Weighted automata and weighted logics

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1 Introduction

In automata theory, Büchi's and Elgot's fundamental theorems [6, 7, 25] established the coincidence of regular and ω-regular languages with languages definable in monadic second-order logic. At the same time, Schützenberger [57] investigated finite automata with weights and characterized their behaviours as rational formal power series. Both of these results have inspired a wealth of extensions and further research, cf. [4, 24, 42, 56, 60] for monographs and surveys as well as the chapters [26, 55] of this handbook, and also led to recent practical applications, e.g. in verification of finite-state programs (model checking, [3,43,46]), in digital image compression [11,33,35,36] and in speech-to-text processing [8,49,51], cf. also chapters [1,29,38,50] of the present handbook [17].

It is the goal of this chapter to introduce a logic with weights taken from an arbirary semiring and to present conditions under which the behaviours of weighted finite automata are precisely the series definable in our weighted monadic second-order logic. We will deal with both finite and infinite words. In comparison to the essential predecessors [13, 14, 20], our logic will be defined in a purely syntactical way, and the results apply to arbitrary (also non-commutative) semirings.

Our motivation for this *weighted logic* is as follows. First, weighted automata and their behaviour can be viewed as a quantitative extension of classical automata. The latter decide whether a given word is accepted or not, whereas weighted automata also compute e.g. the ressources, time or cost used or the probability of its success when executing the word. We would like to have an extension of Büchi's and Elgot's theorems to this setting. Second, classical logic for automata describes whether a certain property (e.g. "there exist three consecutive a's") holds for a given word or not. One could be in-

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terested in knowing how often this property holds, i.e., again in extending the previous qualitative statement to a quantitative one.

Next we describe the syntax of our weighted logics. Its definition incorporates weights taken as elements from a given abstract semiring S, just as done for weighted automata in order to model a variety of applications and situations. Also, our syntax should extend classical (unweighted) MSO logics. The semantics of a weighted logic formula φ should be a formal power series over an extended alphabet and with values in S. It is possible to assign a natural semantics to atomic formulas, to disjunction and conjunction, and to existential and universal quantifications, but a problem arises with negation. It would be natural to define the semantics of $\neg\varphi$ elementwise. But if S is not a Boolean algebra, S does not have a natural complement operation. Therefore we restrict negation to atomic formulas whose semantics will take as values only 0 and 1 in S; then the negation of atomic formulas also has a natural semantics. In comparison to classical MSO-logic, this is not an essential restriction, since the negation of a classical MSO-formula is equivalent (in the sense of defining the same language) to one in which negation is applied only to atomic formulas. This requires us to include conjunction and universal quantifications into our syntax (which we do). In this sense, our weighted MSO-logics then contains the classical MSO-logics which we obtain by letting $S = \mathbb{B}$, the 2-element Boolean algebra.

We define the semantics of sentences φ of our weighted MSO-logic by structural induction over φ. Thus, as usual, we also define the semantics of a formula φ with free variables, here as a formal power series over an extended alphabet. But even for the semiring of natural numbers or the tropical semiring it turns out that neither universal first-order nor universal second-order quantification of formulas preserve recognizability, i.e., representability of their semantics as behaviour of a weighted automaton, and for other (non-commutative) semirings, conjunction does not preserve recognizability. Therefore we have to restrict conjunction and universal quantifications. We show that each formula in our logic which does not contain weights from the semiring (except 0 or 1) has a syntactic representation which is "unambiguous" and so its associated series takes on only 0 or 1 as values. We permit universal second-order quantification only for such syntactically unambiguous formulas, and universal first-order quantification for formulas in the disjunctive-conjunctive closure of arbitrary constants from the semiring and syntactically unambiguous formulas. With an additional restriction of conjunction, we obtain our class of syntactically restricted weighted MSO-formulas. Moreover, if we allow existential set quantifications only to occur at the beginning of a fomula, we arrive at syntactically restricted existential MSO-logic.

Now we give a summary of our results. First we show for any semiring S that the behaviours of weighted automata with values in S are precisely the series definable by sentences of our syntactically restricted MSO-logic, or, equivalently, of our syntactically restricted existential MSO-logic.

Second, if the semiring S is additively locally finite, we can apply universal first-order quantification even to the existential-disjunctive-conjunctive closure of the set of formulas decribed above and still obtain that the semantics of such sentences are representable by weighted automata. Third, if the semiring S is (additively and multiplicatively) locally finite, it suffices to just restrict universal second-order quantification and we still obtain sentences with representable semantics. Locally finite resp. additively locally finite semirings were investigated in [12, 19]; they form large classes of semirings. Fourthly, we also deal with infinite words. As is well-known and customary [10, 24, 26], here one has to impose certain completeness properties on the semiring, i.e., infinite sums and products exist and interact nicely, in order to ensure that the behaviour of weighted automata (and the semantics of weighted formulas) can be defined. Under such suitable completeness assumptions on the semiring, we again obtain that our syntactically restricted MSO-logic (syntactically defined in the same way, but now with semantics on infinite words) is expressively equivalent to a model of weighted Muller automata, and if the semiring is, furthermore, idempotent (like the max-plus- and min-plus-semirings), the same applies to our extension of syntactically restricted MSO-logic described above. We note that we obtain Büchi's and Elgot's theorems for languages of finite and infinite words as particular consequences. Moreover, if the semiring S is given in some effective way, then the constructions in our proofs yield effective conversions of sentences of our weighted logic to weighted automata, and viceversa. If, in addition, S is a field or locally finite, for the case of finite words we also obtain decision procedures.

2 MSO-logic and weighted automata

In this section, we summarize for the convenience of the reader our notation used for classical MSO-logic and basic background of weighted automata acting on finite words. We assume that the reader is familiar with the basics of monadic second-order logic and Büchi's theorem for languages of finite words, cf. [37, 60]. Let Σ be an alphabet. The syntax of formulas of MSO(Σ), the monadic second-order logic over Σ, is given by the grammar

$$\varphi ::= P_a(x) \mid x \leq y \mid x \in X \mid \varphi \vee \varphi \mid \neg\varphi \mid \exists x.\varphi \mid \exists X.\varphi$$

where a ranges over Σ, x, y are first-order variables and X is a set variable. We let Free(φ) be the set of all free variables of φ.

We let Σ^* be the free monoid of all finite words $w = w(1) \dots w(n)$ ($n \geq 0$). If $w \in \Sigma^*$ has length n, we put $\mathrm{dom}(w) = \{1, \dots, n\}$. The word $w \in \Sigma^*$ is usually represented by the structure $(\mathrm{dom}(w), \leq, (R_a)_{a\in\Sigma})$ where $R_a = \{i \in \mathrm{dom}(w) \mid w(i) = a\}$ for $a \in \Sigma$.

Let $\mathcal{V}$ be a finite set of first-order and second-order variables. A $(\mathcal{V}, w)$-assignment σ is a function mapping first-order variables in $\mathcal{V}$ to elements of dom(w) and second-order variables in $\mathcal{V}$ to subsets of dom(w). If x is a first-order variable and $i \in \mathrm{dom}(w)$ then $\sigma[x \to i]$ is the $(\mathcal{V} \cup \{x\}, w)$-assignment which assigns x to i and acts like σ on all other variables. Similarly, $\sigma[X \to I]$ is defined for $I \subseteq \mathrm{dom}(w)$. The definition that (w, σ) satisfies φ, denoted $(w, \sigma) \models \varphi$, is as usual assuming that the domain of σ contains Free(φ). Note that $(w, \sigma) \models \varphi$ only depends on the restriction $\sigma_{|\,\mathrm{Free}(\varphi)}$ of σ to Free(φ).

As usual, a pair (w, σ) where σ is a $(\mathcal{V}, w)$-assignment will be encoded using an extended alphabet $\Sigma_{\mathcal{V}} = \Sigma \times \{0,1\}^{\mathcal{V}}$. More precisely, we will write a word over $\Sigma_{\mathcal{V}}$ as a pair (w, σ) where w is the projection over Σ and σ is the projection over $\{0,1\}^{\mathcal{V}}$. Now, σ represents a *valid* assignment over $\mathcal{V}$ if for each first-order variable $x \in \mathcal{V}$, the x-row of σ contains exactly one 1. In this case, we identify σ with the $(\mathcal{V}, w)$-assignment such that for each first-order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row, and for each second-order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions carrying a 1 on the X-row. Clearly, the language

$$N_{\mathcal{V}} = \{(w, \sigma) \in \Sigma_{\mathcal{V}}^* \mid \sigma \text{ is a valid } (\mathcal{V}, w)\text{-assignment}\}$$

is recognizable. We simply write $\Sigma_\varphi = \Sigma_{\mathrm{Free}(\varphi)}$ and $N_\varphi = N_{\mathrm{Free}(\varphi)}$. By Büchi's theorem, if $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$ then the language

$$\mathcal{L}_{\mathcal{V}}(\varphi) = \{(w, \sigma) \in N_{\mathcal{V}} \mid (w, \sigma) \models \varphi\}$$

defined by φ over $\Sigma_{\mathcal{V}}$ is recognizable. Again, we simply write $\mathcal{L}(\varphi)$ for $\mathcal{L}_{\mathrm{Free}(\varphi)}(\varphi)$. Conversely, each recognizable language L in Σ^* is definable by an MSO-sentence φ, so $L = \mathcal{L}(\varphi)$.

Next, we turn to basic definitions and properties of semirings, formal power series and weighted automata. For background, we refer the reader to [4,42,56] and to [15, 26, 55] in this handbook.

A *semiring* is a structure $(S, +, \cdot, 0, 1)$ where $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \cdot s = s \cdot 0 = 0$ for each $s \in S$. If the multiplication is commutative, we say that S is *commutative*. If the addition is idempotent, then the semiring is called *idempotent*. Important examples include

- the natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ with the usual addition and multiplication,
- the Boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$,
- the tropical semiring $\mathrm{Trop} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ (also known as min-plus semiring), with min and $+$ extended to $\mathbb{N} \cup \{\infty\}$ in the natural way,
- the arctic semiring $\mathrm{Arc} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$,
- the semiring $([0,1], \max, \cdot, 0, 1)$ which can be used to compute probabilities,
- the semirings of languages $(\mathcal{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*)$ and $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$.

Given two subsets A, B of a semiring S, we say that A and B *commute element-wise*, if $a \cdot b = b \cdot a$ for all $a \in A$ and $b \in B$. We let S_A denote the subsemiring of S generated by A. Clearly, due to the distributivity law the elements of S_A can be obtained by taking finite sums of finite products of elements of A. It follows that if $A, B \subseteq S$ and A and B commute element-wise, then S_A and S_B also commute element-wise. If S is a semiring and $n \in \mathbb{N}$, then $S^{n\times n}$ comprises all $(n \times n)$-matrices over S. With usual matrix multiplication and the unit matrix E, $(S^{n\times n}, \cdot, E)$ is a monoid.

A formal power series over a set $\mathcal{Z}$ is a mapping $r : \mathcal{Z} \to S$. In this paper, we will use for $\mathcal{Z}$ either the set Σ^* of finite words, or in Sections 7 and 8 the set Σ^ω of infinite words. It is usual to write (r, w) for $r(w)$. The set $\mathrm{supp}(r) := \{w \in \mathcal{Z} \mid (r, w) \neq 0\}$ is called the *support* of r. The set of all formal power series over S and $\mathcal{Z}$ is denoted by $S\langle\langle \mathcal{Z} \rangle\rangle$. Now let $r, r_1, r_2 \in S\langle\langle \mathcal{Z} \rangle\rangle$ and $s \in S$. The *sum* $r_1 + r_2$, the *Hadamard product* $r_1 \odot r_2$, and the *scalar products* $s \cdot r$ and $r \cdot s$ are each defined pointwise for $w \in \mathcal{Z}$:

$$\begin{aligned}
(r_1 + r_2, w) &:= (r_1, w) + (r_2, w)\\
(r_1 \odot r_2, w) &:= (r_1, w) \cdot (r_2, w)\\
(s \cdot r, w) &:= s \cdot (r, w)\\
(r \cdot s, w) &:= (r, w) \cdot s
\end{aligned}$$

Then $(S\langle\langle \mathcal{Z} \rangle\rangle, +, \odot, 0, 1)$ where 0 and 1 denote the constant series with values 0 resp. 1, is again a semiring.

For $L \subseteq \mathcal{Z}$, we define the *characteristic series* $\mathbb{1}_L : \mathcal{Z} \to S$ by $(\mathbb{1}_L, w) = 1$ if $w \in L$, and $(\mathbb{1}_L, w) = 0$ otherwise. If $S = \mathbb{B}$, the correspondence $L \mapsto \mathbb{1}_L$ gives a useful and natural semiring isomorphism from $(\mathcal{P}(\mathcal{Z}), \cup, \cap, \emptyset, \mathcal{Z})$ onto $(\mathbb{B}\langle\langle \mathcal{Z} \rangle\rangle, +, \odot, 0, 1)$.

Now we turn to weighted automata over finite words. We fix a semiring S and an alphabet Σ. A *weighted finite automaton* over S and Σ is a quadruple $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ where Q is a finite set of states, $\mu : \Sigma \to S^{Q\times Q}$ is the transition weight function and $\lambda, \gamma : Q \to S$ are weight functions for entering and leaving a state, respectively. Here $\mu(a)$ is a $(Q \times Q)$-matrix whose (p, q)-entry $\mu(a)_{p,q} \in S$ indicates the weight (cost) of the transition $p \xrightarrow{a} q$. We also write $\mathrm{wt}(p, a, q) = \mu(a)_{p,q}$. Then μ extends uniquely to a monoid homomorphism (also denoted by μ) from Σ^* into $(S^{Q\times Q}, \cdot, E)$.

The *weight* of a path $P : q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} q_{n-1} \xrightarrow{a_n} q_n$ in $\mathcal{A}$ (where $n \geq 0$) is the product $\mathrm{weight}(P) := \lambda(q_0) \cdot \mu(a_1)_{q_0,q_1} \cdots \mu(a_n)_{q_{n-1},q_n} \cdot \gamma(q_n)$. This path has label $a_1 \dots a_n$. If $n = 0$ and $P = (q_0)$, we have $\mathrm{weight}(P) = \lambda(q_0)\cdot\gamma(q_0)$. The *weight* of a word $w = a_1 \dots a_n \in \Sigma^*$ in $\mathcal{A}$, denoted $(||\mathcal{A}||, w)$, is the sum of $\mathrm{weight}(P)$ over all paths P with label w. One can check that

$$(||\mathcal{A}||, w) = \sum_{p,q\in Q} \lambda(p) \cdot \mu(w)_{pq} \cdot \gamma(q) = \lambda \cdot \mu(w) \cdot \gamma$$

with usual matrix multiplication, considering λ as a row vector and γ as a column vector. If $w = \varepsilon$, we have $(||\mathcal{A}||, \varepsilon) = \lambda \cdot \gamma$. The formal power series $||\mathcal{A}|| : \Sigma^* \to S$ is called the *behavior* of $\mathcal{A}$. A formal power series $r \in S\langle\langle \Sigma^* \rangle\rangle$ is called *recognizable*, if there exists a weighted finite automaton $\mathcal{A}$ such that

$r = ||\mathcal{A}||$. We let $\mathrm{Rec}(S, \Sigma^*)$ be the collection of all recognizable formal power series over S and Σ.

Lemma 2.1 ([24, 55]).

(a) For any recognizable language $L \subseteq \Sigma^*$, the series $\mathbb{1}_L$ is recognizable.
(b) Let $r, r_1, r_2 \in S\langle\langle\Sigma^*\rangle\rangle$ be recognizable, and let $s \in S$. Then $r_1 + r_2$, $s \cdot r$ and $r \cdot s$ are recognizable.
(c) Let $S_1, S_2 \subseteq S$ be two subsemirings such that S_1 and S_2 commute element-wise. Let $r_1 \in \mathrm{Rec}(S_1, \Sigma^*)$ and $r_2 \in \mathrm{Rec}(S_2, \Sigma^*)$. Then $r_1 \odot r_2 \in \mathrm{Rec}(S, \Sigma^*)$.

