Quantum superpositions and projective measurement in the lambda calculus*

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Abstract

We propose an extension of simply typed lambda-calculus to handle some properties of quantum computing. The equiprobable quantum superposition is taken as a commutative pair and the quantum measurement as a non-deterministic projection over it. Destructive interferences are achieved by introducing an inverse symbol with respect to pairs. The no-cloning property is ensured by using a combination of syntactic linearity with linear logic. Indeed, the syntactic linearity is enough for unitary gates, while a function measuring its argument needs to enforce that the argument is used only once.

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1 Introduction

In [7] we have proposed a quantum inspired λ-calculus, where a superposition $u + v$ was formalized as a pair $\langle u, v\rangle$. As the addition is commutative and associative, we considered terms modulo equations

$$\langle u, v\rangle = \langle v, u\rangle \qquad \text{and} \qquad \langle\langle u, v\rangle, w\rangle = \langle u, \langle v, w\rangle\rangle\ ,$$

and consequently types modulo the isomorphisms

$$A \wedge B \equiv B \wedge A \qquad \text{and} \qquad (A \wedge B) \wedge C \equiv A \wedge (B \wedge C)\ .$$

The position based projections π_1 and π_2 had to be replaced by type based projections π_A such that if u has type A and v has type B, the term $\pi_A\langle u, v\rangle$ reduces to u. When A and B are equal, the projection is non deterministic, like the quantum measurement.

If the basis vectors u and v have type A, the superposition $u+v$ has type $A\wedge A$ and not A, thus types permit distinguishing basis vectors from superpositions.

In [7], we have considered all the isomorphisms of simply typed λ-calculus, thus taking distance from the original motivation of quantum computing. In this paper we keep the two first isomorphisms (associativity and commutativity of conjunction), but we drop the two others that do not really make sense in the context of quantum computing and we focus on the two ideas mentioned before: superpositions and basis vectors can be distinguished by their types, and the projection of a pair is the analogous to the quantum measurement.

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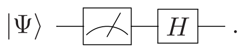
Since our aim is to understand the behaviour of quantum computing, we propose to deal with the no cloning property of quantum computing by combining two approaches seen in the literature: In [4] we proposed that all functions distribute over superpositions, such as linear functions do, e.g. $t\langle u, v\rangle$ will not beta reduce but distribute as $\langle tu, tv\rangle$. This allows, as the CNOT gate does, cloning basis vectors but not general vectors. The second approach is to use a linear type system ensuring that a function will use its argument only once, and so be linear. This way, we can allow functions taking a superposition, but only when the type system ensures that the argument will be used linearly, e.g. $\lambda x^{A\wedge A}\ (H(\pi_A x))$, which represents the quantum algorithm



The general idea is that an operator on basis terms, that is, an operator acting on types which are not conjunctions (so no superpositions), will distribute with respect to pairs, while an operation on superpositions will just beta reduce when a superposition is given to it. That is to say, we have a linear type system which allows weakening and contraction only on conjunction-free types.

In order to mimic the destructive interference from quantum computing we need a symmetric term with respect to pairs. Indeed, given $\langle u, v\rangle$, which is the representation of the superposition $u+v$, we need a w such that $\langle\langle u, v\rangle, w\rangle = u$, that is, a w such that $u+v+w = u$. We will note such a w as $-v$.

Related work.

There are two main trends on the study of functional quantum programming languages: On one hand, one well developed line follows the scheme of quantum-data/classical-control [13]. That is a model where the actual quantum computation runs in a quantum memory [11] while the program controlling which operations to apply and when, runs in a classical computer. This scheme counts with a recent semantical study for higher-order quantum computation [12], as well as several prototypes such as QML [1] or the more scalable and recent Quipper [10]. On the other hand, there is the scheme of quantum data and control. This model, while less suitable to produce a scalable quantum programming language nowadays, may give better insights on the quantum properties and the quantum operations. The present paper is inscribed in this second line. Its origins can be tracked back to Lineal [4] and its type systems such as [3].

Outline of the paper.

In Section 2 we introduce the calculus where superpositions are represented as associative and commutative pairs and projections, base of the measurement, become non-deterministic. In addition, we introduce a minus operator, allowing for destructive interference. In Section 3 we introduce the tensor operator for multiple-qubits systems. Such a tensor is a pair that is neither commutative nor associative. In Section 4 we present two examples in our calculus: the Deutsch algorithm and the Teleportation algorithm. In Section 5 we prove that the resulting system has the Subject Reduction property.

2 No cloning, superpositions and measurement

Notice that it is not useful to distinguish between $(A \wedge A)$ and $(A \wedge A \wedge A)$. Both are superpositions, so they cannot be cloned. What we want to distinguish is when a term is a

$$
\begin{array}{rll}
b := & x \mid \lambda x:A\ t \mid\ |0\rangle\ \mid\ |1\rangle & \text{Basis terms } (\mathcal{B}) \\
v := & b \mid (v+v) \mid 0 \mid\ -v & \text{Values } (\mathcal{V}) \\
t := & v \mid tt \mid (t+t) \mid \pi_A t \mid ?\cdot \mid\ -t & \text{Terms } (\Lambda) \\
A_0 := & \mathbb{Q} \mid A \Rightarrow A & \text{Basis types } (\mathcal{B}) \\
A := & A_0 \mid S(A) & \text{Types } (\mathcal{T})
\end{array}
$$

Figure 1 First grammar of terms and types

basis term from when it is a superposition. Hence, instead of conjunctions or conjunction-free types, we mark as $S(A)$ for a superposition of terms of type A. That is to say, we identify $(A \wedge A) \wedge A$ with $A \wedge A$, but not $A \wedge A$ with A.

Notice that in linear logic we would mark with A the terms that cannot be duplicated while $!A$ types the duplicable terms. In our language A are the terms that cannot be superposed, while $S(A)$ are the terms that can be superposed, and since the superposition forbids duplication, A means that we can duplicate, while $S(A)$ means that we cannot duplicate. So the S is not the same as the bang (!), but the exact opposite. This can be explained by the fact that linear logic is focussed on the possibility of duplication, while we focus on the possibility of superposition, which implies the impossibility of duplication.

As mentioned in the introduction, in order to allow for destructive interferences we need to introduce negatives. Therefore, we introduce an operator "$-$" on terms such that if t has type A, then $-t$ has type $S(A)$.

The first grammars of terms and types are given in Figure 1. The set of free variables of a term t is defined as usual in λ-calculus and denoted by $FV(t)$. Terms are split in three categories. The set of basis terms, denoted by $\mathcal{B}$, includes variables, abstractions and two qubit constants ($|0\rangle$ and $|1\rangle$). We also use $\mathcal{B}$ to denote the set of basis types. The set of values, denoted by $\mathcal{V}$, includes all the basis terms, pairs (noted by $+$), a constant 0 representing the null vector and negative values. We denote by $\mathcal{V}^c$ to the set of closed values and $\mathcal{B}^c$ to the set of closed basis terms, that is, $\mathcal{V}^c = \{t \in \mathcal{V} \mid FV(t) = \emptyset\}$ and $\mathcal{B}^c = \{t \in \mathcal{B} \mid FV(t) = \emptyset\}$. Finally, the set of terms includes all the previous, the pairs on arbitrary terms, and the eliminations: application, projection, an if-then-else construction ($?\cdot$), and negative general terms.

