

# Verification of qualitative $\mathbb{Z}$ constraints

Stéphane Demri and Régis Gascon

LSV, ENS Cachan, CNRS, INRIA  
email: {demri, gascon}@lsv.ens-cachan.fr

**Abstract.** We introduce an LTL-like logic with atomic formulae built over a constraint language interpreting variables in  $\mathbb{Z}$ . The constraint language includes periodicity constraints, comparison constraints of the form  $x = y$  and  $x < y$ , is closed under Boolean operations and admits a restricted form of existential quantification. Such constraints are used for instance in calendar formalisms or abstractions of counter automata by using congruences modulo some power of two. Indeed, various programming languages perform arithmetic operators modulo some integer. We show that the satisfiability and model-checking problems (with respect to an appropriate class of constraint automata) for this logic are decidable in polynomial space improving significantly known results about its strict fragments. This is the largest set of qualitative constraints over  $\mathbb{Z}$  known so far, shown to admit a decidable LTL extension.

## 1 Introduction

*Model-checking infinite-state systems.* The verification of systems with an infinite number of states has benefited from the numerous decidable model-checking problems for infinite-state systems, including timed automata [AD94], infinite transition graphs [MS85, Cau03], or subclasses of counter systems (see e.g. [CJ98]). There exist numerous techniques to prove decidability such as finite partition of the infinite domain, well-structured systems, Presburger definable reachability sets or reduction to the monadic second-order theory of the binary tree... Since it is often possible to find a reduction from the halting problem for Minsky machines [Min67], undecidability is often easy to prove. Decidability can sometimes be regained by naturally restricting the class of models (see e.g. the flatness condition in [CJ98, FL02]) or by considering fragments of the specification language (for instance reachability questions). Symbolic representations of infinite sets of states are often the key argument to get decidability (see e.g. [HMR05]).

*Systems with variables interpreted in  $\mathbb{Z}$ .* Counter machines are operational models that have found numerous applications in the verification of infinite-state systems, including broadcast protocols (see e.g. [EFM99, FL02]) and programs with pointer variables [BFLS06, BBH<sup>+</sup>06]. They consist of a structure with a finite set of control states augmented with a finite set of variables interpreted either in  $\mathbb{Z}$  or  $\mathbb{N}$  (counters). Though this class of automata has numerous undecidable model-checking problems such as the reachability problem, many subclasses have been shown to be decidable:

1. reversal-bounded multcounter machines [Iba78],
2. flat counter systems with affine update functions forming a finite monoid (see e.g. [Boi98,FL02,BFLP03]),
3. flat counter systems [CJ98,BIL06] (weaker class of Presburger guards but no condition on the monoid),
4. admissible Presburger counter systems [DFGvD06],
5. constraint automata with qualitative constraints on  $\mathbb{Z}$  [DD07].

*Our motivation.* Constraint automata with qualitative constraints on  $\mathbb{Z}$  are quite attractive operational models since they can be viewed as abstractions of counter automata where increments and decrements are abstracted by operations modulo some power of two. Indeed, common programming languages perform arithmetic operators for integers modulo  $2^k$  [MOS05], where  $k$  is typically equal to 32 or 64. So, such an abstraction is well-suited to check safety properties about the original counter system. In this paper, we introduce a class of constraint automata with a language of qualitative constraints as rich as possible and a companion LTL-like logic in order to perform model-checking on these models. Our framework should be able to deal both with modulo abstractions (see e.g. [CGL94,LS01]) and with integer periodicity constraints used in logical formalisms to deal with calendars [LM01]. By qualitative constraint, we mean constraints that are interpreted as a non-deterministic binary relation, like  $x < y$  or  $x \equiv_{2^k} y + 5$  (the relationship between  $x$  and  $y$  is not sharp).

*Our contribution.* We introduce CLTL(IPC<sup>\*</sup>) as an extension of LTL over the constraint language IPC<sup>\*</sup>, whose expressions are Boolean combinations of IPC<sup>++</sup> constraints from [Dem06] and constraints of the form  $x < y$ . The language inherits from IPC<sup>++</sup> closure under Boolean operators and first-order quantification. We impose that no constraint of the form  $x < y$  occurs in the scope of a quantifier; otherwise incrementation is definable and this leads to undecidability. In this paper, we show that adding the single type of constraints  $x < y$  leads to many technical complications but not to undecidability. We also introduce the class of IPC<sup>\*</sup>-automata defined as finite-state automata with transitions labelled by CLTL(IPC<sup>\*</sup>) formula à la Wolper [Wol83]. Such structures can be viewed as labelled transition systems obtained by abstraction of counter automata.

Constraint LTL over IPC<sup>++</sup> is already known to be in PSPACE in [Dem06] whereas constraint LTL over constraints of the form either  $x = y$  or  $x < y$  is shown to be also in PSPACE in [DD07]. Though both proofs use reductions to the nonemptiness problem for Büchi automata, following the approach in [VW94], they are of different nature: in [Dem06] the complexity upper bound is obtained by a finite model property argument whereas in [DD07] approximations of classes of symbolic models are considered because some formulae can generate non  $\omega$ -regular classes of symbolic models. We show that the model-checking and satisfiability problems for the logic CLTL(IPC<sup>\*</sup>) are still PSPACE-complete which generalizes and unifies these results. We improve what is done for constraint LTL over the domain  $\langle \mathbb{Z}, <, = \rangle$  by considering both new constraints of the form  $x \leq d$

with  $d \in \mathbb{Z}$  and periodicity constraints. The optimal treatment of the constants introduced in this language is our main technical contribution. As a corollary, we establish that LTL model-checking over integral relational automata [Čer94] is PSPACE-complete. Hence, even though  $\text{IPC}^*$  is a powerful language of qualitative constraints, the PSPACE upper bound is preserved in  $\text{CLTL}(\text{IPC}^*)$ . Moreover, past-time operators can be added for free in our formalism thanks to [GK03] (the PSPACE bound is still preserved).

*Related work.* Reachability problems for subclasses of counter systems have been addressed in numerous works [Iba78,CJ98,FS00,FL02,BFLP03] (see also richer questions in [BEM97,DI02,DPK03,JKMS04]). In this paper, we consider a full LTL-like language used as a specification language which is not restricted to reachability questions, and we have no restriction on the structure of the models unlike [CJ98,DFGvD06]. However, the atomic formulae of the specification language are restricted to qualitative constraints. If we give up the decidability requirement, other extensions of LTL with Presburger constraints can be found in [BEH95,CC00,ID01].

Extensions of LTL over concrete domains, not only restricted to variables interpreted in  $\mathbb{Z}$ , have also been considered in [WZ00,BC02,DD07,GKK<sup>+</sup>03,Dem06] where often PSPACE-completeness results are shown. The idea of building LTL over a language of constraints, although already present in first-order temporal logics, stems from the use of concrete domains for description logics [BH91,Lut04]. The language  $\text{CLTL}(\text{IPC}^*)$  extends the different LTL-like fragments from the works [Čer94,LM01,Dem06].

The underlying constraint language of  $\text{CLTL}(\text{IPC}^*)$  includes integer periodicity constraints, a special class of Presburger constraints that have found applications in many logical formalisms such as abstractions with congruences modulo an integer of the form  $2^k$  (see e.g. [CGL94,MOS05]), logical formalisms dealing with calendars (see e.g. [LM01,Pup06]) and temporal reasoning in database access control [BBFS98]. Such constraints can also be found in real-time logics, see e.g. [AH94]. Our approach of constraint LTL makes explicit the constraints on variables, similarly to the explicit clock approach from [HLP90]. Furthermore, the class of  $\text{IPC}^*$ -automata we introduce generalizes the class of integral relational automata from [Čer94] (see details in Appendix A).

Finally, the concept of symbolic models used in the paper has similarities with untimed languages recognized by some classes of timed machines. In [Bér95], sufficient conditions to get regular untimed languages from timed machines are exhibited.

*Plan of the paper.* The rest of the paper is organized as follows. In Section 2, we introduce the logic  $\text{CLTL}(\text{IPC}^*)$  and the class of  $\text{IPC}^*$ -automata. We present the model-checking and satisfiability problems and discuss expressiveness issues. In Section 3, we analyze the computational complexity of the satisfiability problem of the underlying constraint language  $\text{IPC}^*$ . We also provide a symbolic representation of the valuations that is used later in the decidability proof. Section 4 contains a characterization of the sequences of symbolic valuations that

admit concrete models (valuation sequence). We show that testing the existence of some concrete model is an  $\omega$ -regular property when considering ultimately periodic sequences. In Section 5, we show that given a CLTL(IPC<sup>\*</sup>) formula  $\phi$ , one can build a Büchi automaton  $\mathcal{A}_\phi$  over the alphabet of symbolic valuations such that  $\phi$  is CLTL(IPC<sup>\*</sup>) satisfiable iff  $L(\mathcal{A}_\phi)$  is non-empty. Moreover, we establish that nonemptiness of  $L(\mathcal{A}_\phi)$  can be checked in polynomial space in  $|\phi|$ . Section 6 contains concluding remarks.

This paper is a completed version of [DG05].

## 2 The logic CLTL(IPC<sup>\*</sup>)

### 2.1 Language of constraints

Let  $\text{VAR} = \{x_0, x_1, \dots\}$  be a countably infinite set of variables (in some places for ease of presentation,  $\text{VAR}$  will denote a particular finite set of variables). The language of constraints IPC<sup>\*</sup> is defined by the following grammar:

$$\begin{aligned} \xi ::= & \theta \mid x < y \mid \xi \wedge \xi \mid \neg \xi \\ \theta ::= & x \equiv_k [c_1, c_2] \mid x \equiv_k y + [c_1, c_2] \mid x = y \mid x < d \mid x = d \mid \\ & \theta \wedge \theta \mid \neg \theta \mid \exists x \theta \end{aligned}$$

where  $x, y \in \text{VAR}$ ,  $k \in \mathbb{N} \setminus \{0\}$ ,  $c_1, c_2 \in \mathbb{N}$  and  $d \in \mathbb{Z}$ . In the following, the symbol  $\sim$  is used to mean either  $=$  or  $<$ . We write IPC<sup>++</sup> to denote the restriction of the language to constraints ranged over by  $\theta$ ,  $Z^c$  to constraints of the form either  $x \sim y$  or  $x \sim d$  and  $Z$  to constraints of the form  $x \sim y$  where  $x, y \in \text{VAR}$ ,  $d \in \mathbb{Z}$  and  $\sim \in \{<, =\}$ . A valuation  $v : \text{VAR} \rightarrow \mathbb{Z}$  is a map that associates a value to each variable and the satisfaction relation  $v \models_\star \xi$  is defined in the standard way:

- $v \models_\star x \sim y$  iff  $v(x) \sim v(y)$ ;
- $v \models_\star x \sim d$  iff  $v(x) \sim d$ ;
- $v \models_\star x \equiv_k [c_1, c_2]$  iff  $v(x) - c = kd$  for some  $c_1 \leq c \leq c_2$  and  $d \in \mathbb{Z}$ ;
- $v \models_\star x \equiv_k y + [c_1, c_2]$  iff  $v(x) - v(y) - c = kd$  for some  $c_1 \leq c \leq c_2$  and  $d \in \mathbb{Z}$ ;
- $v \models_\star \xi \wedge \xi'$  iff  $v \models_\star \xi$  and  $v \models_\star \xi'$ ;
- $v \models_\star \neg \xi$  iff  $v \not\models_\star \xi$ ;
- $v \models_\star \exists x \xi$  iff there is  $d \in \mathbb{Z}$  such that  $v[x \leftarrow d] \models_\star \xi$   
where  $v[x \leftarrow d](x') = v(x')$  if  $x \neq x'$  and  $v[x \leftarrow d](x) = d$ .

We will shortly write  $x \equiv_k c$  instead of  $x \equiv_k [c, c]$  and  $x \equiv_k y + c$  instead of  $x \equiv_k y + [c, c]$ . Given a set of IPC<sup>\*</sup>-constraints  $X$ , we note  $v \models_\star X$  whenever  $v \models_\star \xi$  for every  $\xi \in X$ . A constraint  $\xi$  is satisfiable iff there is a valuation  $v$  such that  $v \models_\star \xi$ . Two constraints are equivalent iff they are satisfied by the same valuations.

**Lemma 1.** (I) *The satisfiability problem for  $\text{IPC}^*$  is PSPACE-complete.*  
 (II) *For every constraint in  $\text{IPC}^*$  there exists an equivalent quantifier-free constraint in  $\text{IPC}^*$ .*

*Proof.* (I) Satisfiability for  $\text{IPC}^{++}$  is PSPACE-complete [Dem06] whereas satisfiability for Z is NLOGSPACE-complete. Since constraints in  $\text{IPC}^*$  are Boolean combinations of  $\text{IPC}^{++}$  and Z constraints,  $\text{IPC}^*$  satisfiability is in PSPACE by simply adapting the proof of [Dem06, Theorem 3]. PSPACE-hardness is a consequence of the PSPACE-hardness of  $\text{IPC}^{++}$ .  
 (II)  $\text{IPC}^{++}$  admits quantifier elimination [Dem06] and therefore so does  $\text{IPC}^*$  since Z is quantifier-free.  $\square$

## 2.2 Logical language

We consider the extension of the linear-time temporal logic LTL whose atomic formulae are defined from constraints in  $\text{IPC}^*$  (denoted by  $\text{CLTL}(\text{IPC}^*)$ ). So, the language includes boolean operators as well as the classical temporal operators next (X) and until (U) of LTL. The atomic formulae are of the form  $\xi[x_1 \leftarrow X^{l_1}x_{j_1}, \dots, x_r \leftarrow X^{l_r}x_{j_r}]$ , where  $\xi$  is a constraint of  $\text{IPC}^*$  with free variables  $x_1 \dots x_r$ . We substitute each occurrence of the variable  $x_i$  by  $X^{l_i}x_{j_i}$ , which corresponds to the variable  $x_{j_i}$  preceded by  $l_i$  next symbols. Each expression of the form  $X^l x_j$  is called a term and represents the value of the variable  $x_j$  at the  $l^{\text{th}}$  next state. For example,  $Xy \equiv_{2^{32}} x + 1$  and  $x < Xy$  are atomic formulae of the logic.

