering. Secondly, this ordering, naturally extended on limits of sequences of states is still a well quasi-ordering. Thirdly, we know how to compute the limit of an infinite increasing sequence of states. At last there exists an integer k (which is the number of places of the Petri net) such that the k^{th} "limit" of the reachability set is finite
We extend the definition of a coverability tree to a class of structured transition systems, called **well structured transition systems.** For these systems we show that the quasi-liveness problem is decidable.

2. PRELIMINARIES

We give a definition of a transition system which is equivalent to those in [Keller 72] [Keller 76] using the notion of partial function instead of the notion of binary relation. Our definition is then to compare with those of " a computation system" [Kasai...82].

Definition 1: A **transition system** S is a quadruplet $S = \langle E, T, h, E_o \rangle$ where E is the set of states, T is a finite set of transitions, h is a partial function from $E \times (T \cup \{\lambda\})$ into the power set of E and E_o is a subset of E which elements are called initial states.

In the following we'll put **three hypotheses** about transitions systems:
1) the partial function h is defined from $E \times (T \cup \{\lambda\})$ into E. This restriction allows to apply Koenig's lemma (every infinite tree with a finite degree contains an infinite branch).
2) E_0 has an unique element.
3) E is a countable set.

As in [Kasai...82], the partial function h is changed into a total function h_0 ; we add a new element to E, denoted by $\perp$, which is the least element of E.We suppose now that E contains $\perp$. We define h_0 : $E \times (T \cup \{\lambda\}) \longrightarrow E$ in the following way: $h_0(e,t)$ = **if** h(e,t) is defined **then** h(e,t) **else** $\perp$.
We abuse the notation by denoting h_0 by h. We extend in a natural way the function h to a morphism h: $E \times T^* \longrightarrow E$ such that: $\forall e \in E \ \ \forall x \in T^* \ \ \forall t \in T \quad h(e,xt) = h(h(e,x),t)$.

$e \xrightarrow{x} e'$ or $e \xrightarrow{*} e'$ means that there exists $x \in T^*$ such that h(e,x)=e'.
We say that a transition t of a transition system $S = \langle E, T, h, e \rangle$ is **fireable** from the state e when $h(e,t) \neq \perp$. The state e' = h(e,t) is said to be **reachable** from e by firing t . The **reachability set**, R(S), from the initial state e_o, is defined by $R(S) = \{e \in E \ / \ \exists x \in T^* \ \ h(e_o,x) = e\} \cup \{e_o\}$.We associate to the system S a **reachability tree** RT(S) and a **reachability graph** RG(S). The **language** of a transition system S is defined by $L(S) = \{x \in T^* \ / \ h(e_o,x) \neq \perp\}$. L(S) is the set of words labelling a branch in the tree of states of S. We denote by L(S,e) the language $\{x \in T^* \ / \ h(e,x) \neq \perp\}$. We have $L(S) = L(S,e_o)$.
The analysis of a transition system consists in verifying some properties on this system. The following properties are well known and can be found in [Peterson 81][Brams 83]. A reachable state e of S is a **deadlock state** if in this state no transition is fireable, i.e., $\forall t \in T \quad h(e,t) = \perp$. A transition t is **quasi-live** if it is fireable at least once, i-e, $\exists e \in Acc(S) \quad h(e,t) \neq \perp$. A transition t is **live** if it is quasi-live for every reachable state, i-e,$\forall e \in Acc(S) \ \ \exists e' \in Acc((S,e)) \quad h(e',t) \neq \perp$. The system S is **quasi-live**, respectively **live**, if all transitions are quasi-live, respectively live.

Definition 2: [Birkoff 67] A **quasi-ordering** $\leq$ on a set E is a reflexive and transitive relation. An ordered set is **directed** when it contains a least element and every increasing sequence has a upper bound. A function $f : A \longrightarrow B$ where A and B are two directed ordered sets is **continuous** when it commutes with the upper bounds of increasing sequences.

Petri nets allow to give a finite representation of certain infinite transition systems. Petri nets realize a good compromise between the expression power and the possibility of analysis.