Types are distinguished between conjunction-free types (marked with a subscript 0) and general types. The idea is to distinguish conjunction-free types, which will be used to type basis terms, from general types, which will be used to type superpositions.

Conjunctive types are noted by $S(A)$. We use $(t-t)$ as a shorthand notation for $(t+(-t))$. The term $(t-t)$ has type $S(A)$, and this term will reduce to 0, which is not a basis term.

The operational semantics is given in Figure 2. We use a weak reduction strategy, that is, reduction cannot happen under λ. The reason to use a weak strategy is explained at the end of the section.

Rule (β) is call-by-closed-value. If the argument of the abstraction admits a superposition, the type system will ensure that it is not cloned by imposing that it is used only once. If the type is a basis type, then it can be cloned, and then we must reduce the argument first to ensure that we are cloning a term that can be cloned, for example, a measure over a superposition –which has a basis type but cannot be cloned until it is reduced. A function expecting a basis term but applied to a superposition, is allowed, just that it will first

$$\text{If } u \text{ has type } A \text{ and } u \in \mathcal{V}^c\text{, then } (\lambda x : A\ t)u \longrightarrow (u/x)t \qquad (\beta)$$
$$\text{If } t \text{ has type } A_0 \Rightarrow A\text{, then } t(u+v) \longrightarrow (tu+tv) \qquad (\mathsf{lin_r})$$
$$(t+u)v \longrightarrow (tv+uv) \qquad (\mathsf{lin_l})$$
$$\text{If } t \text{ has type } A\text{, then } \pi_A(t+u) \longrightarrow t \qquad (\mathsf{proj_n})$$
$$\text{If } t \text{ has type } A\text{, then } \pi_A t \longrightarrow t \qquad (\mathsf{proj_1})$$
$$|1\rangle ?u{\cdot}v \longrightarrow u \qquad (\mathsf{if_1})$$
$$|0\rangle ?u{\cdot}v \longrightarrow v \qquad (\mathsf{if_0})$$
$$\text{If } t \text{ has type } A_0 \Rightarrow A \text{ then } t(-u) \longrightarrow -tu \qquad (\mathsf{lineg_r})$$
$$(-t)u \longrightarrow -tu \qquad (\mathsf{lineg_l})$$
$$(0+t) \longrightarrow t \qquad (\mathsf{neutral})$$
$$(t-t) \longrightarrow 0 \qquad (\mathsf{inverse})$$
$$-(-t) \longrightarrow t \qquad (\mathsf{negneg})$$
$$-0 \longrightarrow 0 \qquad (\mathsf{negzero})$$
$$(t+t) \longrightarrow t \qquad (\mathsf{nosup})$$
$$\text{If } t \text{ has type } A_0 \Rightarrow A\text{, then } t0 \longrightarrow 0 \qquad (\mathsf{lin0_r})$$
$$0t \longrightarrow 0 \qquad (\mathsf{lin0_l})$$

Modulo associativity and commutativity of pairs:

$$(u+v) =_{AC} (v+u) \qquad (\mathsf{comm})$$
$$((u+v)+w) =_{AC} (u+(v+w)) \qquad (\mathsf{assoc})$$

Figure 2 First operational semantics

distribute using rule ($\mathsf{lin_r}$), and then do the β-reduction. This way, $(\lambda x : S(\mathbb{Q})\ x)(|0\rangle + |1\rangle)$ reduces to $(|0\rangle + |1\rangle)$ directly, while $(\lambda x : \mathbb{Q}\ x?(-x){\cdot}x)(|0\rangle + |1\rangle)$ must first reduce to $((\lambda x : \mathbb{Q}\ x?(-x){\cdot}x)\,|0\rangle + (\lambda x : \mathbb{Q}\ x?(-x){\cdot}x)\,|1\rangle)$ and then to $(|0\rangle ?(-\,|0\rangle){\cdot}|0\rangle + |1\rangle ?(-\,|1\rangle){\cdot}|1\rangle)$ and so to $(|0\rangle - |1\rangle)$. Notice that in the later term, the argument expected by the abstraction is not a superposition. The beta reduction cannot occur because of the side condition on this rule, but the whole term is typable, as we will see next in Figure 3 (rule $\Rightarrow_{ES}$). The term $(|0\rangle + |1\rangle)$ being a superposition is seen not only in the type but also in the shape of the term, which allows the rule β to be effective. Rule ($\mathsf{lin_l}$) is the analogous to the sum of functions: a pair represents a superposition, which is a sum. Therefore, if an argument is given to a sum of functions, it needs to be given to each function in the sum. Notice that, since we use a weak reduction strategy, the argument v on this rule is closed. Rules ($\mathsf{proj_n}$) and ($\mathsf{proj_1}$) are the projection over associative pairs, that is, the projection over lists (and hence the need of projecting also a single element). The projection, as mentioned in the introduction, projects the term with respect to its type. When more than one term of the expected type is present in the term, due to the commutativity of pairs, the projection ($\mathsf{proj_n}$) will act non-deterministically. Rules ($\mathsf{if_1}$) and ($\mathsf{if_0}$) implement the if-then-else construction. Rules ($\mathsf{lineg_r}$) and ($\mathsf{lineg_l}$) are the analogous to ($\mathsf{lin_r}$) and ($\mathsf{lin_l}$). Rules ($\mathsf{neutral}$), ($\mathsf{inverse}$), (negneg), and ($\mathsf{negzero}$) are self explanatory. Rule (nosup) eliminates fake superpositions such as $(|0\rangle + |0\rangle)$. Rules ($\mathsf{lin0_r}$) and ($\mathsf{lin0_l}$) are analogous to ($\mathsf{lin_r}$) and ($\mathsf{lin_l}$).

The first type system is presented in Figure 3. Rules Ax and $\Rightarrow_I$ are the standard

$$\frac{}{x:A\vdash x:A}\ Ax \qquad \frac{}{\vdash 0:S(A)}\ Ax_0 \qquad \frac{}{\vdash |0\rangle:\mathbb{Q}}\ Ax_{|0\rangle} \qquad \frac{}{\vdash |1\rangle:\mathbb{Q}}\ Ax_{|1\rangle}$$

$$\frac{\Gamma\vdash t:A}{\Gamma\vdash t:S(A)}\ P \qquad \frac{\Gamma\vdash t:S(S(A))}{\Gamma\vdash t:S(A)}\ S \qquad \frac{\Gamma\vdash t:A}{\Gamma\vdash -t:S(A)}\ N$$

$$\frac{}{\vdash ?\cdot:\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}\Rightarrow\mathbb{Q}}\ \mathit{If} \qquad \frac{\Gamma,x:A\vdash t:B}{\Gamma\vdash\lambda x:A\ t:A\Rightarrow B}\ \Rightarrow_I$$

$$\frac{\Gamma\vdash t:A\Rightarrow B \qquad \Delta\vdash u:A}{\Gamma,\Delta\vdash tu:B}\ \Rightarrow_E \qquad \frac{\Gamma\vdash t:S(A\Rightarrow B) \qquad \Delta\vdash u:S(A)}{\Gamma,\Delta\vdash tu:S(B)}\ \Rightarrow_{ES}$$

$$\frac{\Gamma\vdash t:S(A) \qquad \Delta\vdash u:S(A)}{\Gamma,\Delta\vdash (t+u):S(A)}\ \wedge_I \qquad \frac{\Gamma\vdash t:S(A)}{\Gamma\vdash\pi_A t:A}\ \wedge_E$$

$$\frac{\Gamma\vdash t:B}{\Gamma,x:A_0\vdash t:B}\ W \qquad \frac{\Gamma,x:A_0,y:A_0\vdash t:B}{\Gamma,x:A_0\vdash (x/y)t:B}\ C$$