The set of  $\text{CLTL}(\text{IPC}^*)$  formulae  $\phi$  is defined by the grammar below

$$\phi ::= \xi[x_1 \leftarrow X^{i_1}x_{j_1}, \dots, x_r \leftarrow X^{i_r}x_{j_r}] \mid \neg\phi \mid \phi \wedge \phi \mid X\phi \mid \phi U \phi,$$

where  $\xi$  belongs to  $\text{IPC}^*$ . The integers are encoded with a binary representation (this is important for complexity considerations).

A one-step constraint is a constraint where all the terms are of the form  $x$  and  $Xx$  only (for some  $x \in \text{VAR}$ ). Given a set of constraints  $X$  included in  $\text{IPC}^*$ , we write  $\text{CLTL}(X)$  to denote the restriction of  $\text{CLTL}(\text{IPC}^*)$  in which the atomic constraints are built over elements of  $X$ .

A model  $\sigma : \mathbb{N} \times \text{VAR} \rightarrow \mathbb{Z}$  for  $\text{CLTL}(\text{IPC}^*)$  is an  $\omega$ -sequence of valuations. The satisfaction relation is defined as follows:

- $\sigma, i \models \xi[x_1 \leftarrow X^{i_1}x_{j_1}, \dots, x_r \leftarrow X^{i_r}x_{j_r}]$   
 iff  $[x_1 \leftarrow \sigma(i + i_1, x_{j_1}), \dots, x_r \leftarrow \sigma(i + i_r, x_{j_r})] \models_{\star} \xi$ ;
- $\sigma, i \models \phi \wedge \phi'$  iff  $\sigma, i \models \phi$  and  $\sigma, i \models \phi'$ ;
- $\sigma, i \models \neg\phi$  iff  $\sigma, i \not\models \phi$ ;
- $\sigma, i \models X\phi$  iff  $\sigma, i + 1 \models \phi$ ;
- $\sigma, i \models \phi U \phi'$  iff there is  $j \geq i$  s.t.  $\sigma, j \models \phi'$  and for every  $i \leq l < j$ ,  $\sigma, l \models \phi$ .

### 2.3 Satisfiability and model-checking problems

We define below the problems we consider in this paper. The definition of the satisfiability problem is standard.

**Satisfiability problem :**

Given a CLTL(IPC<sup>\*</sup>) formula  $\phi$ , is there a model  $\sigma$  such that  $\sigma, 0 \models \phi$ ?

Note that if we extend IPC<sup>\*</sup> to allow constraints of the form  $x < y$  in the scope of an existential quantifier, then the satisfiability problem for the corresponding extension of CLTL(IPC<sup>\*</sup>) is undecidable since the relation  $x = y + 1$  is then definable and the halting problem for Minsky machines can be easily encoded.

The model-checking problem which is defined in a moment rests on a particular class of constraint automata [Rev02]. An IPC<sup>\*</sup>-automaton is defined as a Büchi automaton over a finite alphabet made of CLTL(IPC<sup>\*</sup>) formulae. In such an automaton, letters on transitions may induce constraints between the variables of the current state and the variables of the next state as done in [CC00]. Hence, guards and update functions are expressed in the same formalism. We are however a bit more general since we allow formulae on transitions as done in [Wol83]. Formally, an IPC<sup>\*</sup>-automaton is a structure  $\mathcal{A} = \langle Q, I, F, \delta \rangle$  such that:

- $Q$  is a finite set of locations,
- $I \subseteq Q$  is a set of initial locations and  $F \subseteq Q$  a set of final locations,
- $\delta \subseteq Q \times \Sigma \times Q$  where  $\Sigma$  is a finite set of IPC<sup>\*</sup>-constraints.

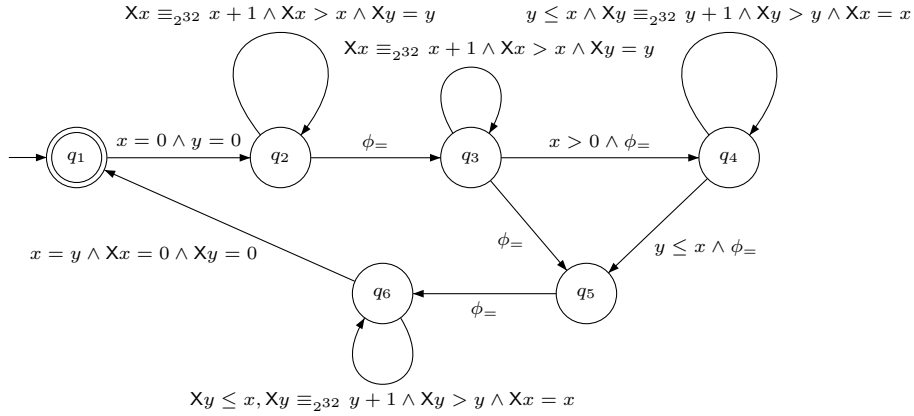
We say that  $\mathcal{A}$  is a restricted IPC<sup>\*</sup>-automaton when  $\Sigma$  is a set of Boolean combination of one-step constraints. A run is an infinite sequence  $q_0 \xrightarrow{\phi_0} q_1 \xrightarrow{\phi_1} q_2 \xrightarrow{\phi_2} \dots$  such that for every  $i \in \mathbb{N}$ ,  $\langle q_i, \phi_i, q_{i+1} \rangle \in \delta$ . Such a run is accepting iff there is a state  $q_f \in F$  such that  $q_i = q_f$  for infinitely many  $i \in \mathbb{N}$ . In this case, the word  $\phi_0 \cdot \phi_1 \cdot \phi_2 \cdot \dots$  is accepted by  $\mathcal{A}$  and we write  $L(\mathcal{A})$  to denote the language recognized by  $\mathcal{A}$  made of  $\omega$ -sequences  $\phi_1 \cdot \phi_2 \cdot \dots \in \Sigma^\omega$  obtained from accepting runs. We say that a valuation sequence  $\sigma$  realizes a word  $\phi_0 \cdot \phi_1 \cdot \phi_2 \cdot \dots$  in  $\Sigma^\omega$  iff for every  $i \geq 0$ ,  $\sigma, i \models \phi_i$ .

As an illustration, we present a (restricted) IPC<sup>\*</sup>-automaton in Figure 1 which is an abstraction of the pay-phone controller from [CC00, Example 1] ( $x$  is the number of quarters that have been inserted and  $y$  measures the total communication time). An increment of a variable  $z$  is abstracted by  $Xz \equiv_{2^{32}} z + 1 \wedge Xz > z$ . The formula  $\phi_ =$  denotes  $Xx = x \wedge Xy = y$ . Messages are omitted because they are irrelevant here (simplifications are then possible).

**Model-checking problem :**

Given an IPC<sup>\*</sup>-automaton  $\mathcal{A}$  and a CLTL(IPC<sup>\*</sup>) formula  $\phi$ , are there an  $\omega$ -word  $\phi_0 \cdot \phi_1 \cdot \dots$  accepted by  $\mathcal{A}$  and a model  $\sigma$  for  $\phi$  that realizes  $\phi_0 \cdot \phi_1 \cdot \dots$ ?

Note that the equivalence problem for Extended Single-String automata [LM01] can be encoded as a model-checking problem for CLTL(IPC<sup>\*</sup>) (see [Dem06]).



**Fig. 1.** A restricted  $\text{IPC}^*$ -automaton

The satisfiability problem and the model-checking problem are reducible to each other in logspace following techniques from [SC85]. Indeed, satisfiability can be seen as a particular case of model-checking problem since one can build an  $\text{IPC}^*$ -automaton such that every valuation sequence realizes some execution of this automaton (consider for instance the automaton that accepts the sequence  $\top^\omega$ ). The converse reduction relies on a standard encoding of the executions of an  $\text{IPC}^*$ -automaton by a  $\text{CLTL}(\text{IPC}^*)$  formula, possibly introducing a new variable to encode the control states of the automaton. In the following, we only refer to the satisfiability problem but the results we prove also hold for the model-checking problem.

Let  $\text{CCTL}^*(\text{IPC}^*)$  denote the  $\text{CTL}^*$  extension of  $\text{CLTL}(\text{IPC}^*)$ . The model-checking problem of the LTL fragment of the logic introduced in [Čer94] against integral relational automata is a subproblem of the model-checking problem for  $\text{CLTL}(\text{IPC}^*)$ . Full  $\text{CCTL}^*(\text{IPC}^*)$  model-checking can be shown to be undecidable by using developments in Appendix A and [Čer94]. This is actually true even for its  $\text{CTL}$ -like fragment. However in [BG06], it has been shown that its existential and universal fragments are decidable even though the proof does not allow to obtain any tight bound on the computational complexity of the problem. Here, we show that the LTL fragment is decidable in polynomial space, a result not captured by these two works.

## 2.4 Expressive power and conciseness of the language

By definition,  $\text{CLTL}(\text{IPC}^*)$ -models interpret variables but not propositional variables. However, it is not difficult to encode propositional variables by using atomic formulae of the form  $x = 0$  where  $x$  is a new variable introduced for this purpose. The model-checking problems for  $\text{CLTL}(\text{IPC}^{++})$  and  $\text{CLTL}(\mathbb{Z})$  are shown to be  $\text{PSPACE}$ -complete respectively in [Dem06] and in [DD07]. However, the proof for  $\text{IPC}^{++}$  uses an  $\omega$ -regular property of the set of models that

does not hold when introducing constraints of the form  $x < y$ . The problem for  $\text{CLTL}(\mathbb{Z}^c)$  is shown to be in  $\text{EXSPACE}$  in [DD07] by translation into  $\text{CLTL}(\mathbb{Z})$  that increases exponentially the size of formulae with a binary encoding of the integers.

Let  $\text{WIPC}^*$  (weak  $\text{IPC}^*$ ) denote the restriction of  $\text{IPC}^*$  to constraints of the form either  $x \sim y$ ,  $x \sim d$  or  $x \equiv_k c$  where  $x, y \in \text{VAR}$ ,  $\sim \in \{<, =\}$ ,  $d \in \mathbb{Z}$  and  $k, c \in \mathbb{N}$ . Though  $\text{WIPC}^*$  is a fragment of  $\text{IPC}^*$ , the logic  $\text{CLTL}(\text{WIPC}^*)$  is as expressive as  $\text{CLTL}(\text{IPC}^*)$ .

**Lemma 2.** *For every  $\phi \in \text{CLTL}(\text{IPC}^*)$ , there exists an equivalent formula  $\psi \in \text{CLTL}(\text{WIPC}^*)$ .*

*Proof.* This is a direct consequence of the facts below:

- $\text{IPC}^*$  admits quantifier-elimination.
- $x \equiv_k [c_1, c_2]$  is equivalent to

$$\bigvee_{c_1 \leq c' \leq c_2} x \equiv_k c$$

- $x \equiv_k y + [c_1, c_2]$  is equivalent to

$$\bigvee_{c_1 \leq c' \leq c_2} \left( \bigvee_{\{(c'_1, c'_2) \in \{0, \dots, k-1\}^2 \mid c'_1 + c'_2 \equiv_k c'\}} (x \equiv_k c'_1 \wedge y \equiv_k c'_2) \right).$$

□

The size of  $\psi$  is exponential in the size of  $\phi$  in the worst case (for an infinite amount of formulae  $\phi$ ). In spite of this exponential blow-up, we shall prove that both  $\text{CLTL}(\text{WIPC}^*)$  and  $\text{CLTL}(\text{IPC}^*)$  have  $\text{PSPACE}$ -complete model-checking problems.

Note also that adding constraints of the form  $ax + by \equiv_k c$  with  $a, b, c \in \mathbb{Z}$  in  $\text{CLTL}(\text{IPC}^*)$  does not add expressiveness since we can translate such constraints in  $\text{CLTL}(\text{IPC}^*)$ . Let  $S = \{\langle c_x, c_y \rangle \in \{0, \dots, k-1\}^2 \mid c_x + c_y \equiv_k c\}$ . We have the following logical equivalence:

$$ax + by \equiv_k c \Leftrightarrow \bigvee_{\langle c_x, c_y \rangle \in S} (ax \equiv_k c_x \wedge by \equiv_k c_y).$$

Then we can translate the constraints of the form  $ax \equiv_k c_x$  into  $\text{CLTL}(\text{IPC}^*)$  by solving a simple diophantine equation. The constraint  $ax \equiv_k c_x$  reduces either to false if  $\text{gcd}(a, k)$  does not divide  $c_x$  or to  $x \equiv_{k'} c'$  with  $k' \times \text{gcd}(a, k) = k$  for some  $c'$  that can be computed in polynomial-time in the respective sizes of  $a$ ,  $k$  and  $c_x$ . The addition of such constraints may cause a gain of conciseness. However, because the sizes of  $k'$  and  $c'$  are bounded by the maximum of the sizes of  $a$ ,  $k$  and  $c$  and  $k$  is a multiple of  $k'$ , the forthcoming  $\text{PSPACE}$  upper bounds for  $\text{CLTL}(\text{IPC}^*)$  problems can be also obtained when constraints of the form  $\sum_i a_i x_i \equiv_k c$  are added.