Definition 3: [Peterson 81] A **Petri net** is a triple R = (P, T, V) where P is a finite set of places, T is a finite set of transitions ($P \cap T = \emptyset$), V is a function from $(P \times T) \cup (T \times P)$ to **N**. We call a **marking** of R every function M from P to N. A marking is often represented as a vector with cardinal(P) components. A marked Petri net is a couple (R, M_0) where M_0 is the initial marking. A marked Petri net defines a transition system S = <E, T, h, M_0> where h is defined if and only if for every place $p \in P$ we have $M(p) \geq V(p, t)$. When h(M,t) is defined, we compute h(M,t) with the following equation: $h(M, t) = M + V(t, .) - V(., t)$ where V(t, .) and V(., t) are vectors whose the i-th component are equal to $V(t, p_i)$ and $V(p_i, t)$.
We often denote, for Petri nets and their extensions (Fifo nets, finite automata communicating by Fifo channels ...), h(M,x)=M' by M(x>M'.

3. REDUCTION OF TRANSITION SYSTEMS

In the general case (when the set of reachable states is infinite) no algorithm does exist verifying traditional properties and using the (infinite) graph of states. But some of these algorithms exist in the framework of particular transition systems with an infinite set of reachable states. For example:
- Petri nets [Peterson 81],[Brams 83],[Hack 75],[Valk...85]
- monogeneous Fifo nets [Finkel 86],[Memmi...85]
- free choice Fifo nets [Finkel 86]
- linear fifo nets [Choquet...87]
- some finite automata communicating by Fifo channels [Rosier...85],[Gouda...86]

We remark that the analysis of the infinite case is often done by reducing the problem to another finite case; then we contract the set of states modulo an equivalence relation. First we are going to illustrate this idea with finite automata; we present two possible reductions of the set of states. In both cases we show that there is a quasi-ordering which induces the two equivalence relations. Let A be the following automaton:

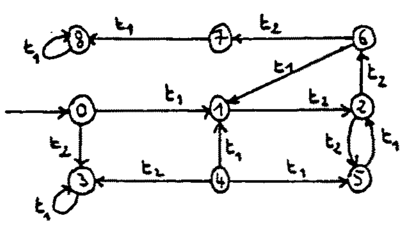

Figure 1

What can we do to reduce the number of states of this automaton and preserve its language ? We remark that: $L(A,0) \subseteq L(A,4)$; $L(A,0) = L(A,6)$; $L(A,3) = L(A,7) = L(A,8)$ where L(A,0) is the set of fireable sequences from the state 0. Let us define $\leq_1$ by: $e \leq_1 e' \Leftrightarrow L(A,e) \subseteq L(A,e')$.
We verify that $\leq_1$ is a quasi-ordering on the set E of states. In defining the equivalence relation $\equiv_1$ naturally associated to $\leq_1$ by: $e \equiv_1 e' \Leftrightarrow e \leq_1 e' \leq_1 e$. We get an equivalence relation such that $E/\equiv_1$ is the set E_1 of states of the reduced finite automaton A_1. We have $E/\equiv_1 = \{0,1,2,3,4,5\}$; the automaton A_1 is the following:

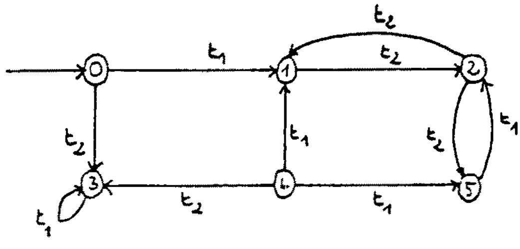

Figure 2

We verify that $L(A) = L(A_1)$. Can we reduce the set of states anymore ? It depends on the properties we want to preserve. Let us suppose we wish to test whether the automaton A_1 is live. Let $\leq_2$ be the quasi-ordering defined by: $e \leq_2 e' \Leftrightarrow$ every word of $L(A_1,e)$ is a subword of a word in $L(A_1,e')$.
Let us recall that $a_1a_2...a_p$ is a subword of any word in $A^*a_1A^*a_2A^*...A^*a_pA^*$. We associate to this quasi-ordering an equivalence relation such that: $e \equiv_2 e' \Leftrightarrow e \leq_2 e' \leq_2 e$. We obtain a new finite automaton A_2 (Figure 3) such that $E_2 = E_1/\equiv_2 = \{0,1,3,4\}$.