Figure 3 First type system

for linear type systems. Rule Ax_0 types the null vector as a non-basis term. Rules $Ax_{|0\rangle}$ and $Ax_{|1\rangle}$ type the basis qubits with the basic type. Rule P says that a basis type can be promoted to a general type. In particular, notice that $((|0\rangle+|0\rangle)-|0\rangle)$ reduces to $|0\rangle$, however it will have type $S(\mathbb{Q})$. Hence, P allows typing $|0\rangle$ as $S(\mathbb{Q})$. Rule S simplifies the superposition of a superposition to a single superposition. Rule N states that a term with a negative mark is not a basis term. Rule *If* types the if-then-else construction, where, in this type system, $(?\cdot)rst$ is a notation for $r?s\cdot t$. Rule $\Rightarrow_E$ is the standard elimination. Rule $\Rightarrow_{ES}$ is the elimination for superpositions, corresponding to both $(\mathsf{lin_r})$ and $(\mathsf{lin_l})$. Rules $\wedge_I$ and $\wedge_E$ are generalisations from conjunction rules: two superpositions superposed are still a superposition, and a superposition projected is not anymore a superposition. Notice that rule $(\mathsf{proj_1})$ implies that we need to be able to preserve the same type if the term is projected with respect to its full type. This typing rule can be derived:

$$\dfrac{\dfrac{\Gamma\vdash t:A}{\Gamma\vdash t:S(A)}\ P}{\Gamma\vdash\pi_A t:A}\ \wedge_e$$

Finally, Rule W and C correspond to weakening and contraction on conjunction-free types. The rationale is that conjunction-free types will type basis terms, so they can be cloned.

Discussion on the weak strategy We consider a weak strategy to avoid cloning free variables. For example, we can derive

$$y:S(\mathbb{Q})\vdash\lambda x:\mathbb{Q}\ \pi_{\mathbb{Q}}(y):\mathbb{Q}\Rightarrow\mathbb{Q}$$

so this term is not a superposition, however it cannot be cloned until y is replaced by a superposition and projected (measured).

3 Multi-qubit system: introducing tensor

A multi-qubit system is represented with the tensor product between single-qubit Hilbert spaces. The tensor product can be seen as a non-commutative pair, hence we want to

$$
\begin{array}{rll}
b := & x \mid \lambda x : A\ t \mid \ |0\rangle \ \mid \ |1\rangle \ \mid b \otimes b & \text{Basis terms } (\mathcal{B}) \\
v := & b \mid (v + v) \mid 0 \mid \ -v \mid v \otimes v & \text{Values } (\mathcal{V}) \\
t := & v \mid tt \mid (t + t) \mid \pi_A t \mid ?\cdot \mid \ -t \mid t \otimes t \mid \mathit{fst}\ t \mid \mathit{snd}\ t & \text{Terms } (\Lambda) \\
A_0 := & \mathbb{Q} \mid A \Rightarrow A \mid A_0 \otimes A_0 & \text{Basis types } (\mathcal{B}) \\
A := & A_0 \mid S(A) \mid A \otimes A & \text{Types } (\mathcal{T})
\end{array}
$$

Figure 4 Grammar of terms and types of the full system

represent the tensor product as a conjunction-like operator. The grammar of terms and types is given in Figure 4.

The full operational semantics is in Figure 5, including all the rules from Figure 2 plus the rules for $\otimes$. Again, the reduction is weak in the sense that no reduction occurs under a λ, and hence all the terms are cosidered closed.

Since $\otimes$ are just pairs, we allow projecting the first or the second element (rules (fst) and (snd)). The null vector, 0, is absorbing with respect $\otimes$ (rules $(\mathsf{abs_r})$ and $(\mathsf{abs_l})$). Rules $(\mathsf{linten_r})$ and $(\mathsf{linten_l})$ are the linearity functions on a pair. Notice that the critical pair $(\mathsf{linten_r})$ and $(\mathsf{linten_l})$ is closed by (assoc).

The full type system is shown in Figure 6. It includes all the typing rules from figure 3, plus the rules for $\otimes$.

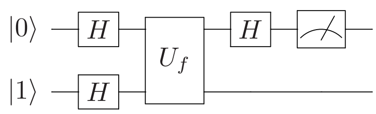
Rule $\otimes_I$ introduces the tensor while $\otimes_{Er}$ and $\otimes_{El}$ eliminate it. Rules P_r and P_l promote a two-qubits system to the status of superposition when one of them is a superposition. Rules S_r and S_l simplifies double superpositions marks.

4 Examples

In this section we show that our language is expressive enough to express the Deutsch algorithm (Section 4.1) and the teleportation algorithm (Section 4.2).

4.1 Deutsch algorithm

We can implement the Deutsch algorithm, which is given by the following circuit.



This algorithm tests whether the binary function f implemented by the oracle U_f is constant or balanced. When the function is constant, the first qubit ends in $|0\rangle$, when it is balanced, it ends in $|1\rangle$.

We need several auxiliary functions. First we define App_1 as the function taking a function to be applied to one qubit and a two qubits system and applying the function to the first.

$$App_1 = \lambda f : \mathbb{Q} \Rightarrow S(\mathbb{Q})\ \lambda x : \mathbb{Q} \otimes \mathbb{Q}\ ((f\ (\mathit{fst}\ x)) \otimes (\mathit{snd}\ x))$$

Similarly, App_{both} receives two functions to apply the first to the first qubit and the second to the second qubit.