As an immediate consequence of Lemma 2.1(c), for any recognizable series $r \in S\langle\langle\Sigma^*\rangle\rangle$ and recognizable language $L \subseteq \Sigma^*$, the series $r \odot \mathbb{1}_L$ is again recognizable

Now let $h : \Sigma^* \to \Gamma^*$ be a homomorphism. For $r \in S\langle\langle\Gamma^*\rangle\rangle$ let $h^{-1}(r) = r \circ h \in S\langle\langle\Sigma^*\rangle\rangle$. That is, $(h^{-1}(r), w) = (r, h(w))$ for all $w \in \Sigma^*$. We call h *length-preserving*, if $|w| = |h(w)|$ for each $w \in \Sigma^*$. We say that h is *non-erasing*, if $h(a) \neq \varepsilon$ for each $a \in \Sigma$, or, equivalently, $|w| \leq |h(w)|$ for each $w \in \Sigma^*$. In this case, for $r \in S\langle\langle\Sigma^*\rangle\rangle$, define $h(r) : \Gamma^* \to S$ by $(h(r), v) := \sum_{w \in h^{-1}(v)} (r, w)$ ($v \in \Gamma^*$), noting that the sum is finite.

Lemma 2.2 ([24, 54]). *Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism.*

(a) $h^{-1} : S\langle\langle\Gamma^*\rangle\rangle \to S\langle\langle\Sigma^*\rangle\rangle$ preserves recognizability.
(b) Let h be non-erasing. Then $h : S\langle\langle\Sigma^*\rangle\rangle \to S\langle\langle\Gamma^*\rangle\rangle$ preserves recognizability.

We say $r : \Sigma^* \to S$ is a *recognizable step function*, if $r = \sum_{i=1}^{n} s_i \cdot \mathbb{1}_{L_i}$ for some $n \in \mathbb{N}$, $s_i \in S$ and recognizable languages $L_i \subseteq \Sigma^*$ ($i = 1, \dots, n$). Then clearly r is a recognizable series by Lemma 2.1(a),(b). The following closure result is easy to see.

Lemma 2.3. *(a) (cf. [12]) The class of all recognizable step functions over Σ and S is closed under sum, scalar products and Hadamard products.*
(b) Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism. Then $h^{-1} : S\langle\langle\Gamma^*\rangle\rangle \to S\langle\langle\Sigma^*\rangle\rangle$ preserves recognizable step functions.

Proof. (b) Let $r = \sum_{i=1}^{n} s_i \cdot \mathbb{1}_{L_i}$ be a recognizable step function with recognizable languages $L_i \subseteq \Gamma^*$. Then each language $h^{-1}(L_i) \subseteq \Sigma^*$ is also recognizable, hence $h^{-1}(r) = \sum_{i=1}^{n} s_i \cdot (\mathbb{1}_{L_i} \circ h) = \sum_{i=1}^{n} s_i \cdot \mathbb{1}_{h^{-1}(L_i)}$ is a recognizable step function. □

3 Weighted logics

In this section, we introduce our weighted logic and study its first properties. We fix a semiring S and an alphabet Σ. For each $a \in \Sigma$, P_a denotes a unary predicate symbol.

Definition 3.1. *The syntax of formulas of the* weighted MSO-logic *is given by the grammar*

$$\begin{aligned}\varphi ::= {} & s \mid P_a(x) \mid \neg P_a(x) \mid x \leq y \mid \neg(x \leq y) \mid x \in X \mid \neg(x \in X) \\ & \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \exists X.\varphi \mid \forall x.\varphi \mid \forall X.\varphi\end{aligned}$$

where $s \in S$ *and* $a \in \Sigma$. *We denote by* $\mathrm{MSO}(S, \Sigma)$ *the collection of all such weighted* MSO-*formulas* φ.

Here, we do not permit negation of general formulas due to difficulties defining then their semantics: The semantics of a weighted logic formula φ should be a formal power series over an extended alphabet and with values in S. It would be natural to define the semantics of $\neg\varphi$ element-wise. In fact, this is possible if S is a bounded distributive lattice with complement function, like, e.g. any Boolean algebra or the semiring $S = ([0,1], \max, \min, 0, 1)$ with complement function $x \mapsto 1 - x$ ($x \in [0,1]$), cf. [16, 54]. But in general, arbitrary semirings as well as many important specific semirings do not have a natural complement function.

Therefore, as noted in the introduction, we restrict negation to atomic formulas whose semantics will take as values only 0 and 1 in S; thus the negation of atomic formulas takes as values 1 and 0. Since the negation of a classical MSO-formula is equivalent (in the sense of defining the same language) to one in which negation is applied only to atomic formulas, in this sense our weighted MSO-logic contains the classical MSO-logic which we obtain by letting $S = \mathbb{B}$. Note that in this case, the constant s in the logic is either 0 (false) or 1 (true).

Now we turn to the definition of the semantics of formulas $\varphi \in \mathrm{MSO}(S, \Sigma)$. As usual, a variable is said to be *free* in φ if there is an occurrence of it in φ not in the scope of a quantifier. A pair (w, σ) where $w \in \Sigma^*$ and σ is a $(\mathcal{V}, w)$-assignment is represented by a word over the extended alphabet $\Sigma_{\mathcal{V}}$ as explained in Section 2. We will define the $\mathcal{V}$-semantics $[\![\varphi]\!]_{\mathcal{V}}$ of φ as a formal power series $[\![\varphi]\!]_{\mathcal{V}} : \Sigma_{\mathcal{V}}^* \to S$. This will enable us to investigate when $[\![\varphi]\!]_{\mathcal{V}}$ is a recognizable series. Also, by letting $S = \mathbb{B}$, the Boolean semiring, we can immediately compare our semantics with the classical one assigning languages to formulas.

Definition 3.2. *Let* $\varphi \in \mathrm{MSO}(S, \Sigma)$ *and* $\mathcal{V}$ *be a finite set of variables containing* $\mathrm{Free}(\varphi)$. *The* $\mathcal{V}$-semantics *of* φ *is a formal power series* $[\![\varphi]\!]_{\mathcal{V}} \in S\langle\langle \Sigma_{\mathcal{V}}^* \rangle\rangle$. *Let* $(w, \sigma) \in \Sigma_{\mathcal{V}}^*$. *If* σ *is not a valid* $(\mathcal{V}, w)$-*assignment, then we put* $[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) = 0$. *Otherwise, we define* $[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) \in S$ *inductively as follows:*

$$[\![s]\!]_{\mathcal{V}}(w, \sigma) = s$$

$$[\![P_a(x)]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$$

$$[\![x \leq y]\!]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$$

$$[\![x \in X]\!]_{\mathcal{V}}(w,\sigma) = \begin{cases} 1 & \textit{if } \sigma(x) \in \sigma(X) \\ 0 & \textit{otherwise} \end{cases}$$

$$[\![\neg\varphi]\!]_{\mathcal{V}}(w,\sigma) = \begin{cases} 1 & \textit{if } [\![\varphi]\!]_{\mathcal{V}}(w,\sigma) = 0 \\ 0 & \textit{if } [\![\varphi]\!]_{\mathcal{V}}(w,\sigma) = 1 \end{cases} \quad \begin{array}{l} \textit{if } \varphi \textit{ is of the form } P_a(x), \\ (x \leq y) \textit{ or } (x \in X). \end{array}$$

$$[\![\varphi \vee \psi]\!]_{\mathcal{V}}(w,\sigma) = [\![\varphi]\!]_{\mathcal{V}}(w,\sigma) + [\![\psi]\!]_{\mathcal{V}}(w,\sigma)$$

$$[\![\varphi \wedge \psi]\!]_{\mathcal{V}}(w,\sigma) = [\![\varphi]\!]_{\mathcal{V}}(w,\sigma) \cdot [\![\psi]\!]_{\mathcal{V}}(w,\sigma)$$

$$[\![\exists x.\varphi]\!]_{\mathcal{V}}(w,\sigma) = \sum_{i \in \mathrm{dom}(w)} [\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(w,\sigma[x \to i])$$

$$[\![\exists X.\varphi]\!]_{\mathcal{V}}(w,\sigma) = \sum_{I \subseteq \mathrm{dom}(w)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(w,\sigma[X \to I])$$

$$[\![\forall x.\varphi]\!]_{\mathcal{V}}(w,\sigma) = \prod_{i \in \mathrm{dom}(w)} [\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(w,\sigma[x \to i])$$

$$[\![\forall X.\varphi]\!]_{\mathcal{V}}(w,\sigma) = \prod_{I \subseteq \mathrm{dom}(w)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(w,\sigma[X \to I])$$

where in the product over $\mathrm{dom}(w)$ *we follow the natural order, and we fix some order on the power set of* $\{1,\ldots,|w|\}$ *so that the last product is defined. We simply write* $[\![\varphi]\!]$ *for* $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$.

Note that if φ is a sentence, i.e., has no free variables, then $[\![\varphi]\!] \in S\langle\langle\Sigma^*\rangle\rangle$. We give several examples of possible interpretations for weighted formulas:

I. Let S be an arbitrary Boolean algebra $(B, \vee, \wedge, \bar{\ }, 0, 1)$. In this case, sums correspond to suprema, and products to infima. Here we can define the semantics of $\neg\varphi$ for an arbitrary formula φ by $[\![\neg\varphi]\!](w,\sigma) := \overline{[\![\varphi]\!](w,\sigma)}$, the complement of $[\![\varphi]\!](w,\sigma)$ in B. Then clearly $[\![\varphi \wedge \psi]\!] = [\![\neg(\neg\varphi \vee \neg\psi)]\!]$, $[\![\forall x.\varphi]\!] = [\![\neg(\exists x.\neg\varphi)]\!]$ and $[\![\forall X.\varphi]\!] = [\![\neg(\exists X.\neg\varphi)]\!]$. This may be interpreted as a multi-valued logic. In particular, if $S = \mathbb{B}$, the 2-valued Boolean algebra, our semantics coincides with the usual semantics of unweighted MSO-formulas, identifying characteristic series with their supports. For the more general case where S is a bounded distributive lattice with complement function, we refer the reader to [54].

II. Let $S = (\mathbb{N}, +, \cdot, 0, 1)$ and assume φ does not contain constants $s \in \mathbb{N}$. We may interpret $[\![\varphi]\!](w,\sigma)$ as the number of proofs or arguments we have that (w,σ) satisfies formula φ. Here, the notion of "proof" should not be considered in an exact proof-theoretic, but in an intuitive sense. Indeed, for atomic formulas the number of proofs should be 0 or 1, depending on whether φ holds for (w,σ) or not. Now if e.g. $[\![\varphi]\!](w,\sigma) = m$ and $[\![\psi]\!](w,\sigma) = n$, the number of proofs that (w,σ) satisfies $\varphi \vee \psi$ should be $m + n$ (since any proof suffices), and for $\varphi \wedge \psi$ it should be $m \cdot n$ (since we may pair the proofs of φ and ψ arbitrarily). Similarly, the semantics of the existential and universal quantifiers can be interpreted.

III. The formula $\exists x.P_a(x)$ counts how often a occurs in the word. Here *how often* depends on the semiring: e.g. natural numbers, Boolean semiring, integers modulo 2, ...
IV. Consider the probability semiring $S = ([0,1], \max, \cdot, 0, 1)$ and the alphabet $\Sigma = \{a_1, \dots, a_n\}$. Assume that each letter a_i has a reliability p_i. Then, the series assigning to a word its reliability can be given by the first-order formula $\forall x. \bigvee_{1\le i\le n}(P_{a_i}(x) \wedge p_i)$.
V. Let $S = ([0,1], \max, \otimes, 0, 1)$ where $x \otimes y = \max(0, x+y-1)$, the semiring occurring in the MV-algebra used to define the semantics of Łukasiewicz multi-valued logic [31]. For this semiring, a restriction of Łukasiewicz logic coincides with our weighted MSO-logic [59].

Observe that if $\varphi \in \mathrm{MSO}(S, \Sigma)$, we have defined a semantics $[\![\varphi]\!]_{\mathcal{V}}$ for each finite set of variables $\mathcal{V}$ containing $\mathrm{Free}(\varphi)$. Now we show that these semantics' are consistent with each other.

Proposition 3.3. *Let* $\varphi \in \mathrm{MSO}(S, \Sigma)$ *and* $\mathcal{V}$ *a finite set of variables containing* $\mathrm{Free}(\varphi)$. *Then*

$$[\![\varphi]\!]_{\mathcal{V}}(w, \sigma) = [\![\varphi]\!](w, \sigma_{|\,\mathrm{Free}(\varphi)})$$

for each $(w, \sigma) \in \Sigma^*_{\mathcal{V}}$ *such that* σ *is a valid* $(\mathcal{V}, w)$*-assignment. In particular,* $[\![\varphi]\!]$ *is recognizable iff* $[\![\varphi]\!]_{\mathcal{V}}$ *is recognizable, and* $[\![\varphi]\!]$ *is a recognizable step function iff* $[\![\varphi]\!]_{\mathcal{V}}$ *is a recognizable step function.*

Proof. The first claim can be shown by induction on the structure of φ.

For the final claim, consider the projection $\pi : \Sigma_{\mathcal{V}} \to \Sigma_{\varphi}$. For $(w, \sigma) \in \Sigma^*_{\mathcal{V}}$, we have $\pi(w, \sigma) = (w, \sigma_{|\,\mathrm{Free}(\varphi)})$. If $[\![\varphi]\!]$ is recognizable then $[\![\varphi]\!]_{\mathcal{V}} = \pi^{-1}([\![\varphi]\!]) \odot \mathbb{1}_{N_{\mathcal{V}}}$ is recognizable by Lemmas 2.1 and 2.2. This also shows that if $[\![\varphi]\!]$ is a recognizable step function, then so is $[\![\varphi]\!]_{\mathcal{V}}$ by Lemma 2.3.

Conversely, let F comprise the empty word and all $(w, \sigma) \in \Sigma^+_{\mathcal{V}}$ such that σ assigns to each variable x (resp. X) in $\mathcal{V} \setminus \mathrm{Free}(\varphi)$ position 1, i.e., $\sigma(x) = 1$ (resp. $\sigma(X) = \{1\}$). Then F is recognizable, and for each $(w, \sigma') \in \Sigma^*_{\varphi}$ there is a unique element $(w, \sigma) \in F$ such that $\pi(w, \sigma) = (w, \sigma')$. Thus $[\![\varphi]\!] = \pi([\![\varphi]\!]_{\mathcal{V}} \odot \mathbb{1}_F)$, as is easy to check. Hence, if $[\![\varphi]\!]_{\mathcal{V}}$ is recognizable then so is $[\![\varphi]\!]$ by Lemmas 2.1 and 2.2. Finally, note that $[\![\varphi]\!]$ assumes the same non-zero values as $[\![\varphi]\!]_{\mathcal{V}}$, and if $s \in S$, then $[\![\varphi]\!]^{-1}(s) = \pi([\![\varphi]\!]^{-1}_{\mathcal{V}}(s))$. Hence, if $[\![\varphi]\!]_{\mathcal{V}}$ is a recognizable step function, so is $[\![\varphi]\!]$. □

Now let $Z \subseteq \mathrm{MSO}(S, \Sigma)$. A series $r : \Sigma^* \to S$ is called *Z-definable*, if there is a sentence $\varphi \in Z$ such that $r = [\![\varphi]\!]$. The main goal of this paper is the comparison of Z-definable with recognizable series, for suitable fragments Z of $\mathrm{MSO}(S, \Sigma)$. Crucial for this will be closure properties of recognizable series under the constructs of our weighted logic. However, it is well-known that $\mathrm{Rec}(S, \Sigma^*)$ is in general not closed under the Hadamard product and hence not under conjunction.

Example 3.4. Let $\Sigma = \{a, b\}$, $S = (\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, and consider the formula $\varphi = \forall x.\big((P_a(x) \wedge \{a\}) \vee (P_b(x) \wedge \{b\})\big)$. Then $(\llbracket\varphi\rrbracket, w) = \{w\}$ for each $w \in \Sigma^*$. Clearly, $\llbracket\varphi\rrbracket$ is recognizable. However, $(\llbracket\varphi\wedge\varphi\rrbracket, w) = \{w\}\cdot\{w\} = \{w^2\}$ for each $w \in \Sigma^*$, and pumping arguments show that $\llbracket\varphi\wedge\varphi\rrbracket$ is not recognizable (cf. [23]).

Next we show that $\mathrm{Rec}(S, \Sigma^*)$ is in general not closed under universal quantification.