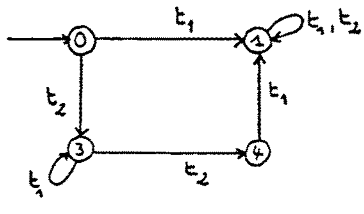

Figure 3
These two finite automata satisfy the following equation: $L(A_1) \subseteq L(A_2) \mid L(A_1)$ where $L|L'$ means that every word x in L is a subword of a word x' in L'.Then A_1 is live if and only if A_2 is live. A_1 is an automaton with nine states and A_2 is an automaton with four states. Let us remark that $\leq_1$ and $\leq_2$ are decidable, so we may compute the two equivalence relations $\equiv_1$ and $\equiv_2$. In the general case this is impossible. Now let us consider an infinite transition system. The Petri net (Figure 4) has an infinite reachability.

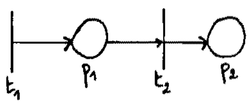

Figure 4
Let us begin the reachability graph (Figure 5) of this Petri net:

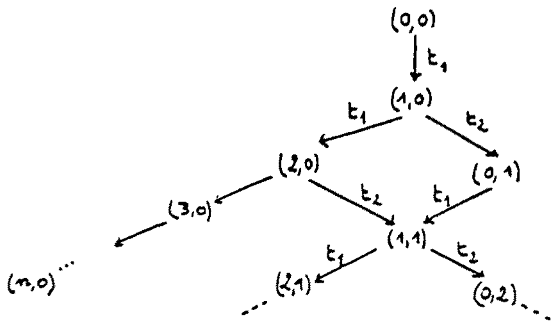

Figure 5
One of the nice properties of Petri nets is that the usual ordering $\leq$ on vectors of integers is included into the relation $\leq_1$ we have defined; we can show that the converse is false by the following couterexample :(0,1) $\leq_1$ (0,0) is verified in the Petri net (Figure 4) but it is not true that $(0,1) \leq (0,0)$. The inclusion of $\leq$ into $\leq_1$ is a nice property because $\leq_1$ is undecidable [Hack 76], [Peterson 81], [Vidal-Naquet 81] but $\leq$ is

decidable. What kind of information is given by the relation $\leq$? Since we can fire t_1 from (0,0) and reach (1,0) the infinite sequence t_1^{ω} is fireable. If we carry on the reachability tree a little bit more, we obtain that $(t_1t_2)^{\omega}$ is also an infinite fireable sequence because we reach a state (0,1) larger than (0,0) by firing t_1t_2. We are going to formalize this notion of monotonicity.

4. STRUCTURED TRANSITION SYSTEMS

One of the reasons which allows to decide the deadlock freedom, quasi-liveness and liveness of a Petri net is the existence of an ordering on vectors of integers; this relation gives a monotonous net in the following sense: from a marking $M' \geq M$ one can fire at least all the sequences of transitions fireable from M. This notion of monotonicity has never been given in general, but only in the particular case of Petri nets [Brams 83]. We propose to distinguish three classes of monotonous transition systems.

Definition 4: An ordered transition system $(S, \leq)$ is a transition system $S = < E, T, h, e_o >$, such that the quasi-ordering $\leq$ on E, admits $\perp$ as the least element of E.

We define six kind of monotonicities which generalize the (unique) notion of monotonicity existing from Petri nets.

Definition 5 : Let $(S, \leq) = (< E, T, h, e_o >, \leq)$ be an ordered transition system . We say that $(S, \leq)$ is

1-monotonous if and only if $\forall e,e' \in E \quad \forall t \in T \quad e \leq e' \Rightarrow h(e,t) \leq h(e',t)$.

1'-monotonous if and only if $\forall e,e' \in E \quad \forall t \in T \quad h(e,t) \neq \perp$ et $e < e' \Rightarrow h(e,t) < h(e',t)$.

2-monotonous if and only if $\forall e,e' \in E \ \forall t \in T \quad e \leq e' \Rightarrow \exists x,y \in T^* \quad h(e,t) \leq h(e',xty)$.