$$App_{both} = \lambda f : \mathbb{Q} \Rightarrow S(\mathbb{Q})\ \lambda g : \mathbb{Q} \Rightarrow S(\mathbb{Q})\ \lambda x : \mathbb{Q} \otimes \mathbb{Q}\ ((f\ (\mathit{fst}\ x)) \otimes (g\ (\mathit{snd}\ x)))$$

$$\text{If } u \text{ has type } A \text{ and } u \in \mathcal{V}^c\text{, then } (\lambda x : A\ t)u \longrightarrow (u/x)t \qquad (\beta)$$
$$\text{If } t \text{ has type } A_0 \Rightarrow A\text{, then } t(u+v) \longrightarrow (tu+tv) \qquad (\mathsf{lin_r})$$
$$(t+u)v \longrightarrow (tv+uv) \qquad (\mathsf{lin_l})$$
$$\text{If } t \text{ has type } A\text{, then } \pi_A(t+u) \longrightarrow t \qquad (\mathsf{proj_n})$$
$$\text{If } t \text{ has type } A\text{, then } \pi_A t \longrightarrow t \qquad (\mathsf{proj_1})$$
$$|1\rangle ?u{\cdot}v \longrightarrow u \qquad (\mathsf{if_1})$$
$$|0\rangle ?u{\cdot}v \longrightarrow v \qquad (\mathsf{if_0})$$
$$\text{If } t \text{ has type } A_0 \Rightarrow A \text{ then } t(-u) \longrightarrow -tu \qquad (\mathsf{lineg_r})$$
$$(-t)u \longrightarrow -tu \qquad (\mathsf{lineg_l})$$
$$(0+t) \longrightarrow t \qquad (\mathsf{neutral})$$
$$(t-t) \longrightarrow 0 \qquad (\mathsf{inverse})$$
$$-(-t) \longrightarrow t \qquad (\mathsf{negneg})$$
$$-0 \longrightarrow 0 \qquad (\mathsf{negzero})$$
$$(t+t) \longrightarrow t \qquad (\mathsf{nosup})$$
$$\text{If } t \text{ has type } A_0 \Rightarrow A\text{, then } t0 \longrightarrow 0 \qquad (\mathsf{lin0_r})$$
$$0t \longrightarrow 0 \qquad (\mathsf{lin0_l})$$
$$\mathit{fst}\ (t \otimes u) \longrightarrow t \qquad (\mathsf{fst})$$
$$\mathit{snd}\ (t \otimes u) \longrightarrow u \qquad (\mathsf{snd})$$
$$t \otimes 0 \longrightarrow 0 \qquad (\mathsf{abs_r})$$
$$0 \otimes t \longrightarrow 0 \qquad (\mathsf{abs_l})$$
$$t((r+s)\otimes u) \longrightarrow (t(r\otimes u) + t(s \otimes u)) \qquad (\mathsf{linten_r})$$
$$t(u\otimes (r+s)) \longrightarrow (t(u\otimes r) + t(u \otimes s)) \qquad (\mathsf{linten_l})$$

Modulo associativity and commutativity of pairs:

$$(u+v) =_{AC} (v+u) \qquad (\mathsf{comm})$$
$$((u+v)+w) =_{AC} (u+(v+w)) \qquad (\mathsf{assoc})$$

Figure 5 Full operational semantics

The Hadamard gate produces $(|0\rangle + |1\rangle)$ when applied to $|0\rangle$ and $(|0\rangle + (-|1\rangle))$ when applied to $|1\rangle$. Hence, it can be implemented with the if-then-else construction:

$$H = \lambda x : \mathbb{Q}\ (|0\rangle + (x?(-|1\rangle){\cdot}|1\rangle))$$

Similarly, for the *not* gate we implement it in the following way:

$$\mathit{not} = \lambda x : \mathbb{Q}\ (x?|0\rangle{\cdot}|1\rangle)$$

The oracle U_f is defined by

$$U_f\,|xy\rangle = |x, y \oplus f(x)\rangle$$

where $\oplus$ is the addition modulo 2. Hence, it can be implemented in the following way:

$$U = \lambda f : \mathbb{Q} \Rightarrow \mathbb{Q}\ \lambda x : \mathbb{Q}\otimes\mathbb{Q}\ ((\mathit{fst}\ x) \otimes ((\mathit{snd}\ x)?(\mathit{not}\ (f\ (\mathit{fst}\ x))){\cdot}(f\ (\mathit{fst}\ x))))$$

$$\frac{}{x : A \vdash x : A}\ Ax \qquad \frac{}{\vdash 0 : S(A)}\ Ax_0 \qquad \frac{}{\vdash |0\rangle : \mathbb{Q}}\ Ax_{|0\rangle} \qquad \frac{}{\vdash |1\rangle : \mathbb{Q}}\ Ax_{|1\rangle}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : S(A)}\ P \qquad \frac{\Gamma \vdash t : S(S(A))}{\Gamma \vdash t : S(A)}\ S \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash -t : S(A)}\ N$$

$$\frac{}{\vdash ?\cdot : \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q}}\ \mathit{If} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A\ t : A \Rightarrow B}\ \Rightarrow_I$$

$$\frac{\Gamma \vdash t : A \Rightarrow B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B}\ \Rightarrow_E \qquad \frac{\Gamma \vdash t : S(A \Rightarrow B) \qquad \Delta \vdash u : S(A)}{\Gamma, \Delta \vdash tu : S(B)}\ \Rightarrow_{ES}$$

$$\frac{\Gamma \vdash t : S(A) \qquad \Delta \vdash u : S(A)}{\Gamma, \Delta \vdash (t + u) : S(A)}\ \wedge_I \qquad \frac{\Gamma \vdash t : S(A)}{\Gamma \vdash \pi_A t : A}\ \wedge_E$$

$$\frac{\Gamma \vdash t : B}{\Gamma, x : A_0 \vdash t : B}\ W \qquad \frac{\Gamma, x : A_0, y : A_0 \vdash t : B}{\Gamma, x : A_0 \vdash (x/y)t : B}\ C$$

$$\frac{\Gamma \vdash t : A \qquad \Delta \vdash r : B}{\Gamma, \Delta \vdash t \otimes r : A \otimes B}\ \otimes_I \qquad \frac{\Gamma \vdash t : A \otimes B}{\Gamma \vdash \mathit{fst}\ t : A}\ \otimes_{Er} \qquad \frac{\Gamma \vdash t : A \otimes B}{\Gamma \vdash \mathsf{snd}\ t : B}\ \otimes_{El}$$

$$\frac{\Gamma \vdash t : S(A) \otimes B}{\Gamma \vdash t : S(A \otimes B)}\ P_r \qquad \frac{\Gamma \vdash t : A \otimes S(B)}{\Gamma \vdash t : S(A \otimes B)}\ P_l$$

$$\frac{\Gamma \vdash t : S(A \otimes S(B))}{\Gamma \vdash t : S(A \otimes B)}\ S_r \qquad \frac{\Gamma \vdash t : S(S(A) \otimes B)}{\Gamma \vdash t : S(A \otimes B)}\ S_l$$

Figure 6 Full type system

and $U_f = Uf$.

In order to measure the first qubit we will project the two-qubits system with respect to $\mathbb{Q} \otimes S(\mathbb{Q})$. However, before projecting we need to factorize the first qubit. For example, if we project with respect to $\mathbb{Q} \otimes S(\mathbb{Q})$ the term $((|0\rangle \otimes |0\rangle) + (|0\rangle \otimes |1\rangle))$, we can obtain either $|0\rangle \otimes |0\rangle$ or $|0\rangle \otimes |1\rangle$, which do not represent the measurement. Instead, if we factorize the first qubit before projecting, we project over $|0\rangle \otimes (|0\rangle + |1\rangle)$, obtaining the same term, as expected when measuring the first qubit.