### 3 Properties of the constraint language

In this section, we establish results about the constraint language underlying the logic CLTL(IPC<sup>\*</sup>). We define a symbolic representation of the valuations in order to build automata that recognize symbolic representations of CLTL(IPC<sup>\*</sup>)-models. Given a finite set  $X$  of IPC<sup>\*</sup> constraints, typically the set of constraints occurring in a given CLTL(IPC<sup>\*</sup>) formula, we introduce the following notations:

- $K$  is the least common multiple of the integers  $k_1, \dots, k_n$  such that periodicity constraints with relations  $\equiv_{k_1}, \dots, \equiv_{k_n}$  occur in  $X$ .
- $\text{CONS}$  is the finite set of constants  $d$  occurring in the constraints of  $X$  of the form  $x \sim d$  (where  $\sim \in \{<, =\}$ ).
- $m$  is the minimal element of  $\text{CONS}$  and  $M$  is its maximal element.
- $\text{CONS}'$  denotes the set of constants  $\{m, m+1, \dots, M\}$ .
- $\text{VAR}$  is the finite set of variables occurring in  $X$ .

In the remaining, we consider that the above objects are always defined (possibly by adding dummy valid constraints in order to make the sets non-empty). In the sequel, we write  $|\mathbf{O}|$  to denote the size of the finite object  $\mathbf{O}$  for some reasonably succinct encoding (in particular binary encoding of integers). Observe that  $|K|$  is in  $\mathcal{O}(|k_1| + \dots + |k_n|)$  and the cardinality of  $\text{CONS}$  is polynomial in the size of  $X$ . The cardinality of  $\text{CONS}'$  is in  $\mathcal{O}(2^{|M|})$  and each element of  $\text{CONS}'$  can be encoded in binary representation with  $\mathcal{O}(|M|)$  bits.

A maximally consistent set  $Y$  of  $\mathbb{Z}^c$  constraints with respect to  $\text{VAR}$  and  $\text{CONS}$  is a set of  $\mathbb{Z}^c$  constraints using only the variables from  $\text{VAR}$  and the constants from  $\text{CONS}$  such that there is a valuation  $v$  verifying  $v \models_\star Y$  and for any proper extension  $Z$  of  $Y$ , there is no valuation  $v'$  verifying  $v' \models_\star Z$ . A valuation is abstracted by three disjoint finite sets of IPC<sup>\*</sup> constraints similar to regions for timed automata.

**Definition 1.** *Given a finite set  $X$  of IPC<sup>\*</sup> constraints, a symbolic valuation  $sv$  is a triple  $\langle Y_1, Y_2, Y_3 \rangle$  such that*

- $Y_1$  is a maximally consistent set of  $\mathbb{Z}^c$  constraints with respect to  $\text{VAR}$  and  $\text{CONS}$ .
- $Y_2$  is a set of constraints of the form  $x = d$  with  $x \in \text{VAR}$  and  $d \in \text{CONS}' \setminus \text{CONS}$ . Moreover, we impose that for every  $x \in \text{VAR}$ ,  $(x = d) \in Y_2$  for some unique  $d \in \text{CONS}' \setminus \text{CONS}$  iff for every  $d' \in \text{CONS}$ ,  $(x = d') \notin Y_1$  and  $\{m < x, x < M\} \subseteq Y_1$ . Note that, each  $x \in \text{VAR}$  occurs at most once in  $Y_2$ .
- $Y_3$  is a set of constraints of the form  $x \equiv_K c$  with  $x \in \text{VAR}$  and  $c \in \{0, \dots, K-1\}$ . Each  $x \in \text{VAR}$  occurs exactly once in  $Y_3$ .

A consequence of Definition 1 is that in a symbolic valuation  $sv = \langle Y_1, Y_2, Y_3 \rangle$ , no constraint occurs in more than one set. Given an IPC<sup>\*</sup> constraint  $\xi$ , we write  $\xi \in sv$  instead of  $\xi \in Y_1 \cup Y_2 \cup Y_3$ . A symbolic valuation is satisfiable iff there is a valuation  $v : \text{VAR} \rightarrow \mathbb{Z}$  such that  $v \models_\star Y_1 \cup Y_2 \cup Y_3$ .

**Lemma 3.** *Let  $X$  be a finite set of IPC\* constraints and  $sv = \langle Y_1, Y_2, Y_3 \rangle$  be a triple composed of IPC\* constraints such that  $Y_1$  is a set of  $Z^c$  constraints built over VAR and CONS,  $Y_2$  is a set of  $Z^c$  constraints of cardinality at most  $|\text{VAR}|$  built over VAR and  $\text{CONS}' \setminus \text{CONS}$ ,  $Y_3$  is a set of constraints of the form  $x \equiv_K c$  of cardinality  $|\text{VAR}|$ . Checking whether  $sv$  is a satisfiable symbolic valuation can be done in polynomial-time in the sum of the respective size of  $X$  and  $sv$ .*

To prove this result, we have to check several things. Maximal consistency of  $Y_1$  can be checked in polynomial-time by using developments from [Čer94, Lemma 5.5]. A set  $Y_1$  of  $Z^c$  constraints is maximally consistent w.r.t. VAR and CONS iff the associated graph  $G_{Y_1} = \langle \text{VAR} \cup \text{CONS}, \bar{\sim}, \leq \rangle$  such that  $n \bar{\sim} n' \stackrel{\text{def}}{\iff} (n \sim n') \in Y_1$  (where  $\sim \in \{<, =\}$ ) satisfies the conditions below:

- (MC1) For all  $n, n'$ , either  $n \bar{\sim} n'$  or  $n' \bar{\sim} n$  for some  $\sim \in \{<, =\}$ .
- (MC2)  $\bar{\sim}$  is a congruence relation compatible with  $\leq$ .
- (MC3) There is no path  $n_0 \xrightarrow{\sim_0} n_1 \xrightarrow{\sim_1} \dots \xrightarrow{\sim_{\alpha-1}} n_\alpha$  such that  $n_0 = n_\alpha$  and one of the symbols  $\sim_0, \dots, \sim_{\alpha-1}$  is equal to  $<$  (no strict cycle).
- (MC4) For all  $d_1, d_2 \in \text{CONS}$ ,  $d_1 \sim d_2$  implies  $d_1 \bar{\sim} d_2$ .
- (MC5) For all  $d_1, d_2$  with  $d_1 \leq d_2$ , there is no path  $n_0 \xrightarrow{\sim_0} n_1 \xrightarrow{\sim_1} \dots \xrightarrow{\sim_{\alpha-1}} n_\alpha$  with  $n_0 = d_1$  and  $n_\alpha = d_2$  such that the number of occurrences of the symbol  $<$  in  $\sim_0, \dots, \sim_{\alpha-1}$  is strictly greater than  $d_2 - d_1$ .

Note that this graph does not take into account the constraints in  $Y_2$  and  $Y_3$ .

Checking that a variable occurs at most once in  $Y_2$  or exactly once in  $Y_3$  can be done in linear time. It is also easy to verify that the equality relations of  $Y_2$  do not contradict the constraints in  $Y_1$ .

Finally, we need to check that the set of congruence relations  $Y_3$  is compatible with the sets of constraints  $Y_1$  and  $Y_2$ :

- (MC6) (a) For all  $n, n'$  such that  $n \bar{\sim} n'$ , for every  $c \in \{0, \dots, K-1\}$ ,  $n \equiv_K c \in Y_3$  iff  $n' \equiv_K c \in Y_3$ .
- (b) For all  $x = d$  in  $Y_2$ , the corresponding constraint  $x \equiv_K c$  belonging to  $Y_3$  is such that  $c \equiv_K d$ .

As illustrated by the following lemma, the symbolic representations of valuations contain the relevant information to evaluate constraints.

**Lemma 4.** *Let  $X$  be a finite set of IPC\* constraints.*

- (I) *For every valuation  $v : \text{VAR} \rightarrow \mathbb{Z}$  there is a unique symbolic valuation  $\langle Y_1, Y_2, Y_3 \rangle$  built w.r.t.  $X$  and denoted by  $sv(v)$  such that  $v \models_\star Y_1 \cup Y_2 \cup Y_3$ .*
- (II) *For all valuations  $v, v'$  such that  $sv(v) = sv(v')$  and for every  $\xi \in X$ ,  $v \models_\star \xi$  iff  $v' \models_\star \xi$ .*

*Proof.* (I) Given a symbolic valuation  $sv$ , let  $V_{sv}$  be the set of tuples  $\langle d_1, \dots, d_n \rangle$  in  $\mathbb{Z}^{|\text{VAR}|}$  (viewed as maps  $\text{VAR} \rightarrow \mathbb{Z}$ ) such that  $\langle d_1, \dots, d_n \rangle \models sv$ . It is easy to show that  $\{V_{sv} : sv \text{ is a symbolic valuation built w.r.t. } X \text{ and } V_{sv} \neq \emptyset\}$  is a

partition of  $\mathbb{Z}^{|\text{VAR}|}$ .

(II) By structural induction on  $\xi$ . The proof is similar to the proof of [Dem06, Lemma 1].  $\square$

Note that by Lemma 4, a symbolic valuation is an equivalence class of valuations. Given a symbolic valuation  $sv$  and a constraint  $\xi$ , we write  $sv \models_{\text{symp}} \xi \stackrel{\text{def}}{\iff}$  for every valuation  $v$  such that  $sv(v) = sv, v \models_{\star} \xi$ .

**Lemma 5.** *Given a symbolic valuation  $sv$  built w.r.t. a set of constraints  $X$  (fixing VAR, CONS and  $K$ ) and a constraint  $\xi \in X$ , checking whether  $sv \models_{\text{symp}} \xi$  is PSPACE-complete.*

*Proof.* PSPACE-hardness can be obtained by reducing QBF. The only constants used are 0 and 1. Each QBF formula  $\phi = Q_1 p_1 Q_2 p_2 \dots Q_n p_n \phi'$  with  $\phi'$  a propositional formula in CNF built over the propositional variables  $\{p_1, \dots, p_n\}$  and  $\{Q_1, \dots, Q_n\} \subseteq \{\forall, \exists\}$  is translated via the map  $t$  as follows

- $t(\exists p_i \phi_i) = \exists x_i (x_i = 0 \vee x_i = 1) \wedge t(\phi_i)$ ,
- $t(\forall p_i \phi_i) = \forall x_i (x_i = 0 \vee x_i = 1) \Rightarrow t(\phi_i)$ ,
- $t$  is homomorphic for Boolean connectives,
- $t(p_i) = (x_i = 1)$ .

One can show that  $\phi$  is QBF satisfiable iff for all symbolic valuations  $sv, sv \models_{\text{symp}} t(\phi)$  which is equivalent to check that an arbitrary symbolic valuation symbolically satisfies  $t(\phi)$ , since  $t(\phi)$  has no free variable.

The proof for the upper bound is similar to the proof of the PSPACE upper bound for first-order model-checking [CM77]. We can define a function  $\text{MC}(sv, \xi', \xi)$  where

- $\xi'$  is a subconstraint occurring in the IPC $^*$  constraint  $\xi$ ,
- $sv$  is a symbolic valuation over the syntactic resources of  $\xi'$  (VAR, CONS,  $K$  defined above),

which returns true iff  $sv \models_{\text{symp}} \xi'$ . Observe that if a variable occurs in  $sv$  but is not free in  $\xi'$  then the satisfaction of  $sv \models_{\text{symp}} \xi'$  is independent of its value. The function MC is defined as a case analysis on the form of  $\xi'$ . For instance,  $\text{MC}(sv, \exists x \xi', \xi)$  returns true iff there is a satisfiable symbolic valuation  $sv'$  extending  $sv$  by addition of  $x$ -constraints, such that  $\text{MC}(sv', \xi', \xi)$  returns true. The symbolic valuations  $sv' = \langle Y'_1, Y'_2, Y'_3 \rangle$  and  $sv = \langle Y_1, Y_2, Y_3 \rangle$  are related as follows:

- $sv'$  is a satisfiable symbolic valuation over the free variables of  $\xi'$ . This can be checked in polynomial-time in  $|\xi|$ .
- $Y_1 \subseteq Y'_1, Y_2 \subseteq Y'_2$  and  $Y_3 \subseteq Y'_3$ .
- The only variable in  $sv'$  but not in  $sv$  is  $x$ .