2'-monotonous if and only if $\forall e,e' \in E \ \forall t \in T \ (h(e,t) \neq \perp$ et $e < e') \Rightarrow \exists x,y \in T^* \quad h(e,t) < h(e',xty)$.

3-monotonous if and only if $\forall e,e' \in E \quad \forall t \in T \quad e \leq e' \Rightarrow \exists x \in T^+ \quad h(e,t) \leq h(e',x)$

3'-monotonous if and only if $\forall e,e' \in E \quad \forall t \in T \quad h(e,t) \neq \perp$ et $e < e' \Rightarrow \exists x \in T^* \quad h(e,t) < h(e',x)$.

1'-monotonicity is the used notion of monotonicity defined for Petri nets. 1-monotonicity is a little bit more general. 2 and 2'-monotonicities mean that a transition t is fireable from a large marking but only after a finite delay. 3 and 3'-monotonicities are the most general definitions that allow to obtain algorithms for testing usual properties. We have the following implications:

$$\begin{array}{ccccc} 1 & \Rightarrow & 2 & \Rightarrow & 3 \\ \Uparrow & & \Uparrow & & \Uparrow \\ 1' & \Rightarrow & 2' & \Rightarrow & 3' \end{array}$$

Example 1 : Let N be the marked Petri net (Figure 6).

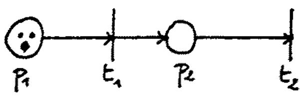

Figure 6

Its associated transition system S(N) is defined by the finite reachability graph (Figure 7)

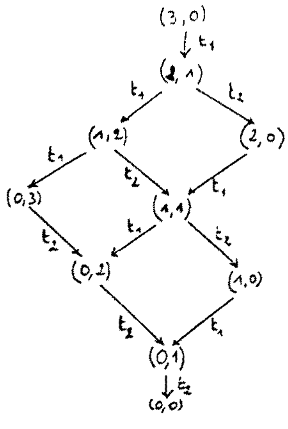

Figure 7

We have $S(N) = <E, T, h, e_o>$ with $E = \{(3, 0), (2, 1), (1, 2), (2, 0), \dots (0, 0)\}$; $T = \{t_1, t_2\}$; $h(e, t) = e' \Leftrightarrow e(t > e'$; $e_o = (3, 0)$. We verify that $(S,\leq)$ is 1'-monotonous for the usual ordering on $\mathbb{N}^2$. In fact, every Petri net is 1'-monotonous for this ordering.

We shall need to be able to decide if a state e' is larger than a state e or if they are equal. We remark that equality is decidable when the quasi-ordering is decidable and when it is an ordering (antisymetric).

Remark : the usual ordering on $\mathbb{N}^p$, $p \leq 1$, is a decidable well-ordering [Dickson 13] . The subword relation is a well-ordering on the set A* of finite words [Higman 52]. However the left factor relation is not a well-ordering on A* (if $|A| \geq 2$). As a matter of fact, for example, the sequence $(a^n b)_{n\geq 0}$ does not contain any increasing subsequence for the left factor ordering. We can now define a structure on ordered transition systems.

Definition 6: An i-structured transition system $(S,\leq)$, $i \in \{1,1',2, 2',3,3'\}$, is an ordered system $(S,\leq)$ such that $(S,\leq)$ is i-monotonous, $\leq$ is a well quasi-ordering, $\leq$ and $=$ are decidable.
When a transition system is i-structured for an $i \in \{1,1',2, 2', 3, 3'\}$, we say that it is **structured**.

Proposition 1: The transition system S(N), associated to the Petri net N with the usual ordering on $\mathbb{N}^p$ (p is the number of places of N), is 1'-structured.

We introduce a new tree, called reduced tree, associated to a structured transition system. This reduced tree allows to decide the finiteness of the reachability set of a 3'-structured transition system and also the finiteness of the language of a 3-structured transition system.

Definition 7: Let $(S,\leq)$ be an ordered transition system. We call **reduced tree** of $(S,\leq)$, denoted by $RT(S,\leq)$ or RT(S), the tree defined in the following way.
1) The root r is labelled by the initial state e_o.