The measurement will be defined as the projection over $Fact_1\ x$. However, to ensure $Fact_1$ is not distributed linearly over x, we use the canon $[r]$ notation for encapsulating a term inside an abstraction:

$$[r] = \lambda x : \mathbb{Q} \Rightarrow \mathbb{Q}\ r \quad \text{whith } x \text{ a fresh variable}$$

and the co-canon $\{r\}$ notation, to free the term from the encapsulation:

$$\{r\} = r\ \lambda x : \mathbb{Q}\ x$$

This way we can stop the linear distribution by encapsulating the term and removing the encapsulation after the term is consumed. In addition, we use the notation $[A]$ for $(\mathbb{Q} \Rightarrow \mathbb{Q}) \Rightarrow A$.

Hence, the measurement is defined by:

$$F_0 = \lambda x : \mathbb{Q} \otimes \mathbb{Q}\ (\mathit{fst}\ x)?0\cdot(\mathit{snd}\ x)$$

$$
\begin{aligned}
F_1 &= \lambda x : \mathbb{Q}\otimes\mathbb{Q}\ (\mathit{fst}\ x)?(\mathit{snd}\ x)\cdot 0\\
\mathit{Fact}_1 &= \lambda x : [S(\mathbb{Q}\otimes\mathbb{Q})]\ ((|0\rangle\otimes(F_0\ \{x\})) + (|1\rangle\otimes(F_1\ \{x\})))\\
\mathit{Meas}_1 &= \lambda x : S(\mathbb{Q}\otimes\mathbb{Q})\ \pi_{\mathbb{Q}\otimes S(\mathbb{Q})}(\mathit{Fact}_1\ [x])
\end{aligned}
$$

Finally, the Deutsch algorithm combines all the previous definitions:

$$
\begin{aligned}
\mathit{Deutsch} = {}& \lambda f : \mathbb{Q}\Rightarrow\mathbb{Q}\\
&(\mathit{Meas}_1\ (\mathit{App}_1\ H\ ((U\ f)\ (\mathit{App}_{both}\ H\ H\ (|0\rangle\otimes|1\rangle)))))
\end{aligned}
$$

This term is typed by

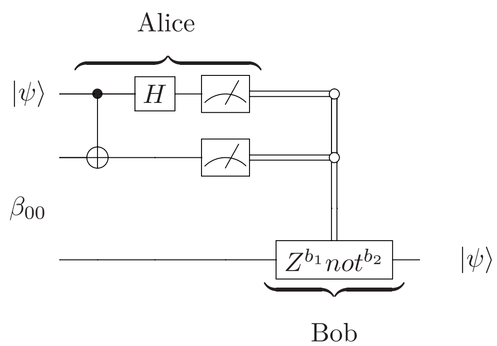
$$\vdash \mathit{Deutsch} : (\mathbb{Q}\Rightarrow\mathbb{Q})\Rightarrow\mathbb{Q}\otimes S(\mathbb{Q})$$

The Deutsch algorithm applied to the identity function reduces, as expected, as follows:

$$\mathit{Deutsch}\ \mathit{id} \longrightarrow^* |1\rangle\otimes(|0\rangle-|1\rangle)$$

4.2 Teleportation algorithm

The previous example does not show the use of the measurement as a state-changing operator. Indeed, the state to be measured is already a basis state. Therefore, we introduce another example, the teleportation algorithm, where the measurement is used as an operator changing the state.



The *cnot* gate will apply a *not* gate to the second qubit only when the first qubit is $|1\rangle$. Hence, it can be implemented with an if-then-else construction as follows:

$$\mathit{cnot} = \lambda x : \mathbb{Q}\otimes\mathbb{Q}\ ((\mathit{fst}\ x)\otimes((\mathit{fst}\ x)?(\mathit{not}\ (\mathit{snd}\ x))\cdot(\mathit{snd}\ x)))$$

We reuse H and *not* from the previous example. The application App_1 is not useful anymore since we need to make a new application for a system with three qubits, App^3_1. In addition, we need to apply *cnot* to the two first qubits, so we define App^3_{12}.

$$
\begin{aligned}
\mathit{App}^3_1 = {}& \lambda f : \mathbb{Q}\Rightarrow S(\mathbb{Q})\\
&\quad \lambda x : (\mathbb{Q}\otimes\mathbb{Q})\otimes\mathbb{Q}\\
&\quad\quad (((f\ (\mathit{fst}\ \mathit{fst}\ x))\otimes(\mathit{snd}\ \mathit{fst}\ x))\otimes(\mathit{snd}\ x))\\
\mathit{App}^3_{12} = {}& \lambda f : \mathbb{Q}\otimes\mathbb{Q}\Rightarrow\mathbb{Q}\otimes\mathbb{Q}\\
&\quad \lambda x : \mathbb{Q}\otimes(\mathbb{Q}\otimes\mathbb{Q})\\
&\quad\quad (f\ (\mathit{fst}\ x\otimes\mathit{fst}\ \mathit{snd}\ x)\otimes\mathit{snd}\ \mathit{snd}\ x)
\end{aligned}
$$

Similar to what we did in the previous example, to define the measurement $Meas^3_{12}$, that is, the function measuring the first two qubits, we need to factorize the first two qubits, and then apply the projection:

$$
\begin{aligned}
F_{00} &= \lambda x : (\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q}\ (\mathit{fst}\ \mathit{fst}\ x)?0{\cdot}((\mathit{snd}\ \mathit{fst}\ x)?0{\cdot}(\mathit{snd}\ x))\\
F_{01} &= \lambda x : (\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q}\ (\mathit{fst}\ \mathit{fst}\ x)?0{\cdot}((\mathit{snd}\ \mathit{fst}\ x)?(\mathit{snd}\ x){\cdot}0)\\
F_{10} &= \lambda x : (\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q}\ (\mathit{fst}\ \mathit{fst}\ x)?((\mathit{snd}\ \mathit{fst}\ x)?0{\cdot}(\mathit{snd}\ x)){\cdot}0\\
F_{11} &= \lambda x : (\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q}\ (\mathit{fst}\ \mathit{fst}\ x)?((\mathit{snd}\ \mathit{fst}\ x)?(\mathit{snd}\ x){\cdot}0){\cdot}0\\
Fact_{12} &= \lambda x : [S((\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q})]\\
&\qquad (((|0\rangle \otimes |0\rangle) \otimes (F_{00}\{x\})+\\
&\qquad\ (|0\rangle \otimes |1\rangle) \otimes (F_{01}\{x\}))+\\
&\qquad ((|1\rangle \otimes |0\rangle) \otimes (F_{10}\{x\})+\\
&\qquad\ (|1\rangle \otimes |1\rangle) \otimes (F_{11}\{x\})))\\
Meas^3_{12} &= \lambda x : S((\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q})\ \pi_{(\mathbb{Q}\otimes\mathbb{Q})\otimes S(\mathbb{Q})}(Fact_{12}[x])
\end{aligned}
$$