Even if the number of symbolic valuations over the free variables of  $\xi'$  is exponential in  $|\xi|$ , it is possible to enumerate them in polynomial space in order to check the existence of some  $sv'$  verifying the above conditions.  $\square$

## 4 Satisfiable $\omega$ -sequences of symbolic valuations

Given a CLTL(IPC<sup>\*</sup>) formula  $\phi$ , we write  $\text{IPC}^*(\phi)$  to denote the set of IPC<sup>\*</sup> constraints  $\xi$  such that some atomic formula of the form  $\xi[x_1 \leftarrow X^{i_1}x_{j_1}, \dots, x_r \leftarrow X^{i_r}x_{j_r}]$  occurs in  $\phi$ . We associate to  $\text{IPC}^*(\phi)$  the objects relative to any finite set of IPC<sup>\*</sup> constraints: the set VAR of variables and the set CONS of constants occurring in  $\phi$ , and  $K$  the least common multiple of the integers  $k_i$  that occur in the congruence relations. This induces a unique set  $\text{CONS}' = \{m, \dots, M\}$  where  $m$  is the minimal element of CONS and  $M$  the maximal element.

We define the X-length of  $\phi$ , denoted by  $|\phi|_X$ , as the maximal number  $i$  such that a term of the form  $X^i x$  occurs in  $\phi$ . In the following, we assume that  $\text{VAR} = \{x_1, \dots, x_s\}$  and  $|\phi|_X = l \geq 1$ . We write  $\text{Terms}(\phi)$  to denote the set of terms of the form  $X^\beta x_\alpha$  with  $\beta \in \{0, \dots, l\}$  and  $\alpha \in \{1, \dots, s\}$ .

Let  $\text{VAR}'$  be a set of fresh variables of cardinality  $|\text{Terms}(\phi)|$ . For technical convenience, we need to introduce a bijection  $f : \text{Terms}(\phi) \rightarrow \text{VAR}'$  such that  $f$  and  $f^{-1}$  can be computed in polynomial time. By extension, for every subformula  $\psi$  of  $\phi$ ,  $f(\psi)$  is obtained from  $\psi$  by replacing each occurrence of  $X^\beta x_\alpha$  by  $f(X^\beta x_\alpha)$ . The map  $f^{-1}$  is used in a similar fashion. A symbolic valuation with respect to  $\phi$  is a symbolic valuation built over the set of variables  $\text{VAR}'$ , CONS and  $K$ .

We say that a pair  $\langle \langle Y_1, Y_2, Y_3 \rangle, \langle Y'_1, Y'_2, Y'_3 \rangle \rangle$  of symbolic valuations with respect to  $\phi$  is one-step consistent iff for every  $j, j' \geq 1$ ,  $x_i, x_{i'} \in \text{VAR}$ ,  $d \in \text{CONS}'$  and  $c \in \{0, \dots, K-1\}$  we have

1.  $f(X^j x_i) \sim f(X^{j'} x_{i'}) \in Y_1$  iff  $f(X^{j-1} x_i) \sim f(X^{j'-1} x_{i'}) \in Y'_1$ ,
2.  $f(X^j x_i) \sim d \in Y_1 \cup Y_2$  iff  $f(X^{j-1} x_i) \sim d \in Y'_1 \cup Y'_2$ ,
3.  $f(X^j x_i) \equiv_K c \in Y_3$  iff  $f(X^{j-1} x_i) \equiv_K c \in Y'_3$ .

An  $\omega$ -sequence  $\rho$  of satisfiable symbolic valuations w.r.t.  $\phi$  is one-step consistent iff for every  $j \in \mathbb{N}$ ,  $\langle \rho(j), \rho(j+1) \rangle$  is one-step consistent, where  $\rho(j)$  denotes the  $j$ th symbolic valuation of the sequence. We say that a symbolic valuation sequence  $\rho$  is satisfied by a CLTL(IPC<sup>\*</sup>)-model  $\sigma$  iff for all  $j \in \mathbb{N}$  and  $\xi \in \rho(j)$ ,  $\sigma, j \models f^{-1}(\xi)$ . In order to simplify the future developments, we will write  $\rho_f$  (or sometimes  $\rho_{f^{-1}}$ ) to denote the  $\omega$ -sequence of IPC<sup>\*</sup>-constraints obtained from  $\rho$  by substituting each occurrence of  $x$  by  $f^{-1}(x)$  for every variable  $x$  used in  $\rho$ .

One-step consistent  $\omega$ -sequences of symbolic valuations w.r.t.  $\phi$  define abstractions of models for  $\phi$ . We represent a one-step consistent sequence  $\rho$  by an infinite labeled structure  $G_\rho = \langle (\text{VAR} \cup \text{CONS}') \times \mathbb{N}, \overset{=}{\rightarrow}, \overset{<}{\rightarrow}, \text{mod} \rangle$  where  $\text{mod} : (\text{VAR} \cup \text{CONS}') \times \mathbb{N} \rightarrow \{0, \dots, K-1\}$  and for all  $\sim \in \{<, =\}$ ,  $x, y \in \text{VAR}$ ,  $d', d_1, d_2 \in \text{CONS}'$  and  $i, j \in \mathbb{N}$  such that  $|i - j| \leq l$  we have:

- $$\begin{aligned} \langle x, i \rangle \overset{\sim}{\rightarrow} \langle y, j \rangle & \text{ iff either } i \leq j \text{ and } x \sim X^{j-i} y \in \rho_f(i) \\ & \text{ or } i > j \text{ and } X^{i-j} x \sim y \in \rho_f(j), \\ \langle x, i \rangle \overset{=}{\rightarrow} \langle d, j \rangle & \text{ iff } x = d \in \rho_f(i), \\ \langle d, i \rangle \overset{=}{\rightarrow} \langle x, j \rangle & \text{ iff } x = d \in \rho_f(j), \\ \langle x, i \rangle \overset{<}{\rightarrow} \langle d, j \rangle & \text{ iff there is } d' \sim d \text{ s.t. } x \sim' d' \in \rho_f(i) \text{ for some } \sim' \in \{<, =\} \\ & \text{ and either } \sim \text{ or } \sim' \text{ is equal to } <, \end{aligned}$$

$$\begin{aligned}
\langle d, i \rangle \xrightarrow{\leq} \langle x, j \rangle & \quad \text{iff there is } d \sim d' \text{ s.t. } d' \sim' x \in \rho_f(j) \text{ for some } \sim' \in \{<, =\} \\
& \quad \text{and either } \sim \text{ or } \sim' \text{ is equal to } <, \\
\langle d_1, i \rangle \xrightarrow{\sim} \langle d_2, j \rangle & \quad \text{iff } d_1 \sim d_2, \\
\text{mod}(\langle x, i \rangle) = c & \quad \text{iff } x \equiv_K c \in \rho_f(i), \\
\text{mod}(\langle d, i \rangle) = c & \quad \text{iff } d \equiv_K c.
\end{aligned}$$

By construction of  $G_\rho$ , the variables and constants are treated in a similar fashion. It is worth observing that  $G_\rho$  is well-defined because  $\rho$  is one-step consistent. Moreover, the construction ensures that the “local” representation of every  $\rho(i)$  verifies the conditions (MC1)–(MC6) introduced Section 3.

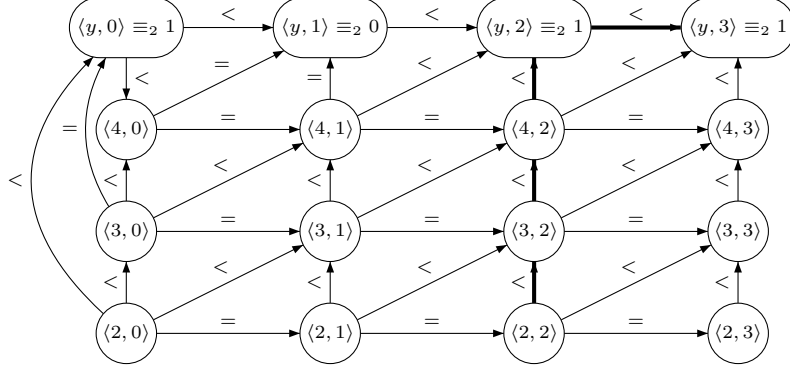
We say that a vertex represents the constant  $d \in \text{CONS}'$  if it is of the form  $\langle d, i \rangle$  for some  $i \in \mathbb{N}$ . The level of a node  $n = \langle a, t \rangle$  in  $G_\rho$  is defined by  $\text{lev}(n) = t$ . There is some redundancy in  $G_\rho$  for the nodes of the form  $\langle d, i \rangle$  but this turn out to be helpful to establish tight relationships between  $\rho$  and  $G_\rho$ .

As an example, assume that  $\text{VAR} = \{y\}$ ,  $\text{CONS} = \{2, 4\}$ ,  $K = 2$ ,  $l = 1$ ,  $\text{VAR}' = \{x, x'\}$  ( $f(y) = x$  and  $f(Xy) = x'$ ) and  $l = 1$ . We consider the sequence  $\rho = sv^0 \cdot (sv^1 \cdot sv^2)^\omega$  where

- $sv^0 = \langle Y_1^0, Y_2^0, Y_3^0 \rangle$  with
  - $Y_1^0 = \{x = x, x' = x', x < x', 2 < x, x < 4, 2 < x', x' = 4\}$ ,
  - $Y_2^0 = \{x = 3\}$ ,
  - $Y_3^0 = \{x \equiv_2 1, x' \equiv_2 0\}$ ,
- $sv^1 = \langle Y_1^1, Y_2^1, Y_3^1 \rangle$  with
  - $Y_1^1 = \{x = x, x' = x', x < x', 2 < x, x = 4, 2 < x', 4 < x'\}$ ,
  - $Y_2^1 = \emptyset$ ,
  - $Y_3^1 = \{x \equiv_2 0, x' \equiv_2 1\}$ ,
- $sv^2 = \langle Y_1^2, Y_2^2, Y_3^2 \rangle$  with
  - $Y_1^2 = \{x = x, x' = x', x < x', 2 < x, 4 < x, 2 < x', 4 < x'\}$ ,
  - $Y_2^2 = \emptyset$ ,
  - $Y_3^2 = \{x \equiv_2 1, x' \equiv_2 1\}$ .

The graph  $G_\rho$  is presented in Figure 2. In order to simplify the representation, closure by transitivity for  $\xrightarrow{\leq}$  and the fact that  $\xrightarrow{=}$  is a congruence are omitted. The function *mod* is directly encoded in the node label.

A path in  $G_\rho$  is a sequence (possible infinite) of the form  $n_0 \xrightarrow{\sim^0} n_1 \xrightarrow{\sim^1} n_2 \xrightarrow{\sim^2} \dots$  (each  $\sim_i$  belongs to  $\{<, =\}$ ). A finite path  $w = n_0 \xrightarrow{\sim^0} n_1 \xrightarrow{\sim^1} n_2 \xrightarrow{\sim^2} \dots \xrightarrow{\sim^{\alpha-1}} n_\alpha$  such that  $n_0 = n_\alpha$  is called a cycle. For any finite path  $w = n_0 \xrightarrow{\sim^0} n_1 \xrightarrow{\sim^1} n_2 \xrightarrow{\sim^2} \dots \xrightarrow{\sim^{\alpha-1}} n_\alpha$ , its strict length  $\text{slen}(w)$  is the number of indices  $i \in \{0, \dots, \alpha - 1\}$  such that  $\sim_i$  is equal to  $<$ . We say that  $w$  is strict if  $\text{slen}(w) > 0$ . The strict length between two nodes  $n_1$  and  $n_2$ , written  $\text{slen}(n_1, n_2)$ , is the least upper bound (possibly equal to  $\omega$ ) of the strict lengths of finite paths between  $n_1$  and  $n_2$ . By convention, if there is no path between from  $n_1$  to  $n_2$ ,  $\text{slen}(n_1, n_2)$  takes the value  $-\infty$ . For example, in Figure 2  $\text{slen}(\langle 2, 2 \rangle, \langle y, 3 \rangle) = 4$  (see the transitions in boldface).



**Fig. 2.** A graph  $G_\rho$

The one-step consistency of  $\rho$  implies global constraints on its graph representation that already hold true locally. By a global constraint, we mean a constraint on the whole graph and not only on the local representation of a single symbolic valuation or on two successive satisfiable symbolic valuations.

**Lemma 6.** *Let  $\rho$  be a one-step consistent symbolic valuation sequence. The following properties hold for  $G_\rho$ .*

- (I)  $G_\rho$  has no strict cycle.
- (II) If there is a finite path  $w$  starting at  $\langle d, i \rangle$  and ending at the node  $n$  of level  $j$ , then: if  $w$  is strict then  $\langle d, j \rangle \lesssim n$ , otherwise  $\langle d, j \rangle \bar{\equiv} n$ .
- (III) If there is a finite path  $w$  starting at the node  $n$  of level  $j$  and ending at  $\langle d, i \rangle$ , then: if  $w$  is strict then  $n \lesssim \langle d, j \rangle$ , otherwise  $n \bar{\equiv} \langle d, j \rangle$ .
- (IV) For every pair of nodes  $n, n'$  in  $G_\rho$  such that  $slen(n, n') = 0$ , we have  $mod(n) = mod(n')$ .