Example 3.5 (cf. [14]). Let $S = (\mathbb{N}, +, \cdot, 0, 1)$. Then $\llbracket\forall x.2\rrbracket(w) = 2^{|w|}$ and $\llbracket\forall y\forall x.2\rrbracket(w) = (2^{|w|})^{|w|} = 2^{|w|^2}$. Clearly, the series $\llbracket\forall x.2\rrbracket$ is recognizable by the weighted automaton $(Q, \lambda, \mu, \gamma)$ with $Q = \{1\}$, $\lambda_1 = \gamma_1 = 1$ and $\mu(a)_{1,1} = 2$ for all $a \in \Sigma$. However, $\llbracket\forall y\forall x.2\rrbracket$ is not recognizable. Suppose there was an automaton $\mathcal{A}' = (Q', \lambda', \mu', \gamma')$ with behavior $\llbracket\forall y\forall x.2\rrbracket$. Let $M = \max\{|\lambda'_p|, |\gamma'_p|, |\mu'(a)_{p,q}| \mid p, q \in Q', a \in \Sigma\}$. Then, for any $w \in \Sigma^*$ and for each path P labeled by w we have $\mathrm{weight}(P) \le M^{|w|+2}$ and since there are $|Q'|^{|w|+1}$ paths labeled w we obtain $(\|\mathcal{A}'\|, w) \le |Q'|^{|w|+1} \cdot M^{|w|+2}$, a contradiction with $(\|\mathcal{A}'\|, w) = 2^{|w|^2}$.

A similar argument applies also for the tropical and the arctic semirings. Observe that in all these cases, $\llbracket\forall x.2\rrbracket$ has infinite image.

Example 3.6 (cf. [19]). Let $S = (\mathbb{N}, +, \cdot, 0, 1)$. Then $(\llbracket\exists x.1\rrbracket, w) = |w|$ and $\llbracket\forall y.\exists x.1\rrbracket, w) = |w|^{|w|}$ for each $w \in \Sigma^*$. Hence $\llbracket\exists x.1\rrbracket$ is recognizable, but $\llbracket\forall y.\exists x.1\rrbracket$ is not, by the argument of the previous example. In contrast, if S is the tropical or arctic semiring (and 1 still the natural number 1), then $\llbracket\exists x.1\rrbracket$ takes on only two values, and $\llbracket\forall y.\exists x.1\rrbracket$ is recognizable.

Example 3.7. Let $S = (\mathbb{N}, +, \cdot, 0, 1)$. Then $\llbracket\forall X.2\rrbracket(w) = 2^{2^{|w|}}$ for any $w \in \Sigma^*$, and as above $\llbracket\forall X.2\rrbracket$ is not recognizable due to its growth. Again, this counterexample also works for the tropical and the arctic semirings.

The examples show that unrestricted conjunction and universal quantification are in general too strong to preserve recognizability. Therefore we will consider fragments of $\mathrm{MSO}(S, \Sigma)$. Their syntactic definition needs a little preparation on unambiguous formulas.

4 Unambiguous Formulas

In all of this section, let S be a semiring and Σ an alphabet. Here we will define our concepts of unambiguous and of syntactically unambiguous MSO-formulas. The idea is that if φ, ψ are formulas whose semantics $\llbracket\varphi\rrbracket, \llbracket\psi\rrbracket$ each takes on only 0 and 1 as values, this is in general no longer true for $\varphi\vee\psi$, $\exists x.\varphi$ and $\exists X.\varphi$ (except if S is idempotent), but we can find "equivalent" constructs assuming only $0, 1$ as values and for these formulas, the Boolean semantics will coincide with the weighted semantics. The unambiguous formulas may be viewed as the logical counterpart of unambiguous rational expressions (and may therefore have independent interest). We let $\mathrm{MSO}^-(S,\Sigma)$ consist of all formulas of $\mathrm{MSO}(S,\Sigma)$ which do not contain constants $s \in S \setminus \{0,1\}$.

Definition 4.1. *The class of* unambiguous *formulas in* $\mathrm{MSO}^-(S,\Sigma)$ *is defined inductively as follows:*

1. *All atomic formulas in* $\mathrm{MSO}^-(S,\Sigma)$ *are unambiguous.*
2. *If* φ, ψ *are unambiguous, then* $\varphi \wedge \psi$*,* $\forall x.\varphi$ *and* $\forall X.\varphi$ *are also unambiguous.*
3. *If* φ, ψ *are unambiguous and* $\mathrm{supp}([\![\varphi]\!]) \cap \mathrm{supp}([\![\psi]\!]) = \emptyset$*, then* $\varphi \vee \psi$ *is unambiguous.*
4. *Let* φ *be unambiguous and* $\mathcal{V} = \mathrm{Free}(\varphi)$*. If for any* $(w,\sigma) \in \Sigma_{\mathcal{V}}^*$ *there is at most one element* $i \in \mathrm{dom}(w)$ *such that* $[\![\varphi]\!]_{\mathcal{V}\cup\{x\}}(w, \sigma[x \to i]) \neq 0$*, then* $\exists x.\varphi$ *is unambiguous.*
5. *Let* φ *be unambiguous and* $\mathcal{V} = \mathrm{Free}(\varphi)$*. If for any* $(w,\sigma) \in \Sigma_{\mathcal{V}}^*$ *there is at most one subset* $I \subseteq \mathrm{dom}(w)$ *such that* $[\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(w, \sigma[X \to I]) \neq 0$*, then* $\exists X.\varphi$ *is unambiguous.*

Note that, as for unambiguous rational expressions, this is not a purely syntactic definition since some restrictions are on the semantics of formulas. First we note:

Proposition 4.2. *Let* $\varphi \in \mathrm{MSO}^-(S,\Sigma)$ *be unambiguous. We may also regard* φ *as a classical* MSO*-formula defining the language* $\mathcal{L}(\varphi) \subseteq \Sigma_\varphi^*$*. Then,* $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ *is a recognizable step function.*

Proof. Let $(w,\sigma) \in \Sigma_\varphi^*$. If $(w,\sigma) \notin N_\varphi$ then $[\![\varphi]\!](w,\sigma) = 0$ and $(w,\sigma) \notin \mathcal{L}(\varphi)$. Assume now that $(w,\sigma) \in N_\varphi$. We show by structural induction on φ that $[\![\varphi]\!](w,\sigma)$ equals 1 if $(w,\sigma) \models \varphi$ and equals 0 otherwise. This is clear for the atomic formulas and their negations. It is also trivial by induction for conjunction and universal quantifications. Using the unambiguity of the formulas, we also get the result by induction for disjunction and existential quantifications. Therefore, $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ and since $\mathcal{L}(\varphi)$ is a recognizable language in Σ_φ^* we obtain that $[\![\varphi]\!]$ is a recognizable step function. $\square$

Next we wish to give a purely syntactic definition of a class of unambiguous formulas and then show that any classical MSO-formula can be effectively transformed into an equivalent one which is syntactically unambiguous. We will proceed by structural induction on the given formula. Here (in contrast to [14]) we will include the case of formulas containing set quantifiers. When dealing with formulas of the form $\exists X.\varphi$ and $\forall X.\varphi$, we employ a linear order on the underlying structure (which is the power set of $\mathrm{dom}(w)$ where $w \in \Sigma^*$). For this, we recall that we identify (in assignments) subsets of $\mathrm{dom}(w)$ with their characteristic functions, and the set $\{0,1\}^{\mathrm{dom}(w)}$ carries the lexicographic order as a natural linear order. Let $y < x = \neg(x \leq y)$.

Definition 4.3. *For any* $\varphi, \psi \in \mathrm{MSO}^-(S,\Sigma)$*, we define inductively formulas* φ^+*,* φ^-*,* $\varphi \xrightarrow{+} \psi$ *and* $\varphi \xleftrightarrow{+} \psi$ *in* $\mathrm{MSO}^-(S,\Sigma)$ *by the following rules:*

1. *If* φ *is atomic, put* $\varphi^+ = \varphi$ *and* $\varphi^- = \neg\varphi$ *with the convention* $\neg\neg\psi = \psi$, *and* $\neg 0 = 1$, $\neg 1 = 0$.
2. $(\varphi \vee \psi)^+ = \varphi^+ \vee (\varphi^- \wedge \psi^+)$ *and* $(\varphi \vee \psi)^- = \varphi^- \wedge \psi^-$
3. $(\varphi \wedge \psi)^- = \varphi^- \vee (\varphi^+ \wedge \psi^-)$ *and* $(\varphi \wedge \psi)^+ = \varphi^+ \wedge \psi^+$
4. $(\exists x.\varphi)^+ = \exists x.(\varphi^+(x) \wedge \forall y.(y < x \wedge \varphi(y))^-)$ *and* $(\exists x.\varphi)^- = \forall x.\varphi^-$
5. $(\forall x.\varphi)^- = \exists x.(\varphi^-(x) \wedge \forall y.(x \leq y \vee \varphi(y))^+)$ *and* $(\forall x.\varphi)^+ = \forall x.\varphi^+$
6. $\varphi \xrightarrow{+} \psi = \varphi^- \vee (\varphi^+ \wedge \psi^+)$ *and* $\varphi \xleftrightarrow{+} \psi = (\varphi^+ \wedge \psi^+) \vee (\varphi^- \wedge \psi^-)$
7. *For set variables* X, Y, *we define the following macros*[3]

$$(X = Y) = \forall z.(z \in X \xleftrightarrow{+} z \in Y)$$
$$(X < Y) = \exists y.((y \in Y) \wedge \neg(y \in X) \wedge \forall z.(z < y \xrightarrow{+} (z \in X \xleftrightarrow{+} z \in Y)))$$
$$(X \leq Y) = (X = Y) \vee (X < Y)$$

8. $(\exists X.\varphi)^+ = \exists X.(\varphi^+(X) \wedge \forall Y.((Y < X) \wedge \varphi(Y))^-)$ *and* $(\exists X.\varphi)^- = \forall X.\varphi^-$
9. $(\forall X.\varphi)^- = \exists X.(\varphi^-(X) \wedge \forall Y.((X \leq Y) \vee \varphi(Y))^+)$ *and* $(\forall X.\varphi)^+ = \forall X.\varphi^+$.

We define the class of *(unweighted) syntactically unambiguous formulas* as the smallest class of formulas containing all formulas of the form

- $\varphi^+, \varphi^-, \varphi \xrightarrow{+} \psi$ or $\varphi \xleftrightarrow{+} \psi$ for $\varphi, \psi \in \mathrm{MSO}^-(S, \Sigma)$, and
- $\forall x.\varphi, \forall X.\varphi$ or $\varphi \wedge \psi$ if it contains φ and ψ.

By induction it is easy to show:

Lemma 4.4. *Let* $\varphi \in \mathrm{MSO}^-(S, \Sigma)$. *Then,*

- $\mathcal{L}(\varphi^+) = \mathcal{L}(\varphi)$ *and* $\mathcal{L}(\varphi^-) = \mathcal{L}(\neg\varphi)$,
- $[\![\varphi^+]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ *and* $[\![\varphi^-]\!] = \mathbb{1}_{\mathcal{L}(\neg\varphi)}$,
- *each syntactically unambiguous formula is unambiguous.*

The following result is a slight improvement of [14, Proposition 5.4].

Proposition 4.5. *For each classical* MSO*-sentence* φ, *we can effectively construct an unweighted syntactically unambiguous* $\mathrm{MSO}(S, \Sigma)$*-sentence* φ' *defining the same language, i.e.,* $[\![\varphi']\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$.

Proof. Using also conjunctions and universal quantifications, transform φ into an equivalent MSO-sentence ψ in which negation is only applied to atomic formulas. Then put $\varphi' = \psi^+$. □

We define $\mathrm{aUMSO}(S, \Sigma)$, the collection of *almost unambiguous* formulas in $\mathrm{MSO}(S, \Sigma)$, to be the smallest subset of $\mathrm{MSO}(S, \Sigma)$ containing all constants s ($s \in S$) and all syntactically unambiguous formulas and which is closed under disjunction and conjunction.

[3] The authors are thankful to Christian Mathissen for this formula $X < Y$ which simplifies an earlier more complicated formula of the authors.

We call two formulas $\varphi, \psi \in \mathrm{MSO}(S, \Sigma)$ *equivalent*, denoted $\varphi \equiv \psi$, if $[\![\varphi]\!] = [\![\psi]\!]$. Now we claim that each almost unambiguous formula ψ is equivalent to a formula ψ' of the form $\psi' = \bigvee_{j=1}^{n}(s_j \wedge \psi_j^+)$ for some $n \in \mathbb{N}$, $s_j \in S$ and $\psi_j \in \mathrm{MSO}^-(S, \Sigma)$ $(i = 1, \dots n)$. Indeed this follows from the following equivalences for any $\varphi, \xi, \zeta \in \mathrm{MSO}(S, \Sigma)$, $\pi, \rho \in \mathrm{MSO}^-(S, \Sigma)$ and $s, t \in S$:

$$\begin{aligned} \varphi \wedge (\xi \vee \zeta) &\equiv (\varphi \wedge \xi) \vee (\varphi \wedge \zeta) \\ \pi^+ \wedge s &\equiv s \wedge \pi^+ \\ \pi^+ &\equiv 1 \wedge \pi^+ \\ s \wedge t &\equiv st \\ \pi &\equiv \pi^+ \text{ if } \pi \text{ is unambiguous.} \end{aligned}$$

Moreover, by forming suitable conjunctions of the formulas ψ_j^+, ψ_j^- in ψ' above, we can obtain that the languages $\mathcal{L}_{\psi'}(\psi_j)$ $(j = 1, \dots, n)$ are pairwise disjoint; then ψ' could be viewed as a "weighted unambiguous" formula similar to Definition 4.1 (we will not need this notion, but it also motivates the notion "almost unambiguous" for ψ).

As a consequence of this description (or Lemma 2.3) and Lemma 4.4, for each $\psi \in \mathrm{aUMSO}(S, \Sigma)$, $[\![\psi]\!]$ is a recognizable step function.

For an arbitrary formula $\varphi \in \mathrm{MSO}(S, \Sigma)$, let $\mathrm{val}(\varphi)$ denote the set containing all values of S occurring in φ.

Next, we define our (weighted) syntactically restricted $\mathrm{MSO}(S, \Sigma)$-formulas:[4]

Definition 4.6. *A formula* $\varphi \in \mathrm{MSO}(S, \Sigma)$ *is called* syntactically restricted*, if it satisfies the following conditions:*

(1) Whenever φ contains a conjunction $\psi \wedge \psi'$ as subformula but not in the scope of a universal first order quantifier, then $\mathrm{val}(\psi)$ *and* $\mathrm{val}(\psi')$ *commute element-wise.*
(2) Whenever φ contains $\forall X.\psi$ as a subformula, then ψ is an unweighted syntactically unambiguous formula.
(3) Whenever φ contains $\forall x.\psi$ as a subformula, then ψ is almost unambiguous.

We let $\mathrm{sRMSO}(S, \Sigma)$ *denote the set of all syntactically restricted formulas of* $\mathrm{MSO}(S, \Sigma)$.

Here condition (1) requires us to be able to check for $x, y \in S$ whether $x \cdot y = y \cdot x$. We assume this basic ability to be given in syntax checks of formulas from $\mathrm{MSO}(S, \Sigma)$. Note that for $\psi, \psi' \in \mathrm{MSO}(S, \Sigma)$, $\mathrm{val}(\psi)$ and $\mathrm{val}(\psi')$ trivially commute element-wise, if S is commutative (which was the general assumption of [14]) or if ψ or ψ' is in $\mathrm{MSO}^-(S, \Sigma)$, thus in particular, if ψ or ψ' is unambiguous. Hence for each $\mathrm{MSO}(S, \Sigma)$-formula φ it can be easily checked effectively whether φ is syntactically restricted or not.

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[4] The authors would like to thank Dietrich Kuske for joint discussions which led to the development of this crucial concept.

A formula $\varphi \in \mathrm{MSO}(S, \Sigma)$ is *existential*, if it is of the form $\varphi = \exists X_1 . \ldots \exists X_n . \psi$ where ψ does not contain any set quantifier. The set of all syntactically restricted and existential formulas of $\mathrm{MSO}(S, \Sigma)$ is denoted $\mathrm{sREMSO}(S, \Sigma)$.