The Z gate return $|0\rangle$ when it receives $|0\rangle$, and $-|1\rangle$ when it receives $|1\rangle$. Hence, it can be implemented by:

$$Z = \lambda x : \mathbb{Q}\ (x?|0\rangle{\cdot}(-|1\rangle))$$

The Bob side of the algorithm will apply Z and/or *not* according to the bits it receives from Alice. Hence, for any $\vdash U : \mathbb{Q} \Rightarrow S(\mathbb{Q})$ or $\vdash U : \mathbb{Q} \Rightarrow \mathbb{Q}$, we define U^b to be the function which depending on the value of a basis qubit will apply a gate or not:

$$U^b = \lambda b : \mathbb{Q}\ \lambda x : \mathbb{Q}\ (b?Ux{\cdot}x)$$

Alice and Bob parts of the algorithm can be defined separately by:

$$
\begin{aligned}
\mathit{Alice} &= \lambda x{:}\ [S(\mathbb{Q} \otimes (\mathbb{Q} \otimes \mathbb{Q}))]\ (Meas^3_{12}(App^3_1\ H\ (App^3_{12}\ \mathit{cnot}\ \{x\})))\\
\mathit{Bob} &= \lambda x : (\mathbb{Q} \otimes \mathbb{Q}) \otimes \mathbb{Q}\ (Z^{(\mathit{fst}\ \mathit{fst}\ x)}\ \mathit{not}^{(\mathit{snd}\ \mathit{fst}\ x)}\ (\mathit{snd}\ x))
\end{aligned}
$$

The teleportation is applied to the state

$$\beta_{00} = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

and it is defined by:

$$Telportation = \lambda q : S(\mathbb{Q})\ (\mathit{Bob}\ (\mathit{Alice}\ [q \otimes \beta_{00}]))$$

This term is typed, as expected, by:

$$\vdash Teleportation : S(\mathbb{Q}) \Rightarrow S(\mathbb{Q})$$

And applying the Teleportation to a superposition $(|0\rangle + |1\rangle)$ will reduce in the following way:

$$
\begin{aligned}
&Teleportation\ (|0\rangle + |1\rangle) \longrightarrow^*\\
&\qquad \mathit{Bob}\ (\pi_{(\mathbb{Q}\otimes\mathbb{Q})\otimes S(\mathbb{Q})}\ (((|0\rangle \otimes |0\rangle) \otimes (|0\rangle + |1\rangle)+\\
&\qquad\qquad (|0\rangle \otimes |1\rangle) \otimes (|1\rangle + |0\rangle))+\\
&\qquad\qquad ((|1\rangle \otimes |0\rangle) \otimes (|0\rangle - |1\rangle)+\\
&\qquad\qquad (|1\rangle \otimes |1\rangle) \otimes (|1\rangle - |0\rangle))))
\end{aligned}
$$

At this point, there are four possible output of the projection:

$$(|0\rangle \otimes |0\rangle) \otimes (|0\rangle + |1\rangle)$$
$$(|0\rangle \otimes |1\rangle) \otimes (|1\rangle + |0\rangle)$$
$$(|1\rangle \otimes |0\rangle) \otimes (|0\rangle - |1\rangle)$$
$$(|1\rangle \otimes |1\rangle) \otimes (|1\rangle - |0\rangle)$$

Assume the projection outputs $(|1\rangle \otimes |1\rangle) \otimes (|1\rangle - |0\rangle)$. Then, the teleportation reduces to

$$\begin{aligned}
&\mathit{Bob}\ ((|1\rangle \otimes |1\rangle) \otimes (|1\rangle - |0\rangle))\\
&\longrightarrow (\mathit{Bob}\ ((|1\rangle \otimes |1\rangle) \otimes |1\rangle) - \mathit{Bob}\ ((|1\rangle \otimes |1\rangle) \otimes |0\rangle))\\
&\longrightarrow (Z^{|1\rangle} \mathit{not}^{|1\rangle}\ |1\rangle - (Z^{|1\rangle} \mathit{not}^{|1\rangle}\ |0\rangle))\\
&\longrightarrow^* (Z^{|1\rangle}\ |0\rangle - (Z^{|1\rangle}\ |1\rangle))\\
&\longrightarrow^* (|0\rangle - (-\ |1\rangle))\\
&\longrightarrow (|0\rangle + |1\rangle)
\end{aligned}$$

The other projections are analogous.

5 Subject reduction

Theorem 5.9 ensures that typing is preserved by weak-reduction. We need some auxiliary lemmas and definitions before proving Subject Reduction.

▶ **Definition 5.1** (Subtyping). The relation $\preceq$ is defined as the least congruence such that

- $A \preceq A$
- $A \preceq S(A)$
- $S(S(C)) \preceq S(C)$
- $S(C) \otimes D \preceq S(C \otimes D)$
- $C \otimes S(D) \preceq S(C \otimes D)$
- $S(C \otimes S(D)) \preceq S(C \otimes D)$
- $S(S(C) \otimes D) \preceq S(C \otimes D)$

▶ **Lemma 5.2.** *If $A \preceq B$, then either $B \notin \mathcal{B}$ or $A = B$.*

Proof. Straightforward case by case analysis of Definition 5.1. ◀

▶ **Lemma 5.3.** *If $A \preceq S(B \Rightarrow C)$, then $A \preceq S(B) \Rightarrow S(E)$.*

Proof. The only possible unification is with $A = S(\dots S(A_1 \Rightarrow A_2)\dots)$ (possible with no S). Hence, the relation happens at the congruence level: $S(\dots S(A_1)\dots) \preceq B$ and $S(\dots S(A_2)\dots) \preceq C$. ◀

▶ **Lemma 5.4.** *If $A \preceq B$ and $\Gamma \vdash t : A$, then $\Gamma \vdash t : B$.*

Proof. Direct consequence of rules P, S, P_r, P_l, S_r and S_l. ◀

Let $|\Gamma|$ be the set of types in Γ. For example, $|x : A, y : B_0| = \{A, B_0\}$. Also, let $\mathcal{B}$ be the set of basis types.

▶ **Lemma 5.5** (Generation lemmas).

- *If* $\Gamma \vdash x : A$, *then* $x : B \vdash x : B$, *with* $x : B \in \Gamma$, $|\Gamma| \setminus \{B\} \subseteq \mathcal{B}$, *and* $B \preceq A$.
- *If* $\Gamma \vdash \lambda x : A\ t : B$, *then* $\Gamma', x : A \vdash t : C$, *with* $\Gamma' \subseteq \Gamma$, $A \Rightarrow C \preceq B$ *and* $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash tu : A$, *then one of the following possibilities happens:*
  - $\Gamma_1 \vdash t : B \Rightarrow C$ *and* $\Gamma_2 \vdash u : B$, *with* $C \preceq A$, *or*
  - $\Gamma_1 \vdash t : S(B \Rightarrow C)$ *and* $\Gamma_2 \vdash u : S(B)$, *with* $S(C) \preceq A$.