*Proof.* (I) We show that for every path  $w = n_0 \xrightarrow{\sim_0} n_1 \xrightarrow{\sim_1} n_2 \xrightarrow{\sim_2} \dots \xrightarrow{\sim_{\alpha-1}} n_\alpha$ ,  $n_0 = n_\alpha$  implies  $slen(w) = 0$ . The proof is by induction on  $\alpha$ . The base case with either  $\alpha = 1$  or  $\alpha = 2$  is by an easy verification since the subgraph corresponding to the symbolic valuation  $\rho(\min\{\text{lev}(n_i) : 1 \leq i \leq \alpha\})$  contains all the nodes of the path and satisfies (MC3). In the induction step, suppose that  $w$  is a cycle with  $\alpha > 2$  and let  $n_\beta$  be a node of the path with the greatest level. Without any loss of generality, we can assume that  $0 < \beta < \alpha$  since one can choose the first node of the cycle. As  $\text{lev}(n_\beta)$  is maximal,  $|\text{lev}(n_{\beta-1}) - \text{lev}(n_{\beta+1})| \leq l$  and so the subgraph corresponding to  $\rho(\min\{\text{lev}(n_{\beta-1}), \text{lev}(n_{\beta+1})\})$  contains the nodes  $n_{\beta-1}, n_\beta$  and  $n_{\beta+1}$ . Since this subgraph satisfies (MC2), this implies that  $n_{\beta-1} \lesssim n_{\beta+1}$  iff  $n_{\beta-1} \lesssim n_\beta$  or  $n_\beta \lesssim n_{\beta+1}$ . As a consequence,  $slen(w) = 0$  iff  $slen(w') = 0$  where the path  $w'$  is obtained from  $w$  by replacing the sub-path from  $n_{\beta-1}$  to  $n_{\beta+1}$  by the edge  $n_{\beta-1} \xrightarrow{\sim} n_{\beta+1}$ . By induction hypothesis,  $slen(w') = 0$  and therefore  $slen(w) = 0$ .

(II) We show that for every path  $w = n_0 \xrightarrow{\sim^0} n_1 \xrightarrow{\sim^1} n_2 \xrightarrow{\sim^2} \dots \xrightarrow{\sim^{\alpha-1}} n_\alpha$  such that  $n_0 = \langle d, i \rangle$ , we have  $\langle d, \text{lev}(n_\alpha) \rangle \xrightarrow{\sim} n_\alpha$  with  $\sim$  equal to  $<$  if  $w$  is strict and to  $=$  otherwise. The proof is by induction on  $\alpha$ . The base case with  $\alpha = 1$  is obvious. Let us consider a path  $w = n_0 \xrightarrow{\sim^0} n_1 \xrightarrow{\sim^1} n_2 \xrightarrow{\sim^2} \dots \xrightarrow{\sim^{\alpha-1}} n_\alpha$  with  $n_0 = \langle d, i \rangle$  and  $\alpha > 1$ . Suppose that  $\langle d, \text{lev}(n_{\alpha-1}) \rangle \xrightarrow{\sim} n_{\alpha-1}$  with  $\sim$  equals to  $<$  if  $n_0 \xrightarrow{\sim^0} n_1 \xrightarrow{\sim^1} n_2 \xrightarrow{\sim^2} \dots \xrightarrow{\sim^{\alpha-2}} n_{\alpha-1}$  is strict and  $\sim$  equals to  $=$  otherwise. We only treat the case  $\text{lev}(n_{\alpha-1}) \leq \text{lev}(n_\alpha)$ , the other case is similar. Since  $\text{lev}(n_\alpha) - \text{lev}(n_{\alpha-1}) \leq l$ , the nodes  $n_{\alpha-1}$  and  $n_\alpha$  belong to the subgraph corresponding to  $\rho(\text{lev}(n_{\alpha-1}))$ . This subgraph satisfies (MC2) and so  $\langle d, \text{lev}(n_{\alpha-1}) \rangle \xrightarrow{\sim'} n_\alpha$  with  $\sim'$  equal to  $<$  if either  $\langle d, \text{lev}(n_{\alpha-1}) \rangle \xrightarrow{\sim} n_{\alpha-1}$  or  $n_{\alpha-1} \xrightarrow{\sim} n_\alpha$ . Hence, by using induction hypothesis we obtain that  $\langle d, \text{lev}(n_{\alpha-1}) \rangle \xrightarrow{\sim'} n_\alpha$  and  $\sim'$  is equal to  $<$  if  $w$  is strict. Since  $\langle d, \text{lev}(n_{\alpha-1}) \rangle \xrightarrow{\sim} \langle d, \text{lev}(n_\alpha) \rangle$  and using the property (MC2), we get  $\langle d, \text{lev}(n_\alpha) \rangle \xrightarrow{\sim'} n_\alpha$  and  $\sim'$  equals  $<$  if  $w$  is strict (and  $=$  otherwise).

(III) Similar to (II).

(IV) Suppose that two nodes  $n$  and  $n'$  are such that  $\text{slen}(n, n') = 0$ . So there is a path,  $w = n \xrightarrow{\sim} n_0 \xrightarrow{\sim} \dots \xrightarrow{\sim} n_{\alpha-1} \xrightarrow{\sim} n_\alpha$  with  $n_\alpha = n'$ . We proceed by induction on  $\alpha$ . If  $\alpha \leq 1$  then the local representation of  $\rho(\min\{\text{lev}(n_0), \text{lev}(n_\alpha)\})$  contains both  $n_0$  and  $n_\alpha$  and we can conclude using (MC6). Otherwise, suppose that  $\alpha > 1$  and  $\text{mod}(n_0) = \text{mod}(n_{\alpha-1})$ . Since  $n_{\alpha-1} \xrightarrow{\sim} n_\alpha$  the local representation of  $\rho(\min\{\text{lev}(n_{\alpha-1}), \text{lev}(n_\alpha)\})$  contains both  $n_{\alpha-1}$  and  $n_\alpha$ . Since, this symbolic valuation satisfies (MC6), we have  $\text{mod}(n_\alpha) = \text{mod}(n_{\alpha-1}) = \text{mod}(n_0)$ .  $\square$

**Corollary 1.** *Let  $\rho$  be a one-step consistent sequence of symbolic valuations and  $G_\rho$  its graph representation. Then, for all nodes  $\langle d_1, i \rangle$  and  $\langle d_2, j \rangle$  in  $G_\rho$  representing constants such that  $d_1 \leq d_2$ ,  $\text{slen}(\langle d_1, i \rangle, \langle d_2, j \rangle) = d_2 - d_1$ .*

*Proof.* Let  $\langle d_1, i \rangle$  and  $\langle d_2, j \rangle$  be vertices of  $G_\rho$  representing respectively the constants  $d_1$  and  $d_2$ . Without any loss of generality, we can assume that  $i \leq j$  (the case  $j > i$  has a similar treatment). Obviously,  $\text{slen}(\langle d_1, i \rangle, \langle d_2, j \rangle) \geq d_2 - d_1$  as witnessed by the path below of strict length  $d_2 - d_1$ :

$$\langle d_1, i \rangle \xrightarrow{\sim} \langle d_1, i+1 \rangle \xrightarrow{\sim} \langle d_1, i+2 \rangle \dots \langle d_1, j \rangle \xrightarrow{\sim} \langle d_1+1, j \rangle \xrightarrow{\sim} \langle d_1+2, j \rangle \dots \xrightarrow{\sim} \langle d_2, j \rangle.$$

Now suppose that there is a path  $w$  between  $\langle d_1, i \rangle$  and  $\langle d_2, j \rangle$  such that  $\text{slen}(w) > d_2 - d_1$ . Consider the restriction of the transitive closure of  $\xrightarrow{\sim}$  to the nodes appearing in  $w$ . This relation is an equivalence relation having exactly  $\text{slen}(w) + 1$  equivalence classes. Let  $X_0, \dots, X_{\text{slen}(w)}$  be an enumeration of these equivalence classes. As a consequence of Lemma 6(II,III), every node  $n$  in  $w$  of level  $j$  is such that  $\langle d_1, j \rangle \xrightarrow{\sim} n \xrightarrow{\sim'} \langle d_2, j \rangle$  for some  $\sim, \sim' \in \{<, =\}$ . By definition of  $\text{CONS}'$  (which contains all the integers of the interval  $[m, M]$ ) and by maximal consistency of the local representations, for every  $i \in \{0, \dots, \text{slen}(w)\}$ , there is  $d'_i \in \text{CONS}'$  such that  $d_1 \leq d'_i \leq d_2$  and every node  $n$  of level  $j$  in  $X_i$  has an outgoing edge  $n \xrightarrow{\sim} \langle d'_i, j \rangle$ . Moreover the constants  $d'_0, \dots, d'_{\text{slen}(w)}$  should be

mutually distinct since all the  $X_i$  represents distinct equivalence classes. This leads to a contradiction since the cardinality of  $\{d_1, \dots, d_2\}$  is  $(d_2 - d_1) + 1$ .  $\square$

So far, we have stated properties about the graph  $G_\rho$ . Below, we establish simple conditions on  $G_\rho$  equivalent to the existence of a CLTL(IPC\*) model satisfying  $\rho$ . An edge-respecting labeling for  $G_\rho$  is a map  $lab : (\text{VAR} \cup \text{CONS}') \times \mathbb{N} \rightarrow \mathbb{Z}$  such that

- for all nodes  $n_1, n_2$  and  $\sim \in \{<, =\}$ , we have  $n_1 \xrightarrow{\sim} n_2$  implies  $lab(n_1) \sim lab(n_2)$ ,
- for every node  $n$ ,  $lab(n) \equiv_K mod(n)$ .

Additionally,  $lab$  is said to be strict if for every  $\langle d, i \rangle$  in  $G_\rho$ ,  $lab(\langle d, i \rangle) = d$ .

**Lemma 7.** *A one-step consistent sequence of symbolic valuations  $\rho$  has a model iff  $G_\rho$  has an edge-respecting labeling.*

*Proof.* Let  $\sigma$  be a model for  $\rho$  and  $lab : (\text{VAR} \cup \text{CONS}') \times \mathbb{N} \rightarrow \mathbb{Z}$  be the map defined as follows:

$$lab(\langle x, i \rangle) = \sigma(i, x); \quad lab(\langle d, i \rangle) = d \quad \text{for all } x \in \text{VAR}, d \in \text{CONS}' \text{ and } i \in \mathbb{N}.$$

It is not difficult to show that  $lab$  is a *strict* edge-respecting labeling for  $G_\rho$ . For instance, we have implications between the propositions below ( $x, y \in \text{VAR}$ ,  $i, j \in \mathbb{N}$ ,  $\sim \in \{<, =\}$ ):

- $\langle x, i \rangle \xrightarrow{\sim} \langle y, j \rangle$  and  $i \leq j$ ,
- $x \sim X^{j-i}y \in \rho_f(i)$  (by definition of  $G_\rho$ ),
- $f(x \sim X^{j-i}y) \in \rho(i)$  (by definition of  $\rho_f$ ),
- $\sigma, i \models f^{-1}(f(x \sim X^{j-i}y))$  (since  $\sigma$  is a model for  $\rho$ ),
- $\sigma, i \models x \sim X^{j-i}y$  ( $f$  is a bijection),
- $\sigma(i, x) \sim \sigma(j, y)$  (by definition of  $\models$ ),
- $lab(\langle x, i \rangle) \sim lab(\langle y, j \rangle)$  (by definition of  $lab$ ).

Hence,  $\langle x, i \rangle \xrightarrow{\sim} \langle y, j \rangle$  and  $i \leq j$  implies  $lab(\langle x, i \rangle) \sim lab(\langle y, j \rangle)$ . Satisfaction of periodicity constraints is based on the same development ( $x \in \text{VAR}$ ,  $c \in \{0, \dots, K-1\}$ ):

- $mod(\langle x, i \rangle) = c$ ,
- $x \equiv_K c \in \rho_f(i)$  (by definition of  $G_\rho$ ),
- $\sigma, i \models x \equiv_K c$  (arguments as above),
- $\sigma(i, x) \equiv_K c$  (by definition of  $\models$ ),
- $lab(\langle x, i \rangle) \equiv_K c$  (by definition of  $lab$ ).

Conversely, let  $lab$  be an edge-respecting labeling of  $G_\rho$ . First, we build from  $lab$  a strict edge-respecting labeling  $lab'$  of  $G_\rho$ . The values greater than  $M$  are divided in consecutive blocks of  $K$  consecutive values in such a way that if  $lab(n) - lab(\langle M, lev(n) \rangle) = \beta > 0$  then  $lab'(n)$  takes its value in the  $\beta$ th block.



$$\overbrace{M}^{\text{block 0}} \overbrace{M+1 \cdots M+K}^{\text{block 1}} \overbrace{M+K+1 \cdots M+2K}^{\text{block 2}} \cdots \overbrace{M+(\gamma-1)K+1 \cdots M+\gamma K}^{\text{block } \gamma} \cdots$$

Then the constraint  $\text{mod}(n)$  insures the unicity of  $\text{lab}'(n)$  such that  $\text{lab}'(n) \equiv_K \text{mod}(n)$ . A similar division is performed for the values smaller than  $m$ .