Our first main result which will be proved in Section 5 is:

Theorem 4.7. *Let S be any semiring and Σ an alphabet. Let $r : \Sigma^* \to S$ be a series. The following are equivalent:*

(1) r is recognizable.
(2) r is definable by some syntactically restricted sentence of $\mathrm{MSO}(S, \Sigma)$.
(3) r is definable by some syntactically restricted existential sentence of $\mathrm{MSO}(S, \Sigma)$.

We note that our proofs will be effective. That is, given a syntactically restricted sentence φ of $\mathrm{MSO}(S, \Sigma)$, we can construct a weighted automaton $\mathcal{A}$ with $||\mathcal{A}|| = [\![\varphi]\!]$ (provided the operations of S are given effectively). For the converse, given $\mathcal{A}$, we will explicitly describe a sentence $\varphi \in \mathrm{sREMSO}(S, \Sigma)$ with $[\![\varphi]\!] = ||\mathcal{A}||$.

Slightly extending [14], we call an $\mathrm{MSO}(S, \Sigma)$-formula φ *restricted*, if

(1) Whenever φ contains a conjunction $\psi \wedge \psi'$ as subformula but not in the scope of a universal first order quantifier, then $\mathrm{val}(\psi)$ and $\mathrm{val}(\psi')$ commute element-wise.
(2) Whenever φ contains $\forall X.\psi$ as a subformula, then ψ is an unambiguous formula.
(3) Whenever φ contains $\forall x.\psi$ as a subformula, then $[\![\psi]\!]$ is a recognizable step function.

Note that in particular conditions (2) and (3) are not purely syntactic, but use the semantics of formulas. In [14] it was shown that if S is a field or a locally finite semiring (cf. Section 6), then it can be effectively checked whether an arbitrary $\mathrm{MSO}(S, \Sigma)$-sentence φ is restricted or not. For the general case, this remained open.

Since, as noted before, the semantics of almost unambiguous formulas are recognizable step functions, we have:

Proposition 4.8. *Each syntactically restricted formula $\varphi \in$* $\mathrm{MSO}(S, \Sigma)$ *is restricted.*

5 Definability equals recognizability

In all of this section, let S be a semiring and Σ an alphabet. We wish to prove Theorem 4.7. For this, we first wish to show that whenever $\varphi \in \mathrm{MSO}(S, \Sigma)$ is restricted, then $[\![\varphi]\!]$ is recognizable. We proceed by induction over the structure of restricted MSO-formulas.

Lemma 5.1. *Let $\varphi \in \mathrm{MSO}(S, \Sigma)$ be atomic. Then $[\![\varphi]\!]$ is a recognizable step function.*

Proof. If $\varphi = s$ with $s \in S$, we have $[\![\varphi]\!] = s \cdot \mathbb{1}_{\Sigma^*}$. If φ is one of the other atomic formulas or their negations, then $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}(\varphi)}$ is immediate from the definition. □

Lemma 5.2. *Let $\varphi, \psi \in \mathrm{MSO}(S, \Sigma)$ such that $[\![\varphi]\!]$ and $[\![\psi]\!]$ are recognizable. Then $[\![\varphi \vee \psi]\!]$, $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are recognizable. Moreover, if $[\![\varphi]\!]$ and $[\![\psi]\!]$ are recognizable step functions, then $[\![\varphi \vee \psi]\!]$ is also a recognizable step function.*

Proof. For the disjunction, let $\mathcal{V} = \mathrm{Free}(\varphi) \cup \mathrm{Free}(\psi)$. By definition, we have $[\![\varphi \vee \psi]\!] = [\![\varphi]\!]_{\mathcal{V}} + [\![\psi]\!]_{\mathcal{V}}$. Hence the result follows from Proposition 3.3 and Lemma 2.1 resp. 2.3.

For the existential quantifiers, let $\mathcal{X}$ be the variable x or X. Let $\mathcal{V} = \mathrm{Free}(\exists \mathcal{X}.\varphi)$ and note that $\mathcal{X} \notin \mathcal{V}$ and $\mathrm{Free}(\varphi) \subseteq \mathcal{V} \cup \{\mathcal{X}\}$. Consider the projection $\pi : \Sigma^*_{\mathcal{V} \cup \{\mathcal{X}\}} \to \Sigma^*_{\mathcal{V}}$ which erases the $\mathcal{X}$-row. One can show that $[\![\exists \mathcal{X}.\varphi]\!] = \pi([\![\varphi]\!]_{V \cup \{\mathcal{X}\}})$. Then Proposition 3.3 and Lemma 2.2(b) show that $[\![\exists \mathcal{X}.\varphi]\!]$ is recognizable. □

Next we deal with conjunction. For any formula $\varphi \in \mathrm{MSO}(S, \Sigma)$, we let $S_\varphi = S_{\mathrm{val}(\varphi)}$, the subsemiring of S generated by all constants occurring in φ.

Lemma 5.3. *Let $\varphi, \psi \in \mathrm{MSO}(S, \Sigma)$.*

(a) Assume that $\mathrm{val}(\varphi)$ and $\mathrm{val}(\psi)$ commute element-wise, and that $[\![\varphi]\!] \in \mathrm{Rec}(S_\varphi, \Sigma^*_\varphi)$ and $[\![\psi]\!] \in \mathrm{Rec}(S_\psi, \Sigma^*_\psi)$. Then $[\![\varphi \wedge \psi]\!]$ is recognizable.
(b) If $[\![\varphi]\!]$ and $[\![\psi]\!]$ are recognizable step functions, so is $[\![\varphi \wedge \psi]\!]$.

Proof. Let $\mathcal{V} = \mathrm{Free}(\varphi) \cup \mathrm{Free}(\psi)$. By definition, we have $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!]_{\mathcal{V}} \odot [\![\psi]\!]_{\mathcal{V}}$.

(a) By Proposition 3.3, we get $[\![\varphi]\!]_{\mathcal{V}} \in \mathrm{Rec}(S_\varphi, \Sigma^*_{\mathcal{V}})$ and $[\![\psi]\!]_{\mathcal{V}} \in \mathrm{Rec}(S_\psi, \Sigma^*_{\mathcal{V}})$. As noted in Section 2, S_φ and S_ψ commute element-wise. Hence the result follows from Lemma 2.1(c).

(b) We apply Proposition 3.3 and Lemma 2.3. □

The most interesting case here arises from universal quantification. In [14], a corresponding result was proved under the assumption that S is commutative. The reason that this assumption can be avoided is due to the following. For a word (over an extended alphabet), the semantics of $\forall x.\varphi$ is evaluated along the sequence of positions, just as the weight of a path in a weighted automaton is computed following the sequence of transitions. This will be crucial in the proof.

Lemma 5.4. *Let $\psi \in \mathrm{MSO}(S, \Sigma)$ such that $[\![\psi]\!]$ is a recognizable step function. Then $[\![\forall x.\psi]\!]$ is recognizable.*

Proof. Let $\mathcal{W} = \mathrm{Free}(\psi) \cup \{x\}$ and $\mathcal{V} = \mathrm{Free}(\forall x.\psi) = \mathcal{W} \setminus \{x\}$. By Proposition 3.3 (in case $x \notin \mathrm{Free}(\psi)$), $[\![\psi]\!]_{\mathcal{W}}$ is a recognizable step function. We may write $[\![\psi]\!]_{\mathcal{W}} = \sum_{j=1}^{n} s_j \cdot \mathbb{1}_{L_j}$ with $n \in \mathbb{N}$, $s_j \in S$ and recognizable languages $L_1, \ldots, L_n \subseteq \Sigma_{\mathcal{W}}^*$ such that $(L_1, \ldots, L_n)$ is a partition of $N_{\mathcal{W}}$. Recall that if $(w, \sigma) \in (\Sigma_{\mathcal{W}})^* \setminus N_{\mathcal{W}}$ then $[\![\psi]\!](w, \sigma) = 0$.

Let $\widetilde{\Sigma} = \Sigma \times \{1, \ldots, n\}$. A word in $(\widetilde{\Sigma}_{\mathcal{V}})^*$ will be written (w, ν, σ) where $(w, \sigma) \in \Sigma_{\mathcal{V}}^*$ and $\nu \in \{1, \ldots, n\}^{|w|}$ is interpreted as a mapping from $\mathrm{dom}(w)$ to $\{1, \ldots, n\}$. Let $\widetilde{L}$ be the set of $(w, \nu, \sigma) \in (\widetilde{\Sigma}_{\mathcal{V}})^*$ such that $(w, \sigma) \in N_{\mathcal{V}}$ and for all $i \in \mathrm{dom}(w)$ and $j \in \{1, \ldots, n\}$ we have

$$\nu(i) = j \quad \text{implies} \quad (w, \sigma[x \to i]) \in L_j.$$

Observe that for each $(w, \sigma) \in N_{\mathcal{V}}$ there is a unique ν such that $(w, \nu, \sigma) \in \widetilde{L}$ since $(L_1, \ldots, L_n)$ is a partition of $N_{\mathcal{W}}$.

We claim that $\widetilde{L}$ is recognizable. In [14, proof of Lemma 4.4], we constructed directly an automaton recognizing $\widetilde{L}$. Here we give an unpublished argument (already developed for [13, 14]) using Büchi's theorem.

First, let $\xi \in \mathrm{MSO}(\Sigma)$ be an arbitrary MSO formula. Define $\tilde{\xi}$ by replacing in ξ any occurrence of $P_a(y)$ by $\bigvee_{1 \le k \le n} P_{(a,k)}(y)$. Then, assuming that $\mathrm{Free}(\xi) \subseteq \mathcal{U}$, it is easy to check by structural induction on ξ that for all $(w, \nu, \sigma) \in (\widetilde{\Sigma}_{\mathcal{U}})^*$ with $(w, \sigma) \in N_{\mathcal{U}}$ we have $(w, \nu, \sigma) \models \tilde{\xi}$ if and only if $(w, \sigma) \models \xi$.

By Büchi's theorem, there is an MSO formula ψ_j with $\mathrm{Free}(\psi_j) \subseteq \mathcal{W}$ such that for all $(w, \tau) \in N_{\mathcal{W}}$ we have $(w, \tau) \in L_j$ if and only if $(w, \tau) \models \psi_j$. Now, we define

$$\zeta = \forall x. \left(\bigwedge_{1 \le j \le n} \bigvee_{a \in \Sigma} P_{(a,j)}(x) \longrightarrow \widetilde{\psi_j} \right).$$

Let $(w, \nu, \sigma) \in (\widetilde{\Sigma}_{\mathcal{V}})^*$ with $(w, \sigma) \in N_{\mathcal{V}}$. We have $(w, \nu, \sigma) \models \zeta$ if and only if for all $i \in \mathrm{dom}(w)$ and $j \in \{1, \ldots, n\}$ we have

$$\nu(i) = j \quad \text{implies} \quad (w, \nu, \sigma[x \mapsto i]) \models \widetilde{\psi_j}$$

and this last statement is equivalent with $(w, \sigma[x \mapsto i]) \models \psi_j$ which in turn is equivalent with $(w, \sigma[x \mapsto i]) \in L_j$. Therefore, the formula ζ defines the language $\widetilde{L}$ and our claim is proved.

Now we proceed similar as in [14] with slight changes as in [19] since here S might not be commutative. There is a deterministic automaton $\widetilde{\mathcal{A}}$ over the alphabet $\widetilde{\Sigma}_{\mathcal{V}}$, recognizing $\widetilde{L}$. Now we obtain a weighted automaton $\mathcal{A}$ with the same state set by adding weights to the transitions of $\widetilde{\mathcal{A}}$ as follows: If $(p, (a, j, s), q)$ is a transition in $\widetilde{\mathcal{A}}$ with $(a, j, s) \in \widetilde{\Sigma}_{\mathcal{V}}$, we let this transition in $\mathcal{A}$ have weight s_j, i.e., $\mu_{\mathcal{A}}(a, j, s)_{p,q} = s_j$. All triples which are not transitions in $\widetilde{\mathcal{A}}$ get weight 0. Also, the initial state of $\widetilde{\mathcal{A}}$ gets initial weight 1 in $\mathcal{A}$, all non-initial states of $\widetilde{\mathcal{A}}$ get initial weight 0, and similarly for the final states and final weights.

Since $\widetilde{\mathcal{A}}$ is deterministic, for each $(w, \nu, \sigma) \in \widetilde{L}$ there is a unique path $P_w = (t_i)_{1 \le i \le |w|}$ in $\widetilde{\mathcal{A}}$ and we have in $\mathcal{A}$

$$(||\mathcal{A}||, (w, \nu, \sigma)) = \text{weight}(P_w) = \prod_{i \in \text{dom}(w)} \text{wt}(t_i)$$

whereas $(||\mathcal{A}||, (w, \nu, \sigma)) = 0$ for each $(w, \nu, \sigma) \in \widetilde{\Sigma}_{\mathcal{V}}^* \setminus \widetilde{L}$. For each $i \in \text{dom}(w)$ note that if $\nu(i) = j$, then $\text{wt}(t_i) = s_j$ by construction of $\mathcal{A}$, and since $(w, \nu, \sigma) \in \widetilde{L}$ we get $(w, \sigma[x \to i]) \in L_j$ and $[\![\psi]\!]_{\mathcal{W}}(w, \sigma[x \to i]) = s_j$.

We consider now the strict alphabetic homomorphism $h : \widetilde{\Sigma}_{\mathcal{V}}^* \to \Sigma_{\mathcal{V}}^*$ defined by $h((a, k, s)) = (a, s)$ for each $(a, k, s) \in \widetilde{\Sigma}_{\mathcal{V}}$. Then for any $(w, \sigma) \in N_V$ and the unique ν such that $(w, \nu, \sigma) \in \widetilde{L}$, we have

$$\begin{aligned}(h(||\mathcal{A}||), (w, \sigma)) = (||\mathcal{A}||, (w, \nu, \sigma)) &= \prod_{i \in \text{dom}(w)} \text{wt}(t_i) \\ &= \prod_{i \in \text{dom}(w)} [\![\psi]\!]_{\mathcal{W}}(w, \sigma[x \to i]) = [\![\forall x.\psi]\!](w, \sigma).\end{aligned}$$

Therefore $[\![\forall x.\psi]\!] = h(||\mathcal{A}||)$ which is recognizable by Lemma 2.2. □

Lemma 5.5. *Let $\psi \in \text{MSO}(S, \Sigma)$ be unambiguous. Then $[\![\forall X.\psi]\!]$ is a recognizable step function.*

Proof. Since ψ is unambiguous, so is $\forall X.\psi$ and by Proposition 4.2 we deduce that $[\![\forall X.\psi]\!]$ is a recognizable step function.

The following result generalizes [14, Theorem 4.5] to non-commutative semirings.