2) A node s, labelled by e, $e \in E$, has no successor if and only if either for every transition t, $h(e,t) = \perp$, or there exists a node s_1 different from s on the branch from r to s , labelled by e_1 with $e_1 \leq e$.

3) If s is a node labelled by $e \in E$, which does not satisfy condition 2) then for every transition t such that $h(e,t) = e'$, there exists a node s', successor of s, labelled by e'. The arc (s,s') is labelled by t.
The **reduced graph** RG(S) is obtained from the reduced tree by identifying nodes which have the same label

Proposition 2: The reduced tree of an ordered transition system $(S,\leq)$ such that $\leq$ is a well quasi-ordering on E is finite.

Proof: Let us suppose the contrary and suppose the reduced tree to be infinite. As the reachability tree of S has a finite degree, it is the same thing for the reduced tree.Then we can apply Koenig's lemma. Let r, s_1, s_2, ..., s_n, ... the nodes of this infinite branch and e_0, e_1, e_2, ..., e_n, ... the sequence of corresponding labels. By hypothesis $\leq$ is a well quasi-ordering on E so it exists an infinite increasing subsequence $\{e_{np}\}$ such that: for each $i \geq o$ $e_{ni} \leq e_{ni+1}$. But there is a contradiction with the definition of the reduced tree of S; so the reduced tree is finite.

Theorem 1: The finiteness of the language of a 3-structured transition system $(S,\leq)$ is a decidable problem.

Proof:

$$L(S) \text{ infinite } \Leftrightarrow \exists e, e' \in E \;\; e \xrightarrow{*} e' \text{ et } e \leq e'.$$

That the right implies the left one is a consequence of 3-monotonicity.For the converse, let us suppose that L(S) is infinite. If E is infinite then there exists a state e such that:

$$e \xrightarrow{*} e \text{ in } RG(S).$$

Let us consider the case where E is infinite; since the reachability tree of S has a finite degree, there is an infinite branch issued from the root r, by Koenig's lemma. By hypothesis, $\leq$ is a well quasi-ordering, hence there exists two elements e_n and e_p such that: $e_o \rightarrow e_n \rightarrow e_p$ and $e_n < e_p$. By definition of the reduced graph we have:

$$\exists\, e, e' \in E \;\; e \xrightarrow{*} e' \text{ and } e \leq e'$$

which is equivalent to

$$\exists e, e' \in RG(S) \;\; e \xrightarrow{*} e' \text{ and } e \leq e'.$$

Then the finiteness of L(S) is a decidable problem.•

Theorem 2: The finiteness of the reachability set of a 3'-structured transition system is a decidable problem.

Proof: we make the same reasonnement than for Theorem 1 [Finkel 86].

The problem of the quasi-liveness of a structured transition system arises naturally. The reduced tree of a structured transition system doesn't allow to decide the quasi-liveness: intuitively speaking, this tree is too small and doesn't contain enough informations about the system. To decide this property we need another tree, a coverability tree.

5. A COVERABILITY GRAPH OF A WELL STRUCTURED TRANSITION SYSTEM

Definition 8: Let $(S,\leq)$ be a structured transition system. A **coverability tree**, CT(S), of $(S,\leq)$ is a tree such that:

1) for every state e such that the word x labels a path from the root e_0 to e, there exists a node s, in the coverability tree, labelled by e', such that $e \leq e'$ and x labels a path from the root of CT(S) to s.

2) for each node in the coverability tree, labelled by e, there exists an increasing sequence of reachable states which converge to e.

We define, in an ordered set, the limit of an infinite increasing sequence as the equivalence class of this sequence for the following equivalence relation:

Definition 9: Let $(E,\leq)$ be an ordered set and $E^{\geq}$ the set of infinite increasing sequences of elements of E. A sequence $\{x_n\}$, $x_n \in E$, is **inferior** to a sequence $\{y_n\}$ if and only if for every $n \geq 0$ there exists $p \in \mathbf{N}$ such that $x_n \leq y_p$. We also denote this relation by $\leq$. Two sequences $\{x_n\}$ and $\{y_n\}$are **equivalent** when

$$\{x_n\} \approx \{y_n\} \;\Leftrightarrow\; \{x_n\} \leq \{y_n\} \text{ et } \{y_n\} \leq \{x_n\}.$$

The relation $\leq$ on $E^{\geq}$ is a quasi-ordering .