- For every  $n = \langle x, i \rangle$  such that  $\langle x, i \rangle \lesssim \langle m, i \rangle$ , then  $\text{lab}'(\langle x, i \rangle) \stackrel{\text{def}}{=} \alpha$  with
    - $\alpha \equiv_K \text{mod}(\langle x, i \rangle)$ ,
    - $m - (\text{lab}(\langle m, i \rangle) - \text{lab}(\langle x, i \rangle)) \times K \leq \alpha \leq m - ((\text{lab}(\langle m, i \rangle) - \text{lab}(\langle x, i \rangle) - 1) \times K - 1)$ .
  - For every  $\langle x, i \rangle$  such that  $\langle M, i \rangle \lesssim \langle x, i \rangle$ , then  $\text{lab}'(\langle x, i \rangle) \stackrel{\text{def}}{=} \alpha$  with
    - $\alpha \equiv_K \text{mod}(\langle x, i \rangle)$ ,
    - $M + ((\text{lab}(\langle x, i \rangle) - \text{lab}(\langle M, i \rangle) - 1) \times K + 1) \leq \alpha \leq M + (\text{lab}(\langle x, i \rangle) - \text{lab}(\langle M, i \rangle)) \times K$ .
- In both above cases,  $\alpha$  is unique since it belongs to an interval of length  $K$  with a periodicity constraint that forces a unique value in this interval.
- For every  $\langle x, i \rangle$  such that  $\langle x, i \rangle \overset{=}{\sim} \langle d, i \rangle$  for some  $d \in \text{CONS}'$ ,  $\text{lab}'(\langle x, i \rangle) = d$ .
  - For every  $\langle d, i \rangle$ ,  $\text{lab}'(\langle d, i \rangle) = d$ .

$\text{lab}'$  is well-defined because  $\rho$  is a sequence of satisfiable symbolic valuations with respect to  $\phi$ . Moreover,  $\text{lab}'$  is a strict edge-respecting labeling. By way of example, suppose that  $n \lesssim n'$ ,  $\text{lev}(n) \leq \text{lev}(n')$ ,  $\langle M, \text{lev}(n) \rangle \lesssim n$  and  $\langle M, \text{lev}(n') \rangle \lesssim n'$ . We have  $\text{lab}(\langle M, \text{lev}(n) \rangle) < \text{lab}(n) < \text{lab}(n')$  because  $\text{lab}$  is edge-respecting and so, since the values of the  $(\text{lab}(n) - \text{lab}(\langle M, \text{lev}(n) \rangle))$ th block after  $M$  are greater than the values of  $(\text{lab}(n') - \text{lab}(\langle M, \text{lev}(n) \rangle))$ th block,  $\text{lab}'(n) < \text{lab}'(n')$ .

Now, we show that the model  $\sigma$  defined by  $\sigma(i, x) = \text{lab}'(\langle x, i \rangle)$  for all  $x \in \text{VAR}$  and  $i \in \mathbb{N}$  satisfies  $\rho$ . By way of example, we show that  $\mathsf{X}^j x \sim \mathsf{X}^k y \in \rho_f(i)$  and  $j \leq k$  implies  $\sigma, i \models \mathsf{X}^j x \sim \mathsf{X}^k y$ . We have implications between the propositions below:

- $\mathsf{X}^j x \sim \mathsf{X}^k y \in \rho_f(i)$  and  $j \leq k$ ,
- $\langle x, i+j \rangle \overset{\sim}{\sim} \langle y, i+k \rangle$  in  $G_\rho$  (by definition of  $G_\rho$ ),
- $\text{lab}(\langle x, i+j \rangle) \sim \text{lab}(\langle y, i+k \rangle)$  ( $\text{lab}'$  is edge-respecting),
- $\sigma(i+j)(x) \sim \sigma(i+k)(y)$  (by definition of  $\sigma$ ),
- $\sigma, i \models \mathsf{X}^j x \sim \mathsf{X}^k y$  (by definition of  $\models$ ).

□

Lemmas 7 states correspondences between  $\rho$  and its graphical representation  $G_\rho$ . We now define a more abstract characterization of the one-step consistent sequences admitting a model.

**Lemma 8.** *Let  $\rho$  be a one-step consistent sequence. The graph  $G_\rho$  has an edge-respecting labeling iff for all nodes  $n_1, n_2$  in  $G_\rho$ ,  $\text{slen}(n_1, n_2) < \omega$ .*

Note that, by construction of  $G_\rho$ , for all nodes  $\langle d_1, i \rangle$  and  $\langle d_2, j \rangle$  representing constants such that  $d_1 \leq d_2$  we have  $\text{slen}(\langle d_1, i \rangle, \langle d_2, j \rangle) = d_2 - d_1$  (see Corollary 1). That is why it suffices to consider nodes  $n_1$  and  $n_2$  that are not both constants.

*Proof.* If  $G_\rho$  has an edge-respecting labeling  $\text{lab}$ , then one can easily show that for all nodes  $n_1, n_2$  in  $G_\rho$   $\text{slen}(n_1, n_2) \leq \text{lab}(n_2) - \text{lab}(n_1)$ .

Conversely, if for all nodes  $n_1, n_2$  in  $G_\rho$ ,  $\text{slen}(n_1, n_2) < \omega$ , we define the following map  $\text{lab} : (\text{VAR} \cup \text{CONS}') \times \mathbb{N} \rightarrow \mathbb{Z}$ :

- $\text{lab}(\langle d, i \rangle) \stackrel{\text{def}}{=} d$ .
- If  $\langle x, i \rangle \xrightarrow{=} \langle d, i \rangle$  then  $\text{lab}(\langle x, i \rangle) \stackrel{\text{def}}{=} d$ .
- Otherwise,
  - If  $\langle x, i \rangle \xrightarrow{<} \langle m, i \rangle$  then  $\text{lab}(\langle x, i \rangle) \stackrel{\text{def}}{=} \alpha$  with
    1.  $\alpha \equiv_K \text{mod}(\langle x, i \rangle)$ .
    2.  $m - \text{slen}(\langle x, i \rangle, \langle m, i \rangle) \times K \leq \alpha \leq m - (\text{slen}(\langle x, i \rangle, \langle m, i \rangle) - 1) \times K - 1$ .
  - If  $\langle M, i \rangle \xrightarrow{<} \langle x, i \rangle$  then  $\text{lab}(\langle x, i \rangle) \stackrel{\text{def}}{=} \alpha$  with
    1.  $\alpha \equiv_K \text{mod}(\langle x, i \rangle)$ .
    2.  $M + (\text{slen}(\langle M, i \rangle, \langle x, i \rangle) - 1) \times K + 1 \leq \alpha \leq M + \text{slen}(\langle M, i \rangle, \langle x, i \rangle) \times K$ .

Similarly to the proof of Lemma 7, each  $\alpha$  is uniquely defined since it belongs to an interval of length  $K$  and the map  $\text{mod}$  forces a unique value in this interval.

We now show that  $\text{lab}$  is a strict edge-respecting labeling of  $G_\rho$ . If the labeling is not edge-respecting, then one of the following cases arises:

- Suppose  $n \xrightarrow{=} n'$ . We treat the case  $\text{lev}(n) < \text{lev}(n')$  (the symmetrical case has an analogous treatment).

*Case 1:*  $\langle M, \text{lev}(n) \rangle \xrightarrow{<} n$  and  $\langle M, \text{lev}(n') \rangle \xrightarrow{<} n'$ .

Since  $n \xrightarrow{=} n'$  and  $\langle M, \text{lev}(n) \rangle \xrightarrow{=} \langle M, \text{lev}(n') \rangle$ , we have

$$\text{slen}(\langle M, \text{lev}(n) \rangle, n) = \text{slen}(\langle M, \text{lev}(n') \rangle, n').$$

Hence  $\text{lab}(n)$  and  $\text{lab}(n')$  belong to the same block of size  $K$  after  $M$ . Moreover,  $\rho(\text{lev}(n))$  is maximally consistent which entails  $\text{mod}(n) = \text{mod}(n')$  (by (MC6)) and  $\text{lab}(n) = \text{lab}(n')$ .

*Case 2:*  $n \xrightarrow{<} \langle m, \text{lev}(n) \rangle$  and  $n' \xrightarrow{<} \langle m, \text{lev}(n') \rangle$ .

Similar to Case 1.

*Case 3:*  $\langle M, \text{lev}(n) \rangle \xrightarrow{<} n$  and  $n' \xrightarrow{\approx} \langle M, \text{lev}(n') \rangle$ .

Since  $\rho(\text{lev}(n))$  is maximally consistent, by (MC2) we have  $n \xrightarrow{\approx} \langle M, \text{lev}(n') \rangle$ .

So we obtain the path

$$\langle M, \text{lev}(n) \rangle \xrightarrow{<} n \xrightarrow{\approx} \langle M, \text{lev}(n') \rangle \xrightarrow{=} \langle M, \text{lev}(n) \rangle,$$

which leads to a contradiction using (MC3).

*Case 4:*  $n \xrightarrow{<} \langle m, \text{lev}(n) \rangle$  and  $\langle m, \text{lev}(n') \rangle \xrightarrow{\approx} n'$ .

Similar to Case 3.

Case 5:  $n \xrightarrow{=} \langle d, \text{lev}(n) \rangle$  and  $n' \xrightarrow{=} \langle d', \text{lev}(n') \rangle$  with  $d < d'$ .

Since  $\rho(\text{lev}(n))$  is maximally consistent, by (MC2) and (MC3) we have  $d = d'$  which leads to a contradiction.

- The case  $n \xrightarrow{<} n'$  can be done in a similar fashion.

□

So we have a characterization of the set of sequences having a model but what we really want is to recognize them with automata. The main difficulty rests on the fact that the set of satisfiable one-step consistent  $\omega$ -sequences of satisfiable symbolic valuations is not  $\omega$ -regular, as shown in [DD07] for the fragment CLTL(Z). However, we can define an  $\omega$ -regular condition such that every one-step consistent ultimately periodic sequence  $\rho$  is satisfiable iff  $G_\rho$  satisfies this condition. An infinite word is ultimately periodic if it is of the form  $\tau \cdot \delta^\omega$  for some finite words  $\tau$  and  $\delta$ . We will see in the following section that this approximation condition is enough for our purpose since satisfiable CLTL(IPC<sup>\*</sup>) formulas always have a ultimately periodic (symbolic) model. Let  $\rho$  be a one-step consistent symbolic valuation sequence and  $G_\rho$  its graphical representation. An infinite forward (resp. backward) path in  $G_\rho$  is defined as a sequence  $w : \mathbb{N} \rightarrow (\text{VAR} \cup \text{CONS}') \times \mathbb{N}$  such that:

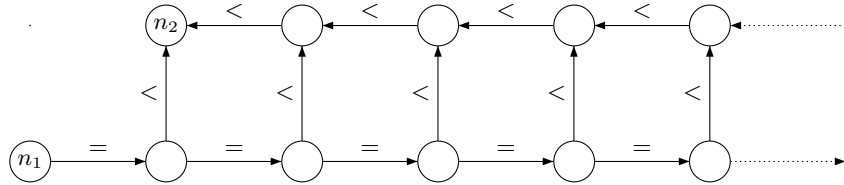
- for every  $i \in \mathbb{N}$ , we have  $w(i) \xrightarrow{\sim} w(i+1)$  (resp.  $w(i+1) \xrightarrow{\sim} w(i)$ ) in  $G_\rho$ ,
- for every  $i \in \mathbb{N}$ , we have  $\text{lev}(w(i)) < \text{lev}(w(i+1))$ .

The path  $w$  is infinitely often strict iff for every  $i \geq 0$ , there is  $j \geq i$  such that  $w(j) \xrightarrow{<} w(j+1)$  (resp.  $w(j+1) \xrightarrow{<} w(j)$ ).

**Definition 2.** A graph  $G_\rho$  satisfies the condition (C) iff there **do not** exist vertices  $n_1$  and  $n_2$  in  $G_\rho$  with  $|\text{lev}(n_1) - \text{lev}(n_2)| \leq l$  satisfying

- (AP1) there is an infinite forward path  $w_{\text{for}}$  from  $n_1$ ,
- (AP2) there is an infinite backward path  $w_{\text{back}}$  from  $n_2$ ,
- (AP3) either  $w_{\text{for}}$  or  $w_{\text{back}}$  is infinitely often strict, and
- (AP4) for all  $i, j \in \mathbb{N}$ , whenever  $|\text{lev}(w_{\text{for}}(i)) - \text{lev}(w_{\text{back}}(j))| \leq l$ ,  $w_{\text{for}}(i) \xrightarrow{<} w_{\text{back}}(j)$  in  $G_\rho$ .

A graph representation of some  $\rho$  not satisfying (C) is presented in Figure 3 where  $n_1$  is a constant node.



**Fig. 3.**  $G_\rho$  does not satisfy (C)

If  $\rho$  admits a model, then necessarily  $G_\rho$  satisfies  $(\mathcal{C})$ . Indeed, if  $G_\rho$  does not satisfy  $(\mathcal{C})$ , then  $\text{slen}(n_1, n_2) = \omega$  which entails that  $\rho$  has no model by Lemma 7 and 8. The converse does not hold in general. However, when  $\rho$  is ultimately periodic, the condition  $(\mathcal{C})$  is sufficient.