Theorem 5.6. *Let S be any semiring, Σ be an alphabet and $\varphi \in \text{MSO}(S, \Sigma)$ be restricted. Then $[\![\varphi]\!] \in \text{Rec}(S, \Sigma_\varphi^*)$.*

Proof. Note that if $\varphi \in \text{MSO}(S, \Sigma)$, then trivially $\varphi \in \text{MSO}(S_\varphi, \Sigma)$. By induction over the structure of φ we show that $[\![\varphi]\!] \in \text{Rec}(S_\varphi, \Sigma_\varphi^*)$. But this is immediate by Lemmas 5.1– 5.5. □

Next we aim at showing that, conversely, recognizable series are definable. First, for $s \in S$, we define

$$(x \in X) \xrightarrow{+} s = \neg(x \in X) \vee ((x \in X) \wedge s).$$

This formula is almost unambiguous, and for any word w and valid assignment σ we have

$$[\![(x \in X) \xrightarrow{+} s]\!](w, \sigma) = \begin{cases} s & \text{if } \sigma(x) \in \sigma(X) \\ 1 & \text{otherwise} \end{cases}.$$

We introduce a few other abbreviations which are all unambiguous formulas. We let $\min(y) := \forall x. y \leq x$, and $\max(z) := \forall x. x \leq z$, and $(y = x+1) := (x \leq y) \wedge \neg(y \leq x) \wedge \forall z.(z \leq x \vee y \leq z)$. If $X_1, \ldots, X_m$ are set variables, put

$$\text{partition}(X_1, \ldots, X_m) := \forall x. \bigvee_{i=1,\ldots,m} \left((x \in X_i) \wedge \bigwedge_{j \neq i} \neg(x \in X_j) \right) .$$

Now we show:

Theorem 5.7. *Let S be any semiring, Σ be an alphabet and $r \in \text{Rec}(S, \Sigma^*)$. Then r is* sREMSO*-definable.*

Proof. Let $\mathcal{A} = (Q, \lambda, \mu, \gamma)$ be a weighted automaton such that $r = ||\,\mathcal{A}\,||$. For each triple $(p, a, q) \in Q \times \Sigma \times Q$ choose a set variable $X_{p,a,q}$, and let $\mathcal{V} = \{X_{p,a,q} \mid p, q \in Q, a \in \Sigma\}$. We choose an enumeration $\overline{X} = (X_1, \ldots, X_m)$ of $\mathcal{V}$ with $m = |Q|^2 \cdot |\Sigma|$. Define the syntactically restricted formula

$$\begin{aligned} \psi(\overline{X}) := \text{partition}(\overline{X}) \wedge \bigwedge_{p,a,q} \forall x.\ (x \in X_{p,a,q}) \xrightarrow{+} P_a(x) \\ \wedge \forall x \forall y.\ (y = x+1) \xrightarrow{+} \bigvee_{p,q,r \in Q, a,b \in \Sigma} (x \in X_{p,a,q}) \wedge (y \in X_{q,b,r}) \ . \end{aligned}$$

Let $w = a_1 \ldots a_n \in \Sigma^+$. If $P = (q_0 \xrightarrow{a_1} q_1 \ldots q_{n-1} \xrightarrow{a_n} q_n)$ is a path in $\mathcal{A}$ over w, we define the $(\mathcal{V}, w)$-assignment σ_P by $\sigma_P(X_{p,a,q}) = \{i \mid (q_{i-1}, a_i, q_i) = (p, a, q)\}$. Clearly, we have $[\![\psi]\!](w, \sigma_P) = 1$. Conversely, let σ be a $(\mathcal{V}, w)$-assignment such that $[\![\psi]\!](w, \sigma) = 1$. For any $i \in \text{dom}(w)$, there are uniquely determined $p_i, q_i \in Q$ such that $i \in \sigma(X_{p_i,a_i,q_i})$ and if $i < n$ then $q_i = p_{i+1}$. Hence, with $q_0 = p_1$ we obtain a unique path $P = (q_0 \xrightarrow{a_1} q_1 \ldots q_{n-1} \xrightarrow{a_n} q_n)$ for w such that $\sigma_P = \sigma$. This gives a bijection between the set of paths in $\mathcal{A}$ over w and the set of $(\mathcal{V}, w)$-assignments σ satisfying ψ, i.e., such that $[\![\psi]\!](w, \sigma) = 1$.

Consider now the formula

$$\begin{aligned} \varphi(\overline{X}) := \psi(\overline{X}) \wedge \exists y. \Big(\min(y) \wedge \bigvee_{p,a,q} (y \in X_{p,a,q}) \wedge \lambda_p \Big) \\ \wedge \forall x. \bigwedge_{p,a,q} (x \in X_{p,a,q}) \xrightarrow{+} \mu(a)_{p,q} \\ \wedge \exists z. \Big(\max(z) \wedge \bigvee_{p,a,q} (z \in X_{p,a,q}) \wedge \gamma_q \Big). \end{aligned}$$

Let $P = (q_0 \xrightarrow{a_1} q_1 \ldots q_{n-1} \xrightarrow{a_n} q_n)$ be a path in $\mathcal{A}$ over w and let σ_P be the associated $(\mathcal{V}, w)$-assignment. We obtain

$$[\![\varphi]\!](w, \sigma_P) = \lambda_{q_0} \cdot \mu(a_1)_{q_0,q_1} \cdots \mu(a_n)_{q_{n-1},q_n} \cdot \gamma_{q_n} = \text{weight}(P) \ .$$

Note that $[\![\varphi(\overline{X})]\!](\varepsilon) = 0$ due to the subformula starting with $\exists y$ in φ. Hence, in order to deal with $w = \varepsilon$, let $\zeta = r(\varepsilon) \wedge \forall x.\neg(x \leq x)$. For $w \in \Sigma^+$ we have $[\![\forall x.\neg(x \leq x)]\!](w) = 0$. Now, $[\![\forall x.\neg(x \leq x)]\!](\varepsilon) = 1$ since an empty product is 1 by convention, hence we get $[\![\zeta]\!](\varepsilon) = r(\varepsilon)$.

Now let $\xi = \exists X_1 \cdots \exists X_m.(\varphi(X_1, \ldots, X_m) \vee \zeta)$. Then $\xi \in \mathrm{MSO}(S, \Sigma)$ is existential, and $[\![\xi]\!](\varepsilon) = [\![\zeta]\!](\varepsilon) = r(\varepsilon)$. Using the bijection above, for $w \in \Sigma^+$ we get

$$
\begin{aligned}
[\![\xi]\!](w) &= \sum_{\sigma\ (\mathcal{V}, w)\text{-assignment}} [\![\varphi]\!](w, \sigma) = \sum_{P \text{ path in } \mathcal{A} \text{ for } w} [\![\varphi]\!](w, \sigma_P) \\
&= \sum_{P \text{ path in } \mathcal{A} \text{ for } w} \mathrm{weight}(P) = (|| \mathcal{A} ||, w).
\end{aligned}
$$

So $[\![\xi]\!] = || \mathcal{A} ||$. In general, φ is not syntactically restricted due to the constants which may not commute. But it is known (cf. [24]) that we may choose $\mathcal{A}$ so that $\lambda(q), \gamma(q) \in \{0, 1\}$ for all $q \in Q$. In this case, φ is syntactically restricted and $\xi \in \mathrm{sREMSO}(S, \Sigma)$. □

Now Theorem 4.7 is immediate by Proposition 4.8 and Theorems 5.6 and 5.7.

Next we consider the effectiveness of our proof of Theorem 4.7 implication (2) → (1). Note that our proof of Theorem 5.6 in general was not effective, since in Lemma 5.4 we may not know the form of the step function $[\![\psi]\!]$. However:

Proposition 5.8. *Let S be an effectively given semiring and Σ an alphabet. Given $\varphi \in \mathrm{sRMSO}(S, \Sigma)$, we can effectively compute a weighted automaton $\mathcal{A}$ for $[\![\varphi]\!]$.*

Proof. We follow the argument for Theorem 5.6 and proceed by induction on the structure of φ. Now, when dealing with a subformula $\forall x.\psi$ of φ, then we know the form of $\psi = \bigvee_{j=1}^{n}(s_j \wedge \psi_j^+)$ with $s_j \in S$ and $\psi_j \in \mathrm{MSO}^-(S, \Sigma)$ for $1 \leq j \leq n$, and we can use these constituents within the proof of Lemma 5.4.

All other lemmas employed are also constructive, meaning that if weighted automata are given for the *arguments*, then weighted automata can be effectively computed for the *results*. □

From this and decidability results for weighted automata, we immediately obtain decidability results for sRMSO-sentences. For instance, if S is an effectively given field (like $\mathbb{Q}$, the rational numbers), for any two sRMSO-sentences φ, ψ, we can decide whether $[\![\varphi]\!] = [\![\psi]\!]$: By Proposition 5.8, construct weighted automata $\mathcal{A}_\varphi$, $\mathcal{A}_\psi$ for φ resp. ψ and then decide whether $|| \mathcal{A}_\varphi || = || \mathcal{A}_\psi ||$ (cf. [4, 42]).

For the implication (1) → (3) of Theorem 4.7, given a weighted automaton $\mathcal{A}$, we can "write down" an sREMSO-sentence φ with $[\![\varphi]\!] = || \mathcal{A} ||$.

Using this, from the theory of formal power series (cf. [4, 42, 56]) we immediately obtain also undecidability results for the semantics of weighted MSO-sentences. For instance, it is undecidable whether a given sREMSO-sentence φ over $\mathbb{Q}$, the field of rational numbers, and an alphabet Σ, satisfies $\mathrm{supp}([\![\varphi]\!]) = \Sigma^*$. Also, by a result of Krob [39], the equality of given recognizable series over the tropical semiring is undecidable. Hence, the equality of two given sREMSO(Trop, Σ)-sentences is also undecidable.

6 Locally finite semirings

Here we will describe two larger classes of syntactically defined sentences which, for more particular semirings, are expressively equivalent to weighted automata.

First let us describe the semirings we will encounter. A monoid M is called *locally finite*, if each finitely generated submonoid of M is finite. Clearly, a commutative monoid M is locally finite iff each cyclic submonoid $\langle a \rangle$ of M is finite. Let us call a semiring S *additively locally finite* if its additive monoid $(S, +, 0)$ is locally finite. This holds iff the cyclic submonoid $\langle 1 \rangle$ of $(S, +, 0)$ is finite. Examples for additively locally finite semirings include:

- all idempotent semirings S (i.e., $x + x = x$ for each $x \in S$), in particular the arctic and the tropical semirings, the semiring $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$ of languages of Σ, and the semiring $([0,1], \max, \cdot, 0, 1)$ useful for describing probabilistic settings;
- all fields of characteristic p, for any prime p;
- all products $S_1 \times \ldots \times S_n$ (with operations defined pointwise) of additively locally finite semirings S_i $(1 \leq i \leq n)$;
- the semiring of polynomials $(S[X], +, \cdot, 0, 1)$ over a variable X and an additively locally finite semiring S;
- all locally finite semirings (see below).

Furthermore, a semiring $(S, +, \cdot, 0, 1)$ is *locally finite* [12], if each finitely generated subsemiring is finite. Clearly, equivalent to this is that both monoids $(S, +, 0)$ and $(S, \cdot, 1)$ are locally finite (cf. [15]). Examples of such semirings include:

- semirings S for which both addition and multiplication are idempotent and commutative; in particular, any bounded distributive lattice $(L, \vee, \wedge, 0, 1)$. Consequently, the chain $([0,1], \max, \min, 0, 1)$ and any Boolean algebra are locally finite;
- the Łukasiewicz semiring $([0,1], \max, \otimes, 0, 1)$ (cf. [15]);
- all matrix semirings $S^{n \times n}$ of $n \times n$-matrices over a locally finite semiring S for any $n \geq 2$, these semirings are non-commutative;
- the algebraic closures of the finite fields $\mathbb{Z}/p\mathbb{Z}$ (p prime) are (infinite) locally finite fields.

Next we turn to the formulas we will consider here. We define $\mathrm{wUMSO}(S,\Sigma)$, the collection of *weakly unambiguous* formulas in $\mathrm{MSO}(S,\Sigma)$, to be the smallest subset of $\mathrm{MSO}(S,\Sigma)$ containing all constants s ($s \in S$) and all syntactically unambiguous formulas φ^+, φ^- ($\varphi \in \mathrm{MSO}^-(S,\Sigma)$) which is closed under disjunction, conjunction and existential quantifications (both first and second order).

Definition 6.1. *A formula $\varphi \in \mathrm{MSO}(S,\Sigma)$ is called* syntactically weakly restricted*, if it satisfies the following conditions:*

(1) Whenever φ contains a conjunction $\psi \wedge \psi'$ as subformula but not in the scope of a universal first order quantifier, then $\mathrm{val}(\psi)$ and $\mathrm{val}(\psi')$ commute element-wise.
(2) Whenever φ contains $\forall X.\psi$ as a subformula, then ψ is an unweighted syntactically unambiguous formula.
(3) Whenever φ contains $\forall x.\psi$ as a subformula, then ψ is weakly unambiguous.

We let $\mathrm{swRMSO}(S,\Sigma)$ denote the set of all syntactically weakly restricted formulas of $\mathrm{MSO}(S,\Sigma)$.

Our first goal will be to show that all syntactically weakly restricted formulas of $\mathrm{MSO}(S,\Sigma)$ have a recognizable semantics, provided S is additively locally finite.

Theorem 6.2. *Let S be any additively locally finite semiring, Σ be an alphabet, and $\varphi \in \mathrm{swRMSO}(S,\Sigma)$. Then $[\![\varphi]\!] \in \mathrm{Rec}(S,\Sigma_\varphi^*)$.*

As in Section 5, we will proceed by induction on the structure of φ. As preparation, first we aim to show that non-deleting homomorphisms preserve recognizable step functions provided S is additively locally finite.

Lemma 6.3 ([4], Cor. III.2.4,2.5). *Let $r : \Sigma^* \to \mathbb{N}$ be a recognizable series over the semiring $\mathbb{N}$. Then, for any $a, b \in \mathbb{N}$ the languages $r^{-1}(a)$ and $r^{-1}(a + b\,\mathbb{N})$ are recognizable.*

Proposition 6.4. *Let S be additively locally finite. Let Σ, Γ be two alphabets and $h : \Sigma^* \to \Gamma^*$ be a non-erasing homomorphism.*

(a) Let $L \subseteq \Sigma^*$ be a recognizable language. Then $h(\mathbb{1}_L) : \Gamma^* \to S$ is a recognizable step function.
(b) $h : S\langle\langle\Sigma^*\rangle\rangle \to S\langle\langle\Gamma^*\rangle\rangle$ preserves recognizable step functions.

Proof. (a) We shall use the same technique as in the proof of [14, Lemma 8.7]. For any $s \in S$ and $n \geq 0$ we define $0 \otimes s = 0$ (of S) and $(n+1) \otimes s = s + (n \otimes s)$. Thus, $n \otimes s = s + \ldots + s$ with n times s. For any $u \in \Gamma^*$, let $m(u) = |h^{-1}(u) \cap L|$. Then $(h(\mathbb{1}_L), u) = m(u) \otimes 1$. The additive monoid $\langle 1 \rangle$ generated by $\{1\}$ is finite. We choose a minimal element $a \in \mathbb{N}$ such that $a \otimes 1 = (a+x) \otimes 1$ for some $x > 0$ and we let b be the smallest such x. Then

$\langle 1\rangle = \{0, 1, \ldots, (a+b-1)\otimes 1\}$. Now for each $u \in \Gamma^*$ we have $m(u)\otimes 1 = d(u)\otimes 1$ for some uniquely determined $d(u) \in \mathbb{N}$ with $0 \le d(u) \le a + b - 1$. Note that if $0 \le d < a$, then $m(u) \otimes 1 = d \otimes 1$ iff $m(u) = d$, and if $a \le d < a + b$, then $m(u) \otimes 1 = d \otimes 1$ iff $m(u) \in d + b\,\mathbb{N}$. For each $0 \le d < a + b$ let $M_d = \{u \in \Gamma^* \mid d(u) = d\}$. Then $h(\mathbb{1}_L) = \sum_{d=0}^{a+b-1} d \cdot \mathbb{1}_{M_d}$.

Also, let $\mathbb{1}'_L \in \mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ be the characteristic series of L over the semiring $\mathbb{N}$. Then by Lemma 2.2 the series $r = h(\mathbb{1}'_L) : \Gamma^* \to \mathbb{N}$ is recognizable, and $(r, u) = \sum_{w\in h^{-1}(u)}(\mathbb{1}'_L, w) = m(u)$ for each $u \in \Gamma^*$. Hence $M_d = \{u \in \Gamma^* \mid m(u) = d\} = r^{-1}(d)$ if $0 \le d < a$, and $M_d = \{u \in \Gamma^* \mid m(u) \in d + b\,\mathbb{N}\} = r^{-1}(d + b\,\mathbb{N})$ if $a \le d < a + b$. In any case, M_d is recognizable by Lemma 6.3. Thus $h(\mathbb{1}_L)$ is a recognizable step function.

(b) Let $r = \sum_{j=1}^{n} s_j \cdot \mathbb{1}_{L_j}$ be a recognizable step function in $S\langle\langle\Sigma^*\rangle\rangle$. Since $h : S\langle\langle\Sigma^*\rangle\rangle \to S\langle\langle\Gamma^*\rangle\rangle$ is a semiring homomorphism, we have $h(r) = \sum_{j=1}^{n} s_j \cdot h(\mathbb{1}_{L_j})$. Now, apply (a) and Lemma 2.3(a). □

Next we consider existential quantifications.