Notation: For every infinite increasing sequence $\{x_n\}$ we denote by $\lim x_n$ the equivalent class of $\{x_n\}$ for the equivalent relation $\approx$. We denote by lim E the quotient $E^{\geq}/_{\approx}$. There is a canonical injection from E into lim E: to an element $e\in E$, we associate the equivalent class of the stationary sequence $\{e_n\}$ defined by : for every $n\geq 0$, $e_n=e$.

Remark : The quasi-ordering $\leq$ defined on E can be extended on lim E in the following way:
1) For every $n \geq 0$ $x_n \leq \lim x_n$.
2) If $\{x_n\} \leq \{y_n\}$ then $\lim x_n \leq \lim y_n$.
3) If $\{\perp\}$ is the stationary sequence always equal to $\perp$ one has, for each element $e\in \lim E$, $\perp \leq e$.

The following result is well known.

Proposition 3: [Birkhoff 67] The set lim E with the quasi-ordering $\leq$ is a directed ordered set.

We put the notation $\lim E = E \cup \lim_{sc} E$ where $\lim_{sc} E$ is the set of limits of strictly increasing sequences of elements of E.

This will allow us to generalize the procedure for constructing a coverability tree first defined by Karp and Miller in the framework of parallel program schemata [Karp...69], to a general class of structured transition systems. To generalize the procedure of Karp and Miller, we need four conditions about the transition system:
1) $(S,\leq)$ is 1'-structured
2) the well quasi-ordering $\leq$ on E is still a well quasi-ordering on $\lim_{sc} E$; hence also on lim E.
3) there exists an integer k such that $\lim_{sc}{}^k E$ is finite ($\lim_{sc}{}^{n+1} E = \lim_{sc}(\lim_{sc}{}^n E)$ and $\lim_{sc}{}^1 E = \lim_{sc} E$)
4) the limit of a strictly increasing sequence is computable ; it means that for all n, the n-th term of this sequence is computable from its two first elements.

Definition 10: A structured transition system $(S,\leq)$ is **well structured** if and only if the four following conditions are satisfied:
1) $(S,\leq)$ is 1'-structured; 2) $\leq$ is a well ordering on $\lim_{sc} E$; 3) there exists an integer k such that $\lim_{sc}{}^k E$ is finite; 4) the limit of a strictly increasing sequence is computable.

We have to use the **continuous extension** of a function h of a 1'-structured transition system $(S,\leq) = (E, T, h, e_o, \leq)$. We still denote this extension by h. $h:(\lim E)x(T\cup\{\lambda\}) \longrightarrow \lim E$ with $h(\lim e_n,t) = \lim h(e_n,t)$.

Definition 11: The **Karp and Miller's tree**, KMT(S), of a well structured transition system is built by the following procedure:
1) The root is labelled by the initial state e_o.
2) A node s, labelled by e, $e\in E$, has no successor if and only if either for every transition t, $h(e,t)=\perp$ or there exists a node $s_1\neq s$ also labelled by e, on the path from r to s.
3) **If** s is a node labelled by e, $e\in E$, which does not satisfy condition 2) **then** for every transition $t\in T$ such that $h(e,t)\neq\perp$ there exists a node s', successor of s and labelled by e' such that the arc (s,s') is labelled by t. **If** there exists a node s_1, on the path from r to s, labelled by e_1 such that $e_1 < h(e,t)$, let us call s_1 the first node from the root satisfying this condition, **then** $e' = \lim u_n$ with $u_1 = e_1$, $u_2 = h(e, x)$, $u_{n+1} = h(u_n,x)$ $\forall n\in \mathbf{N}$ **else** $e' = h(e,t)$.
The **Karp and Miller's graph** ,GKM(S), is obtained by identifying nodes with the same label.

Let us show that the Karp and Miller's tree is a coverability tree.

Theorem 3: Let $(S,\leq)$ be a well structured transition system. Then KMT(S) is a coverability tree of $(S,\leq)$.

Proof: The complete proof is technical and will be done in the full paper.