**Lemma 9.** *Let  $\rho$  be a one-step consistent  $\omega$ -sequence of satisfiable symbolic valuations which is ultimately periodic. Then  $\rho$  admits a model iff  $G_\rho$  satisfies  $(\mathcal{C})$ .*

Thanks to the way  $G_\rho$  is built from  $\rho$ ,  $(\mathcal{C})$  does not explicitly refer to the constants in CONS and the constraints of the form  $x \equiv_K c$ . Hence, Lemma 9 can be proved as [DD07, Lemma 6.2]: the map  $\text{mod}$  in  $G_\rho$  is ignored and a uniform treatment for all nodes in  $(\text{VAR} \cup \text{CONS}') \times \mathbb{N}$  is provided.

Let  $\rho = \tau \cdot \delta^\omega$  be an ultimately periodic one-step consistent  $\omega$ -sequence. If  $\rho$  admits a model then by Lemma 8 it satisfies the condition  $(\mathcal{C})$ . Conversely, if  $\rho$  has no model then by Lemma 8 there exist two vertices  $n_1$  and  $n_2$  such that  $\text{slen}(n_1, n_2) = \omega$ . Intuitively, the proof uses this property to claim the existence between these two nodes of a finite path  $w$  long enough so that two paths  $w_{\text{for}}$  and  $w_{\text{back}}$  satisfying the conditions (AP1)–(AP4) can be deduced. The construction of  $w_{\text{for}}$  and  $w_{\text{back}}$  from  $w$  uses the periodicity of  $\rho$  by repeating infinitely finite subpaths and can be done smoothly by using the properties established in this section (see e.g. Lemma 6). This witnesses that  $G_\rho$  does not satisfy  $(\mathcal{C})$ .

As the proof is not essentially different from [DD07, Lemma 6.2] modulo slight changes mentioned above, we omit it here (see details in [Gas07]).

## 5 Büchi automata and PSPACE upper bound

Based on the previous results and following the approach in [VW94], we show that given a CLTL(IPC<sup>\*</sup>) formula  $\phi$ , one can build a standard Büchi automaton  $\mathcal{A}_\phi$  such that  $\phi$  is CLTL(IPC<sup>\*</sup>) satisfiable iff  $L(\mathcal{A}_\phi)$  is non-empty. Moreover, we establish that nonemptiness of  $L(\mathcal{A}_\phi)$  can be checked in polynomial space in the size of  $\phi$  (denoted by  $|\phi|$ ). The automaton  $\mathcal{A}_\phi$  is precisely the intersection of three Büchi automata and its construction can be done quite smoothly thanks to the previous results. In the following, VAR, VAR', CONS and CONS' are the sets of variables and constants associated to  $\phi$  as defined in Section 4. Moreover,  $K$ ,  $m$  and  $M$  are constants with their usual meaning and we use the map  $f : \text{Terms}(\phi) \rightarrow \text{VAR}'$  as previously.

Unlike LTL, the language recognized by the Büchi automaton  $\mathcal{A}_\phi$  is not a set of models but rather a set of symbolic models. The alphabet  $\Sigma$  of this automaton is the set of symbolic valuations w.r.t.  $\phi$ . As a consequence, a symbolic model for  $\phi$  is an  $\omega$ -sequence  $\rho : \mathbb{N} \rightarrow \Sigma$ . We naturally extend the symbolic satisfaction to sequences. The relation  $\models'$  is defined as  $\models$  except at the atomic level:  $\rho, i \models' \xi \stackrel{\text{def}}{\iff} \rho(i) \models_{\text{symb}} f(\xi)$  where  $\models_{\text{symb}}$  is the satisfaction relation between symbolic valuations and constraints (see Section 3).

By Lemma 5 and using standard techniques for LTL [VW94], checking whether there is a symbolic model  $\rho$  satisfying  $\rho \models' \phi$  can be done in PSPACE (see more details below). Because every model for  $\phi$  generates a unique symbolic model for  $\phi$  (consequence of Lemma 4), we obtain the result below.

**Lemma 10.** *A CLTL(IPC<sup>\*</sup>) formula  $\phi$  is satisfiable iff there is a one-step consistent symbolic valuation  $\rho$  such that  $\rho \models' \phi$  and  $\rho$  has a model.*

*Proof.* Let  $\sigma$  be a model that satisfies  $\phi$ . Consider the symbolic valuation sequence  $\rho$  defined by:  $\rho(i) = sv(v_i)$  where for every  $i \in \mathbb{N}$ ,  $v_i$  is the valuation such that  $v_i(f(X^j x)) = \sigma(i+j)(x)$ . By construction, we have  $\sigma \models \rho$ . Using Lemma 4(II) we can show that for every  $v$  such that  $sv(v) = \rho(i)$  we have  $\sigma(i) \models \xi$  iff  $v \models \xi$  for every atomic subformula  $\xi$  of  $\phi$ . By definition of the symbolic satisfaction relation, this implies that if  $\sigma \models \xi$  then  $\rho \models' \xi$ . Consequently,  $\rho \models' \phi$  (induction on the structure of  $\phi$ ).

Conversely, suppose that  $\rho \models' \phi$  and  $\sigma \models \rho$  for some  $\sigma$  and  $\rho$ . Since for every  $i \in \mathbb{N}$  we have  $\sigma, i \models \rho(i)$ ,  $\rho(i)$  is the symbolic valuation corresponding to the valuation  $v_i$  such that  $v_i(f(X^j x)) = \sigma(i+j)(x)$ . By definition of  $\models'$ , this implies that for every atomic subformula  $\xi$  of  $\phi$ , if  $\rho, i \models' \xi$  then  $\sigma, i \models \xi$ . Thus, we can show that  $\rho \models \phi$  and  $\sigma \models \rho$  imply  $\sigma \models \phi$ .  $\square$

The automaton  $\mathcal{A}_\phi$  is formally defined as the intersection  $\mathcal{A}_{\text{LTL}} \cap \mathcal{A}_{\text{1cons}} \cap \mathcal{A}_{\mathcal{C}}$  of Büchi automata where

- $L(\mathcal{A}_{\text{LTL}})$  is the set of symbolic models satisfying  $\phi$ ,
- $L(\mathcal{A}_{\text{1cons}})$  is the set of one-step consistent sequences of symbolic valuations,
- $L(\mathcal{A}_{\mathcal{C}})$  is the set of sequences of symbolic valuations verifying  $(\mathcal{C})$ .

We briefly explain below how these different automata are built. All of them are built over the alphabet  $\Sigma$  which is of exponential size in  $|\phi|$ . The automaton  $\mathcal{A}_{\text{LTL}}$  is obtained from [VW94] with a difference for atomic formulae. We define  $cl(\phi)$  the closure of  $\phi$  as usual, and an atom of  $\phi$  is a maximally consistent subset of  $cl(\phi)$ . We define  $\mathcal{A}_{\text{LTL}} = (Q, Q_0, \delta, F)$  as the generalized Büchi automaton below:

- $Q$  is the set of atoms of  $\phi$  and  $Q_0 = \{X \in Q : \phi \in X\}$ ,
- $X \xrightarrow{sv} Y$  is in  $\delta$  iff
  - (**atomic constraints**) for every atomic formula  $\xi$  in  $X$ ,  $sv \models_{\text{symp}} f(\xi)$ ,
  - (**one step**) for every  $X\psi \in cl(\phi)$ ,  $X\psi \in X$  iff  $\psi \in Y$ ,
- let  $\{\phi_1 \cup \psi_1, \dots, \phi_r \cup \psi_r\}$  be the set of until formulas in  $cl(\phi)$ . We define  $F$  as the set  $\{F_1, \dots, F_r\}$  such that for every  $i \in \{1, \dots, r\}$ ,

$$F_i = \{X \in Q : \phi_i \cup \psi_i \notin X \text{ or } \psi_i \in X\}.$$

By Lemma 5, the condition about atomic formulae can be checked in PSPACE and so the transition relation can also be computed in PSPACE.

We define  $\mathcal{A}_{\text{1cons}} = \langle Q, Q_0, \delta, F \rangle$  as a Büchi automaton such that  $Q = Q_0 = F = Q = \Sigma$  and the transition relation satisfies:  $sv \xrightarrow{sv''} sv'$  is in  $\delta \stackrel{\text{def}}{\iff} \langle sv, sv' \rangle$  is one-step consistent and  $sv' = sv''$ . Since checking whether a triple of sets of IPC<sup>\*</sup>-constraints is a symbolic valuation and checking whether a pair of symbolic valuations is one-step consistent can both be done in polynomial time (see

Lemma 3), the transition relation of  $\mathcal{A}_{1_{\text{cons}}}$  can be computed in polynomial time.

It remains to define  $\mathcal{A}_{\mathcal{C}}$  that recognizes  $\omega$ -sequences of symbolic valuations satisfying  $(\mathcal{C})$ . As done in [DD07], instead of building  $\mathcal{A}_{\mathcal{C}}$ , it is easier to construct the Büchi automaton  $\mathcal{A}_{\mathcal{C}}^{\neg}$  that recognizes the complement language of  $L(\mathcal{A}_{\mathcal{C}})$ . The automaton  $\mathcal{A}_{\mathcal{C}}^{\neg}$  is essentially the automaton  $\mathcal{B}$  defined in [DD07, Sect.6] except that we work with an extended alphabet. We need to consider vertices in the graph that represent constants in  $\text{CONS}'$  and equality between constants does not need to be explicitly present in the symbolic valuations. Apart from this point, the variables in  $\text{VAR}$  and the constants in  $\text{CONS}$  have a uniform treatment in the definition of  $\mathcal{A}_{\mathcal{C}}^{\neg}$ .

The automaton  $\mathcal{A}_{\mathcal{C}}^{\neg}$  non-deterministically guesses in the first part of the run the vertices  $n_1, n_2$  and which path among  $w_{\text{for}}$  and  $w_{\text{back}}$  is infinitely often strict. Then it checks that the sequence fails to meet  $(\mathcal{C})$ . The Büchi acceptance condition guarantees that  $<$ -labeled edges are infinitely often visited. We store all these pieces of information in the locations. For instance, if the automaton is in the location  $\langle a, i, b, j, \text{for}, 0 \rangle$  at the position  $\alpha \geq 0$  of the run, this means that:

- the position of the current vertex of the forward path is  $\langle a, \alpha + i \rangle$ ,
- the position of the current vertex of the backward path is  $\langle b, \alpha + j \rangle$ ,
- the forward path is infinitely often strict.

The last component is only used to note when the forward path visits a strict edge. It takes the value 1 (respectively 0) when the previous transition is (respectively is not) a  $<$ -transition.

Before defining formally  $\mathcal{A}_{\mathcal{C}}^{\neg}$ , for  $a, a' \in \text{VAR} \cup \text{CONS}'$ ,  $\sim \in \{<, =\}$  and  $i, j \in \{0, \dots, l\}$ , we write  $(X^i a \rightsquigarrow X^j a') \in G_{sv}$  if there is an edge from the node representing  $X^i a$  at the current position to the node representing  $X^j a'$  (according to the definition of the edge relation in Section 4) which means that one of the following cases arises:

- $a, a' \in \text{VAR}$  and  $f(X^i a) \sim f(X^j a') \in sv$  (see definition of  $f$  in Section 4).
- $a \in \text{VAR}$ ,  $a' \in \text{CONS}$  and  $f(X^i a) \sim a' \in sv$ .
- $a' \in \text{VAR}$ ,  $a \in \text{CONS}$  and  $a \sim f(X^j a') \in sv$ .
- $a, a' \in \text{CONS}'$  and  $a \sim a'$ .
- $a \in \text{CONS}' \setminus \text{CONS}$ ,  $a' \in \text{VAR}$  and
  - either  $\sim$  is equality and  $a = f(X^j a') \in sv$
  - or there is  $d \in \text{CONS}$  such that  $a < d$  and  $d \sim f(X^j a') \in sv$
  - or there is  $d \in \text{CONS}'$  such that  $a < d$  and  $d = f(X^j a') \in sv$ .
- $a' \in \text{CONS}' \setminus \text{CONS}$ ,  $a \in \text{VAR}$  and
  - either  $\sim$  is equality and  $f(X^i a) = a' \in sv$
  - or there is  $d \in \text{CONS}$  such that  $d < a'$  and  $f(X^i a) \sim d \in sv$
  - or there is  $d \in \text{CONS}'$  such that  $d < a'$  and  $f(X^i a) = d \in sv$ .

Formally,  $\mathcal{A}_{\mathcal{C}}^{\neg} = \langle Q, Q_0, \rightarrow, F \rangle$  is defined as follows:

- $Q = \{q_0\} \uplus \{(\text{VAR} \cup \text{CONS}') \times \{0, \dots, l\} \times (\text{VAR} \cup \text{CONS}') \times \{0, \dots, l\} \times \{\text{for}, \text{back}\} \times \{0, 1\}\}$  where  $l = |\phi|_X$ ,

- $I = \{q_0\}$ ,
- The transition relation  $\rightarrow$  is defined as follows.
  - (a)  $q_0 \xrightarrow{sv} q_0$  for every  $sv \in \Sigma$ .
  - (b)  $q_0 \xrightarrow{sv} \langle a, i, b, j, for, 0 \rangle$  and  $q_0 \xrightarrow{sv} \langle a, i, b, j, back, 0 \rangle$  for every  $a, b \in \text{VAR} \cup \text{CONS}'$ ,  $i, j \in \{0, \dots, l\}$ ,  $sv \in \Sigma$  and  $X^i a < X^j b \in G_{sv}$ .
  - (c)  $\langle a, i, b, j, p, bin \rangle \xrightarrow{sv} \langle a, i-1, b, j-1, p, bin \rangle$  for every  $p \in \{for, back\}$ ,  $bin \in \{0, 1\}$ ,  $i, j \geq 1$  and  $sv \in \Sigma$ .
  - (d)  $\langle a, 0, b, j, for, bin \rangle \xrightarrow{sv} \langle a', i'-1, b, j-1, for, bin' \rangle$  s.t.  $i' > 0$ ,  $j > 0$  and
    - $(a \xrightarrow{\leq} X^{i'} a') \in G_{sv}$  and  $bin' = 1$ ;
    - or  $(a \xrightarrow{=} X^{i'} a') \in G_{sv}$  and  $bin' = 0$ ;
    - $(X^{i'} a' \xrightarrow{\leq} X^j b) \in G_{sv}$ .