Lemma 6.5. *Let S be additively locally finite and $\varphi \in \mathrm{MSO}(S, \Sigma)$ such that $[\![\varphi]\!]$ is a recognizable step function. Then $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are also recognizable step functions.*

Proof. Let $\mathcal{V} = \mathrm{Free}(\varphi)$ and let $\mathcal{X}$ be x or X. Following the proof of Lemma 5.2, we can write $[\![\exists\mathcal{X}.\varphi]\!]$ as the image under a length-preserving projection of $[\![\varphi]\!]_{\mathcal{V}\cup\{\mathcal{X}\}}$ which is a recognizable step function by assumption and Proposition 3.3. Now apply Proposition 6.4(b). □

Now we can show:

Proof of Theorem 6.2. We proceed by induction over the structure of φ, aiming to show for each subformula ξ of φ that $[\![\xi]\!] \in \mathrm{Rec}(S_\varphi, \Sigma_\varphi^*)$. First we claim that if ξ is weakly unambiguous, then $[\![\xi]\!] : \Sigma_\xi^* \to S_\xi$ is a recognizable step function. For constants and for syntactically unambiguous formulas this is clear by Lemma 4.4. For disjunctions and conjunctions of such formulas we apply Proposition 3.3 and Lemma 2.3(a), and for existential quantifications Lemma 6.5 to obtain our claim. Next we can proceed using Lemmas 5.2– 5.5. □

Next we consider the case where the semiring S is locally finite. First we note:

Proposition 6.6 ([12]). *Let S be locally finite. Then every recognizable series $r \in S\langle\langle\Sigma^*\rangle\rangle$ is a recognizable step function.*

We call a formula $\varphi \in \mathrm{MSO}(S, \Sigma)$ *weakly existential*, if whenever φ contains $\forall X.\psi$ as a subformula, then ψ is syntactically unambiguous. Now we show:

Theorem 6.7. *Let S be any locally finite semiring, Σ an alphabet, and $\varphi \in \mathrm{MSO}(S, \Sigma)$ weakly existential. Then $[\![\varphi]\!]$ is recognizable.*

Proof. We claim that for each subformula ψ of φ, $[\![\psi]\!]$ is a recognizable step function. Due to Proposition 6.6, we only have to show that $[\![\psi]\!]$ is recognizable. Proceeding by induction, this follows from Lemmas 5.1– 5.5. □

7 Weighted Automata on Infinite Words

In this section, we will consider weighted automata $\mathcal{A}$ acting on infinite words. As in the case of weighted automata on finite words, we will define the weight of an infinite path in $\mathcal{A}$ as the product of its – infinitely many – transitions, and the weight of a word w as the sum of all the weights of successful paths realizing w; in general, there might be infinitely (even uncountably) many such paths realizing w. Hence we need to be able to form infinite sums and products in the underlying semiring S. Such complete semirings have already been considered in Conway [10] and Eilenberg [24], see also [32]. For weighted automata on infinite words and characterizations of their behaviors by rational series the reader should consult [26].

Assume that the semiring S is equipped with infinitary sum operations $\sum_I : S^I \to S$, for any index set I, such that for all I and all families $(s_i \mid i \in I)$ of elements of S the following hold:

$$\sum_{i\in\emptyset} s_i = 0, \quad \sum_{i\in\{j\}} s_i = s_j, \quad \sum_{i\in\{j,k\}} s_i = s_j + s_k \text{ for } j \neq k,$$
$$\sum_{j\in J}\Big(\sum_{i\in I_j} s_i\Big) = \sum_{i\in I} s_i, \text{ if } \bigcup_{j\in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for } j \neq j',$$
$$\sum_{i\in I}(c\cdot s_i) = c\cdot\Big(\sum_{i\in I} s_i\Big), \quad \sum_{i\in I}(s_i\cdot c) = \Big(\sum_{i\in I} s_i\Big)\cdot c.$$

Then S together with the operations $\sum_I$ is called *complete* [24, 40].

A complete semiring is said to be *totally complete* [27] if it is endowed with a countably infinite product operation satisfying for all sequences $(s_i \mid i \geq 0)$ of elements of S the following conditions:

$$\prod_{i\geq 0} 1 = 1, \qquad s_0 \cdot \prod_{i\geq 0} s_{i+1} = \prod_{i\geq 0} s_i, \qquad \prod_{i\geq 0} s_i = \prod_{i\geq 0}(s_{n_i}\cdots s_{n_{i+1}-1})$$
$$\prod_{j\geq 0}\sum_{i\in I_j} s_i = \sum_{(i_j)_{j\geq 0}\in\prod_{j\geq 0} I_j} \ \prod_{j\geq 0} s_{i_j},$$

where in the third equation $0 = n_0 < n_1 < n_2 < \ldots$ is a strictly increasing sequence and in the last equation $I_0, I_1, \ldots$ are arbitrary index sets and $\prod_{j\geq 0} I_j$ denotes the full cartesian product of these sets.

Now we say that a totally complete semiring S is *conditionally completely commutative (ccc)*, if whenever $(s_i)_{i\geq 0}$ and $(s'_i)_{i\geq 0}$ are two sequences of elements of S such that $s_i \cdot s'_j = s'_j \cdot s_i$ for all $0 \leq j < i$, then

$$\left(\prod_{i\geq 0} s_i\right)\cdot\left(\prod_{i\geq 0} s'_i\right)=\prod_{i\geq 0}(s_i\cdot s'_i). \tag{1}$$

In [20] the authors considered totally complete semirings S satisfying (1) for *all* sequences $(s_i \mid i\geq 0)$ and $(s'_i \mid i\geq 0)$ in S. Such semirings are necessarily commutative.

Next we wish to show that there is an abundance of conditionally complete commutative semirings which are not commutative. For this, we recall the notions of ordered and continuous semirings (cf. [15]).

A semiring $(S,+,\cdot,0,1)$ with a partial order $\leq$ is called *ordered*, if the partial order is preserved by addition and also by multiplication with elements $s\geq 0$. Now let S be an ordered semiring such that $s\geq 0$ for each $s\in S$. Then S is called *continuous*, if each directed subset D of S has a supremum (least upper bound) $\vee D$ in S, and addition and multiplication preserve suprema of directed subsets, i.e., $s+\vee D=\vee(s+D)$, $s\cdot\vee D=\vee(s\cdot D)$ and $(\vee D)\cdot s=\vee(D\cdot s)$ for each directed subset $D\subseteq S$ and each $s\in S$; here $s+D=\{s+d \mid d\in D\}$, $s\cdot D=\{s\cdot d \mid d\in D\}$ and analogously for $D\cdot s$. We may (and will) equip a continuous semiring with infinitary sum operations given by $\sum_{i\in I}s_i=\vee\{\sum_{i\in F}s_i \mid F\subseteq I \text{ finite}\}$ for any family $(s_i \mid i\in I)$ of elements of S; as is well-known, then S is complete. We refer the reader to [15] for many examples of (both commutative and non-commutative) continuous semirings. For instance, if S is continuous, the matrix semirings $S^{n\times n}$ and the power series semiring $S\langle\langle\Sigma^*\rangle\rangle$ (with addition and Cauchy product) are continuous and clearly non-commutative if $n\geq 2$ resp. $|\Sigma|\geq 2$. Now we show:

Proposition 7.1. *Let S be a continuous semiring and $S'=\{s\in S \mid s\geq 1\}\cup\{0\}$. We define an infinite product operation on S' by letting*

$$\prod_{i\geq 0}s_i=\begin{cases}\bigvee_{n\geq 0}\prod_{i=0}^{n}s_i & \text{if } s_i\neq 0 \text{ for all } i\geq 1\\ 0 & \text{otherwise}\end{cases}$$

for each sequence $(s_i \mid i\geq 0)$ in S'. Then S' is a continuous ccc semiring.

Proof. Clearly S' is a continuous semiring. We claim that S' is totally complete. For this, it suffices to check the infinitary distributivity law. Let I_j $(j\geq 0)$ be index sets and $s_i\in S'$ for $i\in I_j$. We may assume $I_j\neq\emptyset$ for each $j\geq 0$, and that $s_i\neq 0$, thus $s_i\geq 1$, for each $i\in I_j$ $(j\geq 0)$.

By definitions of the infinite sum and product, we have

$$A:=\prod_{j\geq 0}\sum_{i\in I_j}s_i=\bigvee_{n\geq 0}\prod_{j=0}^{n}\bigvee_{\substack{F_j\subseteq I_j\\ F_j\,\text{finite}}}\sum_{i\in F_j}s_i$$

By continuity of multiplication and by distributivity, we obtain

$$A = \bigvee_{n\geq 0} \bigvee_{\substack{F_j\subseteq I_j\\ F_j \text{finite}\\ 0\leq j\leq n}} \prod_{j=0}^{n} \sum_{i\in F_j} s_i = \bigvee_{n\geq 0} \bigvee_{\substack{F_j\subseteq I_j\\ F_j \text{finite}\\ 0\leq j\leq n}} \sum_{(i_0,\ldots,i_n)\in F_0\times\cdots\times F_n} \prod_{j=0}^{n} s_{i_j}$$

We have to show that this quantity equals

$$B := \sum_{(i_j)_{j\geq 0}\in\prod_{j\geq 0} I_j} \prod_{j\geq 0} s_{i_j} = \bigvee_{\substack{F \text{finite}\\ F\subseteq\prod_{j\geq 0} I_j}} \sum_{(i_j)_{j\geq 0}\in F} \bigvee_{n\geq 0} \prod_{j=0}^{n} s_{i_j}$$

By continuity of addition and using a diagonalisation argument we obtain

$$B = \bigvee_{\substack{F \text{finite}\\ F\subseteq\prod_{j\geq 0} I_j}} \bigvee_{n\geq 0} \sum_{(i_j)_{j\geq 0}\in F} \prod_{j=0}^{n} s_{i_j}$$

We first show $A \leq B$. Fix $n \geq 0$ and for $0 \leq j \leq n$ let $F_j \subseteq I_j$ finite. For all $k > n$ choose $i_k \in I_k$ and let $F = F_0 \times \cdots \times F_n \times \prod_{j>n}\{i_k\}$ which is a finite subset of $\prod_{j\geq 0} I_j$. We have

$$\sum_{(i_0,\ldots,i_n)\in F_0\times\cdots\times F_n} \prod_{j=0}^{n} s_{i_j} = \sum_{(i_j)_{j\geq 0}\in F} \prod_{j=0}^{n} s_{i_j}$$

and we deduce that $A \leq B$. Conversely, we show $B \leq A$. Fix a finite subset $F \subseteq \prod_{j\geq 0} I_j$ and some $n \geq 0$. Consider $m \geq n$ such that $|F'| = |F|$ where $F' = \{(i_0,\ldots,i_m) \mid (i_j)_{j\geq 0} \in F\}$. For $0 \leq j \leq m$, let F_j be the j-th projection of F' so that $F' \subseteq F_0 \times \cdots \times F_m \subseteq I_0 \times \cdots \times I_m$. Then, using $s_i \geq 1$ for each $i \in I_j$ and $j \geq 0$ we obtain

$$\sum_{(i_j)_{j\geq 0}\in F} \prod_{j=0}^{n} s_{i_j} = \sum_{(i_0,\ldots,i_m)\in F'} \prod_{j=0}^{n} s_{i_j} \leq \sum_{(i_0,\ldots,i_m)\in F'} \prod_{j=0}^{m} s_{i_j}$$
$$\leq \sum_{(i_0,\ldots,i_m)\in F_0\times\cdots\times F_m} \prod_{j=0}^{m} s_{i_j}$$

and we have shown $B \leq A$.

It remains to show that S' is ccc. Let $(s_i)_{i\geq 0}$ and $(s'_i)_{i\geq 0}$ be two sequences in S' such that $s_i \cdot s'_j = s'_j \cdot s_i$ for all $0 \leq j < i$. Then by continuity of the product, diagonalization, and our commutativity assumption we obtain

$$\left(\prod_{i\geq 0} s_i\right) \cdot \left(\prod_{i\geq 0} s'_i\right) = \bigvee_{m\geq 0} \bigvee_{n\geq 0} \left(\prod_{i=0}^{m} s_i\right) \cdot \left(\prod_{j=0}^{n} s'_j\right) = \bigvee_{n\geq 0} \left(\prod_{i=0}^{n} s_i\right) \cdot \left(\prod_{j=0}^{n} s'_j\right)$$
$$= \bigvee_{n\geq 0} \prod_{i=0}^{n} (s_i \cdot s'_i) = \prod_{i\geq 0} (s_i \cdot s'_i)$$

as required. □

Let S be a totally complete semiring. We call a subsemiring $S' \subseteq S$ a *totally complete subsemiring* of S if S' is closed in S under taking arbitrary sums and countably-infinite products. If $A \subseteq S$, the totally complete subsemiring generated by A is the smallest totally complete subsemiring of S containing A. Due to the infinitary distributivity law, it can be obtained by taking arbitrary sums of the closure A^{cl} of $A \cup \{0,1\}$ under countably-infinite products. To construct A^{cl}, in general it does not suffice to take all countably-infinite products of elements of $A \cup \{0,1\}$, since this set might not be closed under countably-infinite products; the process of taking countably-infinite products has to be iterated transfinitely (ω_1 steps suffice).

Lemma 7.2. *Let S be a ccc semiring and $A, B \subseteq S$ such that A and B commute element-wise. Let S_A^{tc} and S_B^{tc} be the totally complete subsemirings of S generated by A resp. B. Then S_A^{tc} and S_B^{tc} commute element-wise.*

Proof. Choose any $s \in S_B^{\mathrm{tc}}$. First we show:

(1) If $(a_i)_{i \geq 1}$ is a sequence in S such that all a_i $(i \geq 1)$ commute with s, then $\prod_{i \geq 1} a_i$ commutes with s.
Indeed, put $a_0 = s_0 = s_i = 1$ for $i \geq 2$ and $s_1 = s$. Since S is ccc, we obtain:

$$\left(\prod_{i \geq 1} a_i\right) \cdot s = \left(\prod_{i \geq 0} a_i\right) \cdot \left(\prod_{i \geq 0} s_i\right) = \prod_{i \geq 0} (a_i \cdot s_i) = \prod_{i \geq 0} (s_i \cdot a_i)$$
$$= \left(\prod_{i \geq 0} s_i\right) \cdot \left(\prod_{i \geq 0} a_i\right) = s \cdot \prod_{i \geq 1} a_i.$$

(2) If $(a_i)_{i \in I}$ is a family in S such that all a_i $(i \in I)$ commute with s, then $\sum_I a_i$ commutes with s. Clearly, this holds in any complete semiring.

Now assume $s \in B$. Let A^{cl} be the closure of $A \cup \{0,1\}$ under countably infinite products. By the description of A^{cl} given above, by rule (1) and transfinite induction we obtain that each element of A^{cl} commutes with s. Now S_A^{tc} consists of all sums of elements from A^{cl}. Hence rule (2) implies that each element of S_A^{tc} commutes with s.

So S_A^{tc} and B commute element-wise. By a dual argument applied to S_B^{tc}, we obtain that S_A^{tc} and S_B^{tc} commute element-wise. □

We also note:

Lemma 7.3. *Let S be a totally complete and idempotent semiring. Then $\Sigma_I 1 = 1$ for each set I of size at most continuum.*

Proof. By distributivity, we have $1 = \prod_{i \geq 0}(1+1) = \sum_{f \in 2^\omega} 1$. Now let $\leq$ be the natural partial order on the idempotent semiring S; i.e., for $x, y \in S$ we have $x \leq y$ iff $x + z = y$ for some $z \in S$. It follows that $1 \leq \Sigma_I 1 \leq \Sigma_{2^\omega} 1 = 1$ for each non-empty subset $I \subseteq 2^\omega$. Hence $1 = \sum_I 1$. □

For the rest of this section, let S be a totally complete semiring. Now we present two weighted automata models acting on infinite words. We denote by Σ^ω the set of infinite words over Σ. Recall that a formal power series over infinite words is a mapping $r : \Sigma^\omega \to S$ and that we denote by $S\langle\langle\Sigma^\omega\rangle\rangle$ the set of formal power series over S and Σ^ω.

Definition 7.4.

(a) A weighted Muller automaton *(WMA for short) over S and Σ is a quadruple $\mathcal{A} = (Q, \lambda, \mu, \mathcal{F})$ where Q is a finite set of states, $\mu : \Sigma \to S^{Q\times Q}$ is the transition weight function, $\lambda : Q \to S$ is the weight function for entering a state, and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the family of final state sets.*

(b) A WMA $\mathcal{A}$ is a weighted Büchi automaton *(WBA for short) if there is a set $F \subseteq Q$ such that $\mathcal{F} = \{S \subseteq Q \mid S \cap F \neq \emptyset\}$.*

As for weighted finite automata, the value $\mu(a)_{p,q} \in S$ indicates the weight of the transition $p \xrightarrow{a} q$. We also write $\mathrm{wt}(p, a, q) = \mu(a)_{p,q}$.