  In both cases, $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$ *and* $|\Gamma| \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash (t+u) : A$, *then* $\Gamma_1 \vdash t : S(B)$ *and* $\Gamma_2 \vdash u : S(B)$, *with* $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, $|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| \subseteq \mathcal{B}$ *and* $S(B) \preceq A$.
- *If* $\Gamma \vdash \pi_B t : A$, *then* $\Gamma' \vdash t : S(B)$ *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $B \preceq A$.
- *If* $\Gamma \vdash |0\rangle : A$, *then* $\mathbb{Q} \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash |1\rangle : A$, *then* $\mathbb{Q} \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash ?\cdot : A$, *then* $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash 0 : A$, *then exists* $B \notin \mathcal{B}$ *such that* $B \preceq A$ *and* $|\Gamma| \subseteq \mathcal{B}$.
- *If* $\Gamma \vdash -t : A$, *then* $\Gamma' \vdash t : B$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $S(B) \preceq A$.
- *If* $\Gamma \vdash t \otimes u : A$, *then* $\Gamma_1 \vdash t : B$ *and* $\Gamma_2 \vdash u : C$, *with* $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma$, $|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| \subseteq \mathcal{B}$ *and* $B \otimes C \preceq A$.
- *If* $\Gamma \vdash \mathit{fst}\ t : A$, *then* $\Gamma' \vdash t : B \otimes C$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $B \preceq A$.
- *If* $\Gamma \vdash \mathit{snd}\ t : A$, *then* $\Gamma' \vdash t : B \otimes C$, *with* $\Gamma' \subseteq \Gamma$, $|\Gamma \setminus \Gamma'| \subseteq \mathcal{B}$ *and* $C \preceq A$.

Proof. First notice that if $\Gamma \vdash t : A$ is derivable, then $\Delta \vdash t : B$ is derivable, with $\Gamma \subseteq \Delta$ and $|\Delta \setminus \Gamma| \subseteq \mathcal{B}$ (because of rule W) and $A \preceq B$, (because of rules P, S, P_r, P_l, S_r and S_l). Notice also that those are the only typing rules changing the sequent without changing the term on the sequent. All the other rules — except for $\Rightarrow_E$ and $\Rightarrow_{ES}$, which anyway are straightforward to check — are syntax directed: one rule for each term. Therefore, the lemma is proven by a straightforward rule by rule analysis. ◀

▶ **Lemma 5.6.** *If* $\Gamma, x : A \vdash t : B$ *with* $A \notin \mathcal{B}$, *then* x *appears free exactly once in* t.

Proof. The only way to have a variable in the context appearing more than once in a term is by the contraction rule (C), however, the variable need to have a basis type. The only way to have a variable in the context not appearing in a term is by the weakening rule (W), however, the variable also need to have a basis type. Therefore, a variable in the context with a general type must appear exactly once in the term. ◀

▶ **Lemma 5.7.** *If* $\Gamma \vdash t : A$ *and* $FV(t) = \emptyset$, *then* $|\Gamma| \subseteq \mathcal{B}$.

Proof. If $FV(t) = \emptyset$ then $\vdash t : A$. If $\Gamma \neq \emptyset$, the only way to derive $\Gamma \vdash t : A$ is by using rule W to form Γ, hence $|\Gamma| \subseteq \mathcal{B}$. ◀

The following lemma is standard in proofs of subject reduction and can be found for example in [6, prop. 3.1.11]. It ensures that when substituting terms for term variables, in an adequate manner, then the type derived remains valid.

▶ **Lemma 5.8** (Substitution lemma). *For any terms* t *and* u, *any types* A *and* B *and any context* Γ *such that* $u \in \mathcal{V}^c$, *then if* $\Gamma, x : A \vdash t : B$ *and* $\Delta \vdash u : A$ *we have* $\Gamma, \Delta \vdash (u/x)t : B$.

Proof. First, notice that due to Lemma 5.7, $|\Delta| \subseteq \mathcal{B}$. Without lost of generality, we consider $\Delta = \emptyset$. We proceed by structural induction on t. We give here only one case, the full proof is developed in Appendix A.

Let $t = vw$. By Lemma 5.5 there are two cases:

1. $\Gamma_1 \vdash v : C \Rightarrow D$ and $\Gamma_2 \vdash w : C$, with $D \preceq B$.
2. $\Gamma_1 \vdash v : S(C \Rightarrow D)$ and $\Gamma_2 \vdash w : S(C)$, with $S(D) \preceq B$.

In any case, with $(\Gamma_1 \cup \Gamma_2) \subseteq \Gamma \cup \{x : A\}$ and $(|\Gamma| \cup \{A\}) \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$. We consider the two cases:

- $A \notin \mathcal{B}$ and $u \in \mathcal{V}^c$. Then, by Lemma 5.6, x appears free exactly once in vw. Cases:
  - Let x appear in v and not in w. Then $x : A \in \Gamma_1$ and so, by the induction hypothesis, in case 1, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D$, in case 2, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D)$. Hence, one of the following derivations is valid:

$$\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D \quad \Gamma_2 \vdash w : C}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \vdash (u/x)vw : D} \Rightarrow_E}{\Gamma \vdash (u/x)vw : D} W$$

  or

$$\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D) \quad \Gamma_2 \vdash w : S(C)}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \vdash (u/x)vw : S(D)} \Rightarrow_{ES}}{\Gamma \vdash (u/x)vw : S(D)} W$$

  In any case, by Lemma 5.4, $\Gamma \vdash (u/x)vw : B$. Notice that $(u/x)vw = (u/x)(vw)$.
  - Let x appear in w and not in v. Then $x : A \in \Gamma_2$ and so, by the induction hypothesis, in case 1, $\Gamma_2 \setminus \{x : A\} \vdash (u/x)w : C$, in case 2, $\Gamma_2 \setminus \{x : A\} \vdash (u/x)w : S(C)$. Hence, one of the following derivations is valid:

$$\dfrac{\dfrac{\Gamma_1 \vdash v : C \Rightarrow D \quad \Gamma_2 \setminus \{x : A\} \vdash (u/x)w : C}{\Gamma_1, \Gamma_2 \setminus \{x : A\} \vdash v(u/x)w : D} \Rightarrow_E}{\Gamma \vdash v(u/x)w : D} W$$

  or

$$\dfrac{\dfrac{\Gamma_1 \vdash v : S(C \Rightarrow D) \quad \Gamma_2 \setminus \{x : A\} \vdash (u/x)w : S(C)}{\Gamma_1, \Gamma_2 \setminus \{x : A\} \vdash v(u/x)w : S(D)} \Rightarrow_{ES}}{\Gamma \vdash v(u/x)w : S(D)} W$$

  In any case, by Lemma 5.4, $\Gamma \vdash v(u/x)w : B$. Notice that $v(u/x)w = (u/x)(vw)$.
- $A \in \mathcal{B}$ and $u \in \mathcal{B}^c$. Then if x appears free exactly once in vw, this case is analogous to the previous item. Let $x \in FV(v) \cap FV(w)$. Then, by the induction hypothesis, in case 1, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D$ and $\Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : C$, in case 2, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D)$ and $\Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : S(C)$. Therefore, one of the following derivations are valid:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D \quad \Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : C}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u'/x)w : D} \Rightarrow_E}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u/x)w : D} C}{\Gamma \vdash (u/x)v(u/x)w : D} W$$

  or

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D) \quad \Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : S(C)}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u'/x)w : S(D)} \Rightarrow_{ES}}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u/x)w : S(D)} C}{\Gamma \vdash (u/x)v(u/x)w : S(D)} W$$

  In any case, by Lemma 5.4, $\Gamma \vdash (u/x)v(u/x)w : B$. Notice that $(u/x)v(u/x)w = (u/x)(vw)$. ◀

Since the strategy is weak, subject reduction is proven for closed terms.