We abuse the definition in calling coverability tree, the Karp and Miller's tree.

Example 2: consider the net of Figure 8, which is "almost" a Petri net except that f is a Fifo queue.

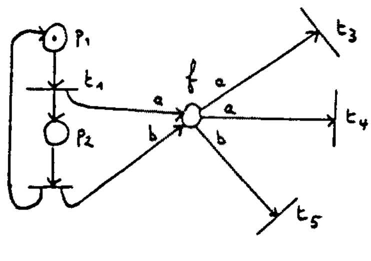

Figure 8

A state of this net is here a triple e=(integer,integer,word). Let << be the ordering defined by $(n_1,n_2,m) = e \ll e' = (n'_1,n'_2,m')$ if and only if $n_1=n'_1$, $n_2=n'_2$ and $\forall x \in L(S,e)$ $m\varphi_f(x) \leq m'\varphi_f(x)$ where $\varphi_f(t)=V(t,f)$.

One can then verify that (S,<<) is well structured. Let us draw its coverability tree (Figure 9).

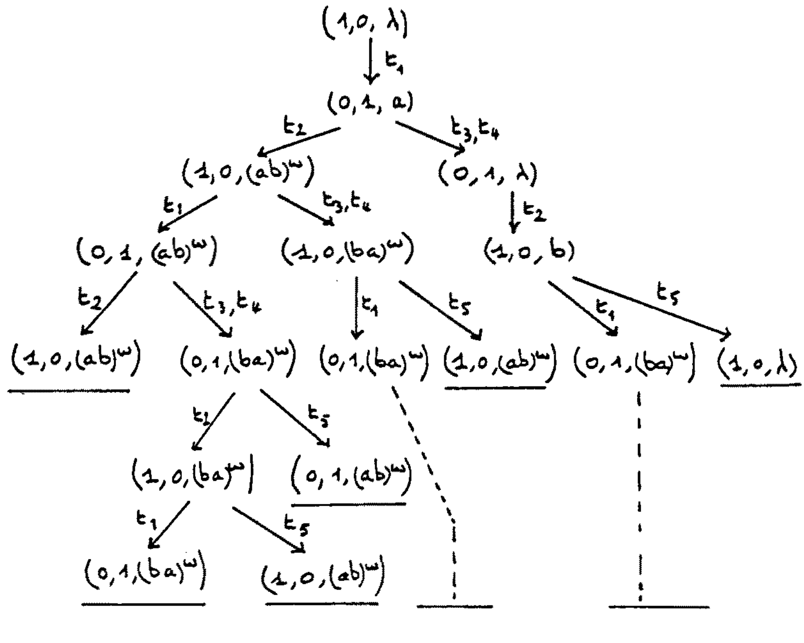

Figure 9

Proposition 4: The coverability tree and then the coverability graph of a well structured transition system are finite.

Let us formulate the two results about structured transition systems by using the coverability graph.

Theorem 1': Let $(S,\leq)$ be a well structured transition system. The reachability set E of S is infinite if and only if there exists at least an element of $\lim_{sc} E$ in CG(S).

Theorem 2': The language L(S) of a well structured transition system is infinite if and only if there exists a circuit in CG(S).

Theorem 4: The quasi-liveness of a well structured transition system is a decidable problem.

Proof: We show that a transition $t \in T$ is fireable at least once if and only if it appears in the coverability graph of the system. If there exists a state e such that $h(e, t) \neq \perp$ then part 1) of theorem 3 provids with $f \in \lim E$ such that $h(f, t) \neq \perp$. Let us now consider an element $e \in \lim E$ such that $h(e, t) \neq \perp$. If $e \in E$, then $h(e, t) \neq \perp$. If $e \in \lim_{sc} E$, part 2) of theorem 3 shows that there is a strictly increasing infinite sequence $\{e_n\}$ such that $\lim e_n = e$ then

$$h(e, t) \neq \perp \Leftrightarrow h(\lim e_n, t) \neq \perp \Leftrightarrow \lim h(e_n, t) \neq \perp \Rightarrow \exists k \in \mathbf{N} \quad h(e_k, t) \neq \perp .$$

and t is fireable from the reachable state e_k.

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