The *weight* of an infinite path $P : q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \to \cdots$ in $\mathcal{A}$ is the product $\mathrm{weight}(P) := \lambda(q_0) \cdot \prod_{i\geq 0} \mathrm{wt}(q_i, a_i, q_{i+1})$. This path has label $a_0a_1\ldots$ and it is *successful*, if $\{q \in Q \mid q = q_i \text{ for infinitely many } i\} \in \mathcal{F}$. The *weight* of a word $w = a_0a_1 \ldots \in \Sigma^\omega$ in $\mathcal{A}$, denoted $(||\,\mathcal{A}\,||, w)$, is the sum of $\mathrm{weight}(P)$ over all successful paths P with label w. The formal power series $||\,\mathcal{A}\,|| : \Sigma^\omega \to S$ is called the ω-*behavior* of $\mathcal{A}$.

A series $r : \Sigma^\omega \to S$ is called *Muller recognizable* (resp. *Büchi recognizable* or ω-*recognizable*) if there is a WMA (resp. WBA) $\mathcal{A}$ such that $S = \|\mathcal{A}\|$. The class of all Muller recognizable (resp. ω-recognizable series) over S and Σ is denoted by $\text{M-Rec}(S, \Sigma^\omega)$ (resp. $\omega\text{-Rec}(S, \Sigma^\omega)$).

The following result was proved in [20].

Theorem 7.5. $\text{M-Rec}(S, \Sigma^\omega) = \omega\text{-Rec}(S, \Sigma^\omega)$.

In the sequel, we wish to provide a logical characterization of the class of ω-recognizable series in our weighted MSO logics interpreted over infinite words. For this goal we shall need closure properties of ω-recognizable series which we recall in the following.

Lemma 7.6.

(a) For any ω-recognizable language $L \subseteq \Sigma^\omega$, the series $\mathbb{1}_L$ is ω-recognizable.

(b) Let $r, r_1, r_2 \in S\langle\langle\Sigma^\omega\rangle\rangle$ be ω-recognizable, and let $s \in S$. Then $r_1 + r_2$, $s \cdot r$ and $r \cdot s$ are ω-recognizable.

Next we show:

Lemma 7.7. *Let S be a ccc semiring. Let $S_1, S_2 \subseteq S$ be two totally complete subsemirings such that S_1 and S_2 commute element-wise. Let $r_1 \in \omega\text{-Rec}(S_1, \Sigma^\omega)$ and $r_2 \in \omega\text{-Rec}(S_2, \Sigma^\omega)$. Then $r_1 \odot r_2 \in \omega\text{-Rec}(S, \Sigma^\omega)$.*

Proof. We show that a classical construction of a weighted Muller automaton for $r_1 \odot r_2$ (cf. [20]) works under the present assumptions on S.

Let $\mathcal{A}_1 = (Q_1, \lambda_1, \mu_1, \mathcal{F}_1)$ and $\mathcal{A}_2 = (Q_2, \lambda_2, \mu_2, \mathcal{F}_2)$ be two WMA. We construct the WMA $\mathcal{A} = (Q, \lambda, \mu, \mathcal{F})$ in the following way. Its state set is $Q = Q_1 \times Q_2$, and the initial distribution is given by $\lambda(q, q') = \lambda_1(q)\lambda_2(q')$ for all $(q, q') \in Q$. Its weight transition mapping is specified by $\mathrm{wt}((q, q'), a, (p, p')) = \mathrm{wt}_1(q, a, p)\,\mathrm{wt}_2(q', a, p')$ for all $(q, q'), (p, p') \in Q$, $a \in A$. Finally, the family $\mathcal{F}$ is constructed as follows: $\mathcal{F} = \{F \mid \pi_1(F) \in \mathcal{F}_1,\ \pi_2(F) \in \mathcal{F}_2\}$ where $\pi_i : Q \to Q_i$, is the projection of Q on Q_i $(i = 1, 2)$. Now let $w = a_0 a_1 \ldots \in \Sigma^\omega$, and let $P_i = (q_0^i \xrightarrow{a_0} q_1^i \xrightarrow{a_1} q_2^i \to \ldots)$ be a path for w in $\mathcal{A}_i$ $(i = 1, 2)$. Then $P = ((q_0^1, q_0^2) \xrightarrow{a_0} (q_1^1, q_1^2) \xrightarrow{a_1} (q_2^1, q_2^2) \to \ldots)$ is a path for w in $\mathcal{A}$. Clearly, P is successful in $\mathcal{A}$ iff both P_1 and P_2 are successful in $\mathcal{A}_1$ resp. $\mathcal{A}_2$. Moreover, since S is ccc and S_1 and S_2 commute element-wise, we obtain

$$\begin{aligned}
\mathrm{weight}(P) &= \lambda_1(q_0^1)\lambda_2(q_0^2) \prod_{i \geq 0} (\mathrm{wt}_1(q_i^1, a_i, q_{i+1}^1) \cdot \mathrm{wt}_2(q_i^2, a_i, q_{i+1}^2)) \\
&= \Big(\lambda_1(q_0^1) \prod_{i \geq 0} \mathrm{wt}_1(q_i^1, a_i, q_{i+1}^1)\Big) \cdot \Big(\lambda_2(q_0^2) \prod_{i \geq 0} \mathrm{wt}_2(q_i^2, a_i, q_{i+1}^2)\Big) \\
&= \mathrm{weight}(P_1) \cdot \mathrm{weight}(P_2).
\end{aligned}$$

From this it easily follows that $(\|\mathcal{A}\|, w) = (\|\mathcal{A}_1\|, w) \cdot (\|\mathcal{A}_2\|, w)$. Hence $\|\mathcal{A}\| = \|\mathcal{A}_1\| \odot \|\mathcal{A}_2\| = r_1 \odot r_2$. □

Now let $h : \Sigma^* \to \Gamma^*$ be a length-preserving homomorphism. Then h can be extended to a mapping $h : \Sigma^\omega \to \Gamma^\omega$ by letting $h(w) = h(w(0))h(w(1))\ldots$.

For $r \in S\langle\langle \Gamma^\omega \rangle\rangle$ let $h^{-1}(r) = r \circ h \in S\langle\langle \Sigma^\omega \rangle\rangle$. For $r \in S\langle\langle \Sigma^\omega \rangle\rangle$, define $h(r) : \Gamma^\omega \to S$ by $(h(r), v) := \sum_{w \in h^{-1}(v)} (r, w)$ for $v \in \Gamma^\omega$.

Lemma 7.8 ([20]). *Let $h : \Sigma^* \to \Gamma^*$ be a length-preserving homomorphism. Then $h^{-1} : S\langle\langle \Gamma^\omega \rangle\rangle \to S\langle\langle \Sigma^\omega \rangle\rangle$ and $h : S\langle\langle \Sigma^\omega \rangle\rangle \to S\langle\langle \Gamma^\omega \rangle\rangle$ preserve ω-recognizability.*

We say that $r \in S\langle\langle \Sigma^\omega \rangle\rangle$ is an *ω-recognizable step function*, if $r = \sum_{i=1}^n s_i \cdot \mathbb{1}_{L_i}$ for some $n \in \mathbb{N}, s_i \in S$ and ω-recognizable languages $L_i \subseteq \Sigma^\omega$ $(i = 1, \ldots, n)$. Then clearly r is an ω-recognizable series by Lemma 7.6. The following closure result is easy to see.

Lemma 7.9. *(a) The class of all ω-recognizable step functions over Σ and S is closed under sum, scalar products and Hadamard products.*

(b) Let $h : \Sigma^* \to \Gamma^*$ be a length-preserving homomorphism. Then $h^{-1} : S\langle\langle \Gamma^\omega \rangle\rangle \to S\langle\langle \Sigma^\omega \rangle\rangle$ preserves ω-recognizable step functions.

(c) Let $h : \Sigma^\omega \to \Gamma^\omega$ be a length-preserving homomorphism and assume that S is idempotent. Then $h : S\langle\langle \Sigma^\omega \rangle\rangle \to S\langle\langle \Gamma^\omega \rangle\rangle$ preserves ω-recognizable step functions.

Proof. (a) Straightforward.

(b) We follow the proof of Lemma 2.3(b) and note that the class of ω-recognizable languages is closed under inverses of length-preserving homomorphisms (cf. [52]).

(c) For any language $L \subseteq \Sigma^\omega$, we have $h(\mathbb{1}_L) = \mathbb{1}_{h(L)}$ by Lemma 7.3. Now follow the argument for Proposition 6.4(b). □

8 Weighted Logics on Infinite Words

In this section, we wish to develop weighted logics for infinite words which are expressively equivalent to weighted Büchi automata. In particular, we will derive analogues of Theorems 4.7 and 6.2 for infinite words.

MSO-logic over infinite words is defined as in Section 2. The only difference is that the domain of an infinite word is now $\mathbb{N}$. Again, the language

$$N_{\mathcal{V}}^\omega = \{(w,\sigma) \in \Sigma_{\mathcal{V}}^\omega \mid \sigma \text{ is a valid } (\mathcal{V},w)\text{-assignment}\}$$

is recognizable and by Büchi's theorem, if $\mathrm{Free}(\varphi) \subseteq \mathcal{V}$, the language

$$\mathcal{L}_{\mathcal{V}}^\omega(\varphi) = \{(w,\sigma) \in N_{\mathcal{V}}^\omega \mid (w,\sigma) \models \varphi\}$$

defined by φ over $\Sigma_{\mathcal{V}}$ is recognizable. We simply write $\mathcal{L}^\omega(\varphi)$ for $\mathcal{L}^\omega_{\mathrm{Free}(\varphi)}(\varphi)$.

In all of this section, let S be a totally complete semiring and Σ an alphabet. Given weighted MSO-formulas as in Definition 3.1, we first have to define their semantics for infinite words.

Definition 8.1. *Let $\varphi \in \mathrm{MSO}(S,\Sigma)$ and $\mathcal{V}$ be a finite set of variables containing* $\mathrm{Free}(\varphi)$*. The* ω-$\mathcal{V}$-semantics *of φ is a formal power series $[\![\varphi]\!]_{\mathcal{V}}^\omega \in S\langle\langle \Sigma_{\mathcal{V}}^\omega \rangle\rangle$. For short, in this section we write $[\![\varphi]\!]_{\mathcal{V}}$ for $[\![\varphi]\!]_{\mathcal{V}}^\omega$. Let $(w,\sigma) \in \Sigma_{\mathcal{V}}^\omega$. If σ is not a valid $(\mathcal{V},w)$-assignment, then we put $[\![\varphi]\!]_{\mathcal{V}}(w,\sigma) = 0$. Otherwise, we define $[\![\varphi]\!]_{\mathcal{V}}(w,\sigma) \in S$ inductively just as in Definition 3.2.*

To define the semantics of $\forall X.\varphi$ we assume that in S products over index sets of size continuum exist. Then we put

$$[\![\forall X.\varphi]\!]_{\mathcal{V}}(w,\sigma) = \prod_{I \subseteq \mathrm{dom}(w)} [\![\varphi]\!]_{\mathcal{V}\cup\{X\}}(w,\sigma[X \to I]).$$

We simply write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathrm{Free}(\varphi)}$.

We note that the additional assumption here on products in S can be lifted again in a moment, since we will only consider formulas $\varphi \in \mathrm{MSO}(S,\Sigma)$ in which universal set quantification is only applied to syntactically unambiguous formulas, and we define uncountable products of the elements 0, 1 in the obvious way.

Indeed, from now on we will consider syntactically unambiguous, almost unambiguous, syntactically restricted and weakly unambiguous formulas in $\mathrm{MSO}(S,\Sigma)$, precisely as defined before. Our two main results will be the following.

Theorem 8.2. *Let S be a totally complete semiring which is ccc, let Σ be an alphabet, and let $r : \Sigma^\omega \to S$ be a series. The following are equivalent:*

(1) r is ω-recognizable.
(2) r is definable by some syntactically restricted sentence of $\mathrm{MSO}(S, \Sigma)$.
(3) r is definable by some syntactically restricted existential sentence of $\mathrm{MSO}(S, \Sigma)$.

Theorem 8.3. *Let S be a totally complete semiring which is ccc and idempotent, and let Σ be an alphabet. Let $\varphi \in \mathrm{swRMSO}(S, \Sigma)$. Then $[\![\varphi]\!] \in \omega\text{-}\mathrm{Rec}(S, \Sigma_\varphi^\omega)$.*

For the proof of these results, we proceed almost exactly as before. For the convenience of the reader, we just indicate the main steps below where we assume that S is a totally complete semiring which is ccc.

As in the finitary case, the definition of the ω-semantics of a weighted MSO-formula $\varphi \in \mathrm{MSO}(S, \Sigma)$ depends on the set $\mathcal{V}$. In the following, we show that $[\![\varphi]\!]_\mathcal{V}$ in fact depends only on $\mathrm{Free}(\varphi)$.

Proposition 8.4. *Let $\varphi \in \mathrm{MSO}(S, \Sigma)$ and $\mathcal{V}$ a finite set of variables containing* $\mathrm{Free}(\varphi)$. *Then*

$$[\![\varphi]\!]_\mathcal{V}(w, \sigma) = [\![\varphi]\!](w, \sigma_{|\,\mathrm{Free}(\varphi)})$$

for each $(w, \sigma) \in \Sigma_\mathcal{V}^\omega$ such that σ is a valid $(\mathcal{V}, w)$-assignment. In particular, $[\![\varphi]\!]$ is ω-recognizable iff $[\![\varphi]\!]_\mathcal{V}$ is ω-recognizable, and $[\![\varphi]\!]$ is an ω-recognizable step function iff $[\![\varphi]\!]_\mathcal{V}$ is an ω-recognizable step function.

Proof. We can follow the proof of Proposition 3.3 taking into account Lemmas 7.6(a), 7.7, 7.8 and 7.9(a),(b). □

We define the notion of *unambiguous* formulas (but now with respect to infinite words) as in Definition 4.1. Then we have:

Proposition 8.5. *Let $\varphi \in \mathrm{MSO}(S, \Sigma)$ be unambiguous. We may also regard φ as a classical* MSO*-formula defining the language $\mathcal{L}^\omega(\varphi) \subseteq \Sigma_\varphi^\omega$. Then, $[\![\varphi]\!] = \mathbb{1}_{\mathcal{L}^\omega(\varphi)}$ is an ω-recognizable step function.*

Now we obtain:

Lemma 8.6. *Let $\varphi \in \mathrm{MSO}^-(S, \Sigma)$. Then:*

- $\mathcal{L}^\omega(\varphi^+) = \mathcal{L}^\omega(\varphi)$ *and* $\mathcal{L}^\omega(\varphi^-) = \mathcal{L}^\omega(\neg\varphi)$,
- $[\![\varphi^+]\!] = \mathbb{1}_{\mathcal{L}^\omega(\varphi)}$ *and* $[\![\varphi^-]\!] = \mathbb{1}_{\mathcal{L}^\omega(\neg\varphi)}$,
- φ^+ *and* φ^- *are unambiguous.*

As a by-product, we have:

Proposition 8.7. *For each classical* MSO*-sentence φ, we can effectively construct an unweighted syntactically unambiguous* $\mathrm{MSO}(S, \Sigma)$*-sentence φ' defining the same language, i.e., $[\![\varphi']\!] = \mathbb{1}_{\mathcal{L}^\omega(\varphi)}$.*

The definition of ω-restricted formulas is precisely as for restricted formulas, just replacing recognizable step functions by ω-recognizable step functions.

Now we proceed as in Section 5:

Lemma 8.8. *Let $\varphi \in \mathrm{MSO}(S, \Sigma)$ be atomic. Then $[\![\varphi]\!]$ is an ω-recognizable step function.*

Lemma 8.9. *Let $\varphi, \psi \in \mathrm{MSO}(S, \Sigma)$ such that $[\![\varphi]\!]$ and $[\![\psi]\!]$ are ω-recognizable. Then $[\![\varphi \vee \psi]\!]$, $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are ω-recognizable. Moreover, if $[\![\varphi]\!]$ and $[\![\psi]\!]$ are ω-recognizable step functions, then $[\![\varphi \vee \psi]\!]$ is also an ω-recognizable step function.*

Proof. Analogously to Lemma 5.2, now using Proposition 8.4 and Lemmas 7.6, 7.8 and 7.9. □

Next we deal with conjunction. If $\varphi \in \mathrm{MSO}(S, \Sigma)$, we let S_φ^{tc} be the totally complete subsemiring of S generated by $\mathrm{val}(\varphi)$.