▶ **Theorem 5.9** (Subject reduction on closed terms). *For any closed terms t and u and type A, if $t \longrightarrow u$ and $\vdash t : A$, then $\vdash u : A$.*

Proof. By induction on the rewrite relation. Developed in full details in Appendix A. ◀

6 Conclusion

In this paper we have introduced an extension to simply typed lambda calculus to handle some aspects of quantum computing, namely destructive superpositions, measurement and no cloning. No cloning is treated, following Lineal [4], by ensuring linearity on the applications (particularly through rewrite rules $(\mathsf{lin_r})$ and $(\mathsf{lineg_r})$). However, an application implementing a measurement cannot be linear in this sense. Hence, in such a case, a type system based on linear logic [9] is used to ensure that the function will use its arguments just once. As stated in [8] the two meanings of linearity differ: No-cloning means that the functions are linear on superpositions, in the sense of Lineal $(f(u + v) = (fu + fv))$, while the linearity in the sense of linear logic means that the argument have to be used only once. Indeed, not allowing using a basis term more than once in the body of an abstraction would prevent defining *cnot* as we did:

$$\mathit{cnot} = \lambda x : \mathbb{Q} \otimes \mathbb{Q}\ ((\mathit{fst}\ x) \otimes ((\mathit{fst}\ x)?\mathit{not}(\mathit{snd}\ x)\cdot(\mathit{snd}\ x)))$$

Notice that $\mathit{cnot}(|0\rangle + |1\rangle) = (\mathit{cnot}\,|0\rangle + \mathit{cnot}\,|1\rangle)$. On the other hand, linear logic allows us to prevent a superposition to be cloned by imposing that an abstraction receiving a superposition cannot use it more than once. This allows us to define the measurement as:

$$\mathit{meas} = \lambda x : S(\mathbb{Q})\ (\pi_{\mathbb{Q}}(x))$$

In particular, $\mathit{meas}\ (|0\rangle+|1\rangle)$ must reduce to $\pi_{\mathbb{Q}}(|0\rangle+|1\rangle)$ and not to $(\mathit{meas}\,|0\rangle+\mathit{meas}\,|1\rangle)$, which reduces to $(\pi_{\mathbb{Q}}\,|0\rangle + \pi_{\mathbb{Q}}\,|1\rangle)$ and so to $(|0\rangle + |1\rangle)$.

A well known problem in λ-calculus with a linear logic type system including modalities is the following example:

$$y : S(\mathbb{Q}) \vdash (\lambda x : \mathbb{Q} \Rightarrow S(\mathbb{Q})\ (x \otimes x))(\lambda z : \mathbb{Q}\ y) : S(\mathbb{Q}) \otimes S(\mathbb{Q})$$

If we allow β-reducing this term, we would obtain $(\lambda z : \mathbb{Q}\ y) \otimes (\lambda z : \mathbb{Q}\ y)$ which is not typable in context $y : S(\mathbb{Q})$. One solution to this counter-example is by the so-called Dual Intuitionistic Linear Logic [5], where the terms that can be cloned are distinguished by a mark, and used in a *let* construction, while non-clonable terms are used in λ abstractions. In our case we remove the counter-example by requiring the argument to be closed. Notice that, for example, both $(\lambda x : \mathbb{Q} \Rightarrow S(\mathbb{Q})\ (x \otimes x))(\lambda z : \mathbb{Q}\ (|0\rangle + |1\rangle))$ and $(\lambda z : \mathbb{Q}\ (|0\rangle + |1\rangle)) \otimes (\lambda z : \mathbb{Q}\ (|0\rangle + |1\rangle))$ are typable (with empty context).

As future work we are willing to re-introduce scalars to the calculus following [2, 3].

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A Omitted proofs

Proof of Lemma 5.8 (Substitution lemma). First, notice that due to Lemma 5.7, $|\Delta| \subseteq \mathcal{B}$. Without lost of generality, we consider $\Delta = \emptyset$. We proceed by structural induction on t.

$\boldsymbol{t = x}$**.** By Lemma 5.5, $A \preceq B$ and $|\Gamma| \subseteq \mathcal{B}$. Since $(u/x)x = u$, we have $\vdash (u/x)x : A$. Hence, since $A \preceq B$, by Lemma 5.4, $\vdash (u/x)x : B$. Finally, since $|\Gamma| \subseteq \mathcal{B}$, by rule W, we have $\Gamma \vdash (u/x)x : B$.

$\boldsymbol{t = y \neq x}$**.** By Lemma 5.5, $y : C \vdash y : C$ with $y : C \in \Gamma \cup \{x : A\}$, $(|\Gamma| \cup \{A\}) \setminus \{C\} \subseteq \mathcal{B}$ and $C \preceq B$. Since $C \preceq B$, by Lemma 5.4, $y : C \vdash y : B$. Since $(|\Gamma| \cup \{A\}) \setminus \{C\} \subseteq \mathcal{B}$ and $y : C \in \Gamma \cup \{x : A\}$, by rule W, $\Gamma, x : A \vdash y : B$. Finally, since $(u/x)y = y$, we have $\Gamma, x : A \vdash (u/x)y : B$.

$\boldsymbol{t = \lambda y : C\ v}$**.** Without lost of generality, assume y is does not appear free in u. By Lemma 5.5, $\Gamma', y : C \vdash v : D$, with $\Gamma' \subseteq \Gamma \cup \{x : A\}$, $C \Rightarrow D \preceq B$ and $(|\Gamma| \cup \{A\}) \setminus |\Gamma'| \subseteq \mathcal{B}$. By the induction hypothesis, $\Gamma', y : C \vdash (u/x)v : D$. Hence, by rule $\Rightarrow_I$, $\Gamma' \vdash \lambda y : C\ (u/x)v : C \Rightarrow D$. Since $C \Rightarrow D \preceq B$, by Lemma 5.4, $\Gamma' \vdash \lambda y : C\ (u/x)v : B$. Hence, since $|\Gamma| \setminus |\Gamma'| \subseteq \mathcal{B}$, by rule W, $\Gamma \vdash \lambda y : C\ (u/x)v : B$. Since y is does not appear free in u, $\lambda y : C\ (u/x)v = (u/x)(\lambda y : C\ v)$. Therefore, $\Gamma \vdash (u/x)(\lambda y : C\ v) : B$.

$\boldsymbol{t = vw}$**.** By Lemma 5.5 there are two cases:

1. $\Gamma_1 \vdash v : C \Rightarrow D$ and $\Gamma_2 \vdash w : C$, with $D \preceq B$.

2. $\Gamma_1