These rules just check that there is a forward edge from the current node to the next node of the forward path.

- (e)  $\langle a, i, b, 0, for, bin \rangle \xrightarrow{sv} \langle a, i-1, b, j'-1, for, bin' \rangle$  if  $i > 0$ ,  $j' > 0$  and the conditions of (d) are verified when doing the following substitutions

$$a \leftarrow b, \quad a' \leftarrow b', \quad b \leftarrow a.$$

This corresponds to check that there is a backward edge from the current node to the next node of the backward path.

- (f)  $\langle a, 0, b, 0, for, bin \rangle \xrightarrow{sv} \langle a', i', b', j', for, bin' \rangle$  if the obvious combination of the constraints (d) and (e) is verified.
  - (g) Similar conditions are needed to consider the case where the backward path is infinitely often strict.
- $F$  is the set of states of the form  $\langle a, i, b, j, p, 1 \rangle$  for every  $a, b \in \text{VAR} \cup \text{CONS}'$ ,  $i, j \in \{0, \dots, l\}$  and  $p \in \{for, back\}$ .

**Lemma 11.** *A CLTL(IPC<sup>\*</sup>) formula  $\phi$  is satisfiable iff  $L(\mathcal{A}_\phi)$  is nonempty.*

The proof of this lemma is similar to [DD07, Lemma 6.3]. The main trick is to observe that if  $L(\mathcal{A}_\phi)$  is nonempty then  $\mathcal{A}_\phi$  accepts an ultimately periodic  $\omega$ -sequence so that Lemma 9 can be applied. Since given a formula  $\phi$  we can effectively construct  $\mathcal{A}_\phi$  and check whether  $L(\mathcal{A}_\phi)$  is nonempty, the model-checking and satisfiability problems for CLTL(IPC<sup>\*</sup>) are decidable. We also have all the arguments to establish the PSPACE upper bound by using arguments from [Saf88].

**Theorem 1.** *The satisfiability problem for CLTL(IPC<sup>\*</sup>) is PSPACE-complete.*

*Proof.* PSPACE-hardness is a consequence of the PSPACE-hardness of LTL [SC85]. As far as the PSPACE upper bound is concerned, the automata  $\mathcal{A}_{\text{LTL}}$ ,  $\mathcal{A}_{\text{1cons}}$  and  $\mathcal{A}_{\mathcal{C}}^-$  are of exponential size in  $|\phi|$  and can be built in polynomial space in  $|\phi|$ .

The automaton  $\mathcal{A}_{\mathcal{C}}$  is obtained from  $\mathcal{A}_{\mathcal{C}}^-$  by Safra's construction [Saf88] to complement a Büchi automaton.  $\mathcal{A}_{\mathcal{C}}^-$  has a number of states polynomial in  $|\phi|$ ,

say  $P(|\phi|)$ . From this, we can build a deterministic Streett automaton which accepts the complement of the language accepted by  $\mathcal{A}_C^-$  and has  $\mathcal{O}(2^{P(|\phi|) \times \log(|\phi|)})$  states. These are the same arguments as in the proof of [DD07, Theorem 6.6]. This automaton can be converted to an equivalent Büchi automaton  $\mathcal{A}_C$  with the same order of states. Hence,  $\mathcal{A}_C$  can be built in polynomial space in  $|\phi|$ .

So, computing the intersection automaton  $\mathcal{A}_\phi = \mathcal{A}_{LTL} \cap \mathcal{A}_{1\text{cons}} \cap \mathcal{A}_C$  can be done in polynomial space in  $|\phi|$ . Since the emptiness problem for Büchi automata is NLOGSPACE-complete, by [BDG88, Corollary 3.36], we finally get that testing emptiness of  $L(\mathcal{A}_\phi)$  can be done non-deterministically in polynomial space in  $|\phi|$ . As usual, by Savitch's theorem we get the PSPACE upper bound.  $\square$

Note that, all the temporal operators in  $\text{CLTL}(\text{IPC}^*)$  are definable in monadic second order logic (MSO). By using [GK03], it is immediate that any extension of  $\text{CLTL}(\text{IPC}^*)$  obtained by adding a finite amount of MSO-definable temporal operators remains in PSPACE. Only the automaton  $\mathcal{A}_{LTL}$  needs to be updated.

Another corollary is that the model-checking of the linear-time fragment of the logic of [Čer94] against integral relational automata is in PSPACE (only decidability is established by [Čer94]).

**Corollary 2.** *The model-checking problem for integral relational automata restricted to the LTL fragment of  $\text{CCTL}^*$  introduced in [Čer94] is in PSPACE.*

## 6 Conclusion

In this paper, we have introduced the logic  $\text{CLTL}(\text{IPC}^*)$  extending formalisms in [Čer94, LM01, DD07, Dem06] and we have shown that both model-checking over  $\text{IPC}^*$ -automata and satisfiability are decidable in polynomial space. The proof heavily relies on a translation into the nonemptiness problem for standard Büchi automata and on the approximation of non  $\omega$ -regular sets of symbolic models. As a by-product, the model-checking problem over the integral relational automata defined in [Čer94] is also PSPACE-complete when restricted to its LTL fragment. The logic  $\text{CLTL}(\text{IPC}^*)$  supports a rich class of constraints including those of the form  $x < y$  unlike periodicity constraints from [Dem06] (which are quite useful to compare absolute dates) and comparison with constants unlike logics shown in PSPACE in [DD07]. Abstraction of counter automata by performing reasoning modulo can be encoded in  $\text{CLTL}(\text{IPC}^*)$  thanks to the presence of integer periodicity constraints.

To conclude, we mention a few open problems that are worth investigating.

- The model checking of  $\text{CTL}^*$  for integral relational automata is undecidable [Čer94] whereas we have shown that its LTL fragment is PSPACE-complete. Moreover, it is shown in [BG06] that the existential and universal fragments have a decidable model checking problem (complexity is not known). It would be interesting to design other decidable branching-time extensions of  $\text{CLTL}(\text{IPC}^*)$ .
- The decidability status of the satisfiability problem of the full  $\text{CTL}^*$  extension is also an open question.



- The decidability status of constraint LTL over the domain  $\langle\{0,1\}^*, \subseteq\rangle$  is open either with the subword relation or with the prefix relation. Note that constraint LTL over the domain  $\langle\{0\}^*, \subseteq\rangle$  is already equivalent to constraint LTL over  $\langle\mathbb{N}, <, =\rangle$  that is a strict fragment of CLTL(IPC\*).

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## A Integral relational automata are restricted IPC\*-automata

An integral relational automata (RA) is defined in [Čer94] as a program with a finite amount of variables interpreted in  $\mathbb{Z}$ . The set OP of operations is composed of the following instructions and guards:

- comparisons of the form  $x < y$ ,  $x < d$ ,  $d < y$ ,
- assignments of the form  $x \leftarrow y$ ,  $x \leftarrow d$ ,
- input value  $?x$ ,
- output value  $!x$  or  $!d$ ,
- dummy operation NOP,

where  $x, y$  are variables and  $d$  is a constant in  $\mathbb{Z}$ . A relational automaton  $\mathcal{A} = \langle Q, \delta, op, g \rangle$  [Čer94] is a finite directed graph where

- $Q$  is a finite set of control states,
- $\delta \subseteq Q \times Q$ ,
- $op : Q \rightarrow \text{OP}$ ,
- $g : \delta \rightarrow \{+, -\}$ .

Let  $Var(\mathcal{A})$  and  $Cons(\mathcal{A})$  be respectively the sets of variables and constants occurring in  $op$ . A configuration of  $\mathcal{A}$  is a pair  $\langle n, v \rangle$  where  $n \in Q$  and  $v$  is a map  $v : Var(\mathcal{A}) \cup Cons(\mathcal{A}) \rightarrow \mathbb{Z}$  equal to identity for its restriction to  $Cons(\mathcal{A})$ . The configuration graph of  $\mathcal{A}$  is defined as the pair  $\langle S, \rightarrow \rangle$  where  $S$  is the set of configurations of  $\mathcal{A}$  and  $\langle n, v \rangle \rightarrow \langle n', v' \rangle$  iff there exists  $e = \langle n, n' \rangle \in \delta$  such that

1.  $v$  and  $v'$  are related depending on the nature of  $op(n)$ :
    - if  $op(n)$  is  $?x$ , then for every  $y \in Var(\mathcal{A}) \setminus \{x\}$ ,  $v'(y) = v(y)$ ,
    - if  $op(n)$  is an output value or the dummy operation, then  $v' = v$ ,
    - if  $op(n)$  is  $x \leftarrow a$ , then  $v' = v[x \leftarrow v(a)]$ ,
    - if  $op(n)$  is  $a < b$ , then  $v = v'$  and
      - either  $g(e) = +$  and  $v(a) < v(b)$ ,
      - or  $g(e) = -$  and  $v(a) \geq v(b)$ ,
- where  $a, b \in Var(\mathcal{A}) \cup Cons(\mathcal{A})$ ,  $v(x)$  denotes the value of  $x$  in  $v$  and  $v[x \leftarrow a]$  is such that  $v[x \leftarrow z](y) = z$  if  $x = y$  and  $v[x \leftarrow z](y) = v(y)$  otherwise.

Observe that equality between variables can be tested by performing two negative tests on  $a < b$  and  $b < a$ .

From a relational automaton  $\mathcal{A} = \langle Q, \delta, op, g \rangle$ , we build a restricted IPC\*-automaton  $\mathcal{A}'$ , having an isomorphic configuration graph in a sense to be precised below. Given  $\mathcal{A} = \langle Q, \delta, op, g \rangle$ , the restricted IPC\*-automaton  $\mathcal{A}' = \langle Q', Q'_0, \delta', F' \rangle$  is defined as follows:

1.  $Q' = Q'_0 = F' = Q$ . Without any loss of generality, we can assume that  $Q$  is a finite set of  $\mathbb{Z}$  constants not occurring in  $op$ .
2. To each  $e = \langle n, n' \rangle$  in  $\mathcal{A}$ , we associate  $n \xrightarrow{\phi_e} n'$  in  $\delta'$ , where  $\phi_e$  is a conjunction of constraints defined as follows:

- $\mathbf{X}ic = n'$  is a conjunct of  $\phi_e$  where  $ic$  is a new variable taking care of the instruction counter,
- if  $op(n) = ?x$  then  $\bigwedge_{y \in \text{VAR}(\mathcal{A}) \setminus \{x\}} y = \mathbf{X}y$  belongs to  $\phi_e$ ,
- if  $op(n)$  is an output or a dummy operation, then  $\bigwedge_{y \in \text{VAR}(\mathcal{A})} y = \mathbf{X}y$  belongs to  $\phi_e$ ,
- if  $op(n) = x \leftarrow a$  then  $\bigwedge_{y \in \text{VAR}(\mathcal{A}) \setminus \{x\}} y = \mathbf{X}y \wedge \mathbf{X}x = a$  belongs to  $\phi_e$ ,
- if  $op(n) = a < b$  then
  - if  $g(e) = +$  then  $a < b \wedge \bigwedge_{y \in \text{VAR}(\mathcal{A})} y = \mathbf{X}y$  belongs to  $\phi_e$ ,
  - $g(e) = -$  then  $a \geq b \wedge \bigwedge_{y \in \text{VAR}(\mathcal{A})} y = \mathbf{X}y$  belongs to  $\phi_e$ .

The configuration graphs of  $\mathcal{A}$  and  $\mathcal{A}'$  are isomorphic with the following property. The transition  $\langle n, v \rangle \rightarrow \langle n', v' \rangle$  belongs to the configuration graph of  $\mathcal{A}$  iff  $\langle n, \bar{v} \rangle \rightarrow \langle n', \bar{v}' \rangle$  belongs to the configuration graph of  $\mathcal{A}'$  where  $\bar{v} : \text{Var}(\mathcal{A}) \cup \text{Cons}(\mathcal{A}) \cup \{ic\} \rightarrow \mathbb{Z}$  is a conservative extension of  $v : \text{Var}(\mathcal{A}) \cup \text{Cons}(\mathcal{A}) \rightarrow \mathbb{Z}$  and  $\bar{v}(ic) = n$ . The map  $\bar{v}'$  is defined similarly. As a corollary, the LTL fragment of CLTL\* defined in [Čer94] where the atomic formulae are of the form  $n$ ,  $x < y$  and  $x = y$ , can be reduced to the model-checking problem for restricted IPC\*-automata (just replace  $n$  by  $ic = n$  to obtain CLTL(IPC\*) formulae).