Lemma 8.10. *Let $\varphi, \psi \in \mathrm{MSO}(S, \Sigma)$.*

(a) Assume that $\mathrm{val}(\varphi)$ and $\mathrm{val}(\psi)$ commute element-wise, and that $[\![\varphi]\!] \in \omega\text{-Rec}(S_\varphi^{\mathrm{tc}}, \Sigma_\varphi^\omega)$ and $[\![\psi]\!] \in \omega\text{-Rec}(S_\psi^{\mathrm{tc}}, \Sigma_\psi^\omega)$. Then $[\![\varphi \wedge \psi]\!]$ is ω-recognizable.
(b) If $[\![\varphi]\!]$ and $[\![\psi]\!]$ are ω-recognizable step functions, so is $[\![\varphi \wedge \psi]\!]$.

Proof. (a) As shown in Lemma 7.2, S_φ^{tc} and S_ψ^{tc} commute element-wise. Now apply Proposition 8.4 and Lemma 7.7.

(b) We apply Proposition 8.4 and Lemma 7.9(a). □

Next we turn to universal quantification.

Lemma 8.11. *Let $\psi \in \mathrm{MSO}(S, \Sigma)$ such that $[\![\psi]\!]$ is an ω-recognizable step function. Then $[\![\forall x.\psi]\!]$ is ω-recognizable.*

Proof. We proceed as for Lemma 5.4, utilizing that the class of ω-recognizable languages is closed under Boolean operations. Then the corresponding ω-recognizable language $\widetilde{L}$ can be accepted by a deterministic Muller automaton $\widetilde{\mathcal{A}}$. We can transform $\widetilde{\mathcal{A}}$ into a weighted Muller automaton $\mathcal{A}$ by keeping its state set and the set of final states and defining initial weights and weights of transitions as before. Proceeding as before, we obtain $[\![\forall x.\varphi]\!] = h(||\,\mathcal{A}\,||)$ which is ω-recognizable by Lemma 7.8. □

Now we can give the

Proof of Theorem 8.2. (3)→(2): Trivial.

(2)→(1): Combine Proposition 8.5 and Lemmas 8.8 – 8.11.

(1)→(3): (Here we only need that S is totally complete.) Let $\mathcal{A} = (Q, \lambda, \mu, F)$ be a weighted Büchi automaton with $r = ||\,\mathcal{A}\,||$. By possibly adding a new initial state, we may assume that $\lambda(q) \in \{0, 1\}$ for each $q \in Q$.

We define the formula $\psi(\overline{X})$ as in the proof of Theorem 5.7. Consider now the formula

$$\varphi(\overline{X}) := \psi(\overline{X}) \wedge \exists y. \Big(\min(y) \wedge \bigvee_{p,a,q} (y \in X_{p,a,q}) \wedge \lambda_p\Big)$$
$$\wedge \forall x. \bigwedge_{p,a,q} (x \in X_{p,a,q}) \xrightarrow{+} \mu(a)_{p,q}$$
$$\wedge \left(\bigvee_{(p,a,q) \in F \times \Sigma \times Q} \forall x.\ \exists y.\ (x < y \wedge (y \in X_{p,a,q})) \right)^{+}$$

Intuitively, the last conjunct ensures that the considered paths are accepting. The proof is now similar to the finitary case (Theorem 5.7). □

Next we turn to the proof of Theorem 8.3. We will need

Lemma 8.12. *Let S be idempotent and $\varphi \in \mathrm{MSO}(S, \Sigma)$ such that $[\![\varphi]\!]$ is an ω-recognizable step function. Then $[\![\exists x.\varphi]\!]$ and $[\![\exists X.\varphi]\!]$ are also ω-recognizable step functions.*

Proof. We follow the proof of Lemma 6.5, applying Proposition 8.4 and Lemma 7.9(c). □

Now we can show:

Proof of Theorem 8.3. Following the proof of Theorem 6.2, we proceed by induction on the structure of φ. Here we apply Lemmas 8.6, 7.9(a), 8.8 – 8.12 and Proposition 8.4 and 8.5. □

Finally, we note that all constructions for the proofs of Theorems 8.2 and 8.3 are again effective (if S is given effectively).

9 Conclusions and Open Problems

In this chapter, we have presented a weighted logic which is expressively equivalent to weighted automata, both if interpreted over finite or infinite words, respectively. In the case of finite words, together with Schützenberger's theorem [55, 57] we thus obtain for arbitrary semirings an equivalence between weighted automata, rational expressions for formal power series, and our logical formalism by syntactically restricted MSO-logic. In the case of infinite words, we needed completeness assumptions on the semiring. Further equivalences were obtained in case the semiring is additively locally finite or locally finite or (for infinite words) idempotent.

In [14], we also investigated weighted first-order logic and could show an equivalence result to a concept of aperiodic series (cf. [12]), thus also extending the classical equivalence result between aperiodic and first-order definable languages into a weighted setting. This needed the semiring to be bi-aperiodic and commutative (in which case it is also locally finite, but not conversely). We refer the reader to [14] for these results.

Weighted automata with discounting have been investigated in [18]. Discounting is a well-known concept in mathematical economics as well as systems theory in which later events get less value than earlier ones, cf. e.g. [2]. In [18], the possible behaviors of such weighted automata with discounting were characterized by rational resp. ω-rational expressions, also see [41] for further results on this. In [19] they were further characterized by a discounted restricted weighted logic. Somewhat surprizingly, the discounting only had to be reflected in the semantics of the universal first-order quantifier.

In [21], also cf. [30], an equivalence result for weighted automata and a weighted logic over ranked trees was obtained for all commutative semirings. In up-coming work [22], our present approach will be applied to unranked trees, a syntactically defined weighted logic and arbitrary semirings.

Our approach has also been extended to pictures [28], traces [48], distributed processes [5], also cf. [29] in this handbook, and very recently to texts, sp-biposets and nested words with an application to algebraic formal power series, see [44, 45]. In each case, crucial differences occur when dealing with the universal first-order quantifier. In [53], weighted automata and weighted logics for infinite trees were investigated. In [16], weighted logics with values in bounded distributive lattices were considered, cf. also [54].

These results show the robustness of our approach. One could also try to define weighted temporal logics and study not only expressiveness but also decidability and complexity of natural problems such as quantitative model checking.

Open Problems:

1. Given any signature $\mathcal{S}$ of predicate calculus and a semiring S, we might define the syntax of a weighted logic as in Definition 3.1, employing the new atomic formulas and their negations. The semantics can then be defined similarly as in Definition 3.2 for arbitrary finite $\mathcal{S}$-structures, and for arbitrary $\mathcal{S}$-structures assuming S is totally complete. Which results of model theory [9,34] can be developed for such a general weighted logic?
2. Find a model of weighted automata which is expressively equivalent to the full logic $\mathrm{MSO}(S, \Sigma)$.
3. Find a weighted temporal logic which is expressively equivalent to suitable fragments of $\mathrm{MSO}(S, \Sigma)$.
4. Find applications.

References

1. J. Albert and J. Kari. Digital image compression. In [17], chapter 11. 2009.
2. L. de Alfaro, T.A. Henzinger and R. Majumda. Discounting the future in systems theory. In Proc. of: *Automata, Languages and Programming (30th ICALP, Eindhoven)*, Lecture Notes in Computer Science, vol. 2719, pages 1022-1037, Springer-Verlag, 2003.
3. A. Arnold. *Finite Transition Systems.* International Series in Computer Science. Prentice Hall, 1994.
4. J. Berstel and Ch. Reutenauer. *Rational Series and Their Languages*, volume 12 of *EATCS Monographs in Theoretical Computer Science.* Springer-Verlag, 1988.
5. B. Bollig and I. Meinecke. Weighted distributed systems and their logics. In Proc. of: *Logical Foundations of Computer Science (LFCS, New York)*, Lecture Notes in Computer Science, vol. 4514, pages 54–68, Springer-Verlag, 2007.
6. J.R. Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik Grundlagen Math.*, 6:66–92, 1960.
7. J.R. Büchi On a decision method in restricted second order arithmetic. In *Proc. 1960 Int. Congress for Logic, Methodology and Philosophy of Science*, pages 1-11, 1962.
8. A.L. Buchsbaum, R. Giancarlo, and J.R. Westbrook. On the determinization of weighted finite automata. *SIAM Journal on Computing*, 30:1502–1531, 2000.
9. C.C. Chang and H.J. Keisler. *Model Theory.* North Holland, Amsterdam, 1990.
10. J.H. Conway. *Regular Algebra and Finite Machines.* Chapman & Hall, 1971.
11. K. Culik and J. Kari. Image compression using weighted finite automata. *Computer and Graphics*, 17:305–313, 1993.
12. M. Droste and P. Gastin. On aperiodic and star-free formal power series in partially commuting variables. *Theory of Computing Systems*, 42:608–631, 2008.
13. M. Droste and P. Gastin. Weighted automata and weighted logics. In Proc. of: *Automata, Languages and Programming (32nd ICALP, Lisbon)*, Lecture Notes in Computer Science, vol. 3580, pages 513–525. Springer-Verlag, 2005.
14. M. Droste and P. Gastin. Weighted automata and weighted logics. *Theoretical Computer Science*, 380:69–86, 2007.
15. M. Droste and W. Kuich. Semirings and formal power series. In [17], chapter 1. 2009.
16. M. Droste, W. Kuich and G. Rahonis. Multi-valued MSO logics over words and trees. *Fundamenta Informaticae*, 84:305–327, 2008.
17. M. Droste, W. Kuich and H. Vogler (eds.). *Handbook of Weighted Automata.* Springer-Verlag, 2009.
18. M. Droste and D. Kuske. Skew and infinitary formal power series. *Theoretical Computer Science*, 366: 189–227, 2006; extended abstract in Proc. of: *Automata, Languages and Programming (30th ICALP, Eindhoven)*, Lecture Notes in Computer Science, vol. 2719, pages 426-438, Springer-Verlag, 2003.
19. M. Droste and G. Rahonis. Weighted automata and weighted logics with discounting. In Proc. of: *Implementation and Application of Automata (12th CIAA, Prague)*, Lecture Notes in Computer Science, vol. 4783, pages 73–84, Springer-Verlag, 2007.
20. M. Droste and G. Rahonis. Weighted automata and weighted logics on infinite words. In *Izvestiya VUZ. Matematika*, 2008.
21. M. Droste and H. Vogler. Weighted tree automata and weighted logics. *Theoretical Computer Science*, 366:228-247, 2006.

22. M. Droste and H. Vogler. *Weighted logic for automata on unranked trees*, manuscript.
23. M. Droste and G.-Q. Zhang. On transformations of formal power series. *Information and Computation*, 184:369–383, 2003.
24. S. Eilenberg. *Automata, Languages and Machines*, volume A. Academic Press, New York, 1974.
25. C.C. Elgot. Decision problems of finite automata design and related arithmetics. *Trans. Amer. Math. Soc.*, 98:21–52, 1961.
26. Z. Ésik and W. Kuich. Finite automata. In [17], chapter 3, 2009.
27. Z. Ésik, W. Kuich, On iteration semiring-semimodule pairs. *Semigroup Forum*, 75:129–159, 2007.
28. I. Fichtner. Weighted picture automata and weighted logics. *Theory of Computing Systems*, to appear.
29. I. Fichtner, D. Kuske and I. Meinecke. Traces, pomsets, and pictures. In [17], chapter 10. 2009.
30. Z. Fülöp and H. Vogler. Weighted tree automata and tree transducers. In [17], chapter 9. 2009.
31. B. Gerla. Automata over MV-algebras. In Proc. of: *IEEE International Symposium on Multiple-Valued Logic (34th ISMVL, Toronto)*, pages 49–54. IEEE Computer Society Press, 2004.
32. S. Golan. *Semirings and their Applications.* Kluwer Academic Publisher, 1999.
33. U. Hafner. *Low Bit-Rate Image and Video Coding with Weighted Finite Automata.* PhD thesis, Universität Würzburg, Germany, 1999.
34. W. Hodges. *Model Theory.* Cambridge University Press, 1993.
35. Z. Jiang, B. Litow, and O. de Vel. Similarity enrichment in image compression through weighted finite automata. In Proc. of: *Computing and Combinatorics (6th COCOON, Sydney)*, Lecture Notes in Computer Science, vol. 1858, pages 447–456. Springer-Verlag, 2000.
36. F. Katritzke. *Refinements of Data Compression using Weighted Finite Automata.* PhD thesis, Universität Siegen, Germany, 2001.
37. B. Khoussainov and A. Nerode. *Automata Theory and its Applications.* Birkhäuser, Boston, 2001.
38. K. Knight and J. May. Applications in natural language processing. In [17], chapter 15. 2009.
39. D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. *International Journal of Algebra and Computation*, 4:405–425, 1994.
40. W. Kuich. Semirings and formal power series. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages, Vol. 1*, chapter 9, pages 609–677. Springer-Verlag, 1997.
41. W. Kuich. On skew formal power series. In *Proceedings of the Conference on Algebraic Informatics (S. Bozapalidis, A. Kalampakas, G. Rahonis, eds.)*, Thessaloniki 2005, pages 7–30.
42. W. Kuich and A. Salomaa. *Semirings, Automata, Languages*, volume 6 of *EATCS Monographs in Theoretical Computer Science.* Springer-Verlag, 1986.
43. R.P. Kurshan. *Computer-Aided Verification of Coordinating Processes.* Princeton Series in Computer Science. Princeton University Press, 1994.
44. Ch. Mathissen. Definable transductions and weighted logics for texts. In Proc. of: *Developments in Language Theory (11th DLT, Turku)*, Lecture Notes in Computer Science, vol. 4588, pages 324–336, Springer-Verlag, 2007.

45. Ch. Mathissen. Weighted logics for nested words and algebraic formal power series. In Proc. of: *Automata, Languages and Programming (35th ICALP, Reykjavik)*, Part II, Lecture Notes in Computer Science, vol. 5126, pages 221–232, Springer-Verlag, 2008.
46. K. McMillan. *Symbolic Model Checking.* Kluwer Academic Publishers, 1993.
47. R. McNaughton and S. Papert. *Counter-free Automata.* MIT Press, Cambridge, 1971.
48. I. Meinecke. Weighted logics for traces. In Proc. of: *Computer Science - Theory and Applications (1st CSR, St. Petersburg)*, Lecture Notes in Computer Science, vol. 3967, pages 235–246, Springer-Verlag, 2006.
49. M. Mohri. Finite-state transducers in language and speech processing. *Computational Linguistics*, 23:269–311, 1997.
50. M. Mohri. Algorithms for weighted automata. In [17], chapter 6. 2009.
51. M. Mohri, F. Pereira, and M. Riley. The design principles of a weighted finite-state transducer library. *Theoretical Computer Science*, 231:17–32, 2000.
52. J.-É Pin and D. Perrin. *Infinite Words.* Elsevier, 2004.
53. G. Rahonis. Weighted Muller tree automata and weighted logics. *Journal of Automata, Languages, and Combinatorics*, 12:455–483, 2007.
54. G. Rahonis. Fuzzy languages. In [17], chapter 13. 2009.
55. J. Sakarovitch. Rational and recognisable power series. In [17], chapter 4. 2009.
56. A. Salomaa and M. Soittola. *Automata-Theoretic Aspects of Formal Power Series.* Texts and Monographs in Computer Science. Springer-Verlag, 1978.
57. M.P. Schützenberger. On the definition of a family of automata. *Information and Control*, 4:245–270, 1961.
58. M.P. Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8:190–194, 1965.
59. S. Schwarz. Łukasiewicz logics and weighted logics over MV-semirings. *Journal of Automata, Languages and Combinatorics*, 12:485–499, 2007.
60. W. Thomas. Languages, automata and logic. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages, Vol. 3*, chapter 7, pages 389–485. Springer-Verlag, 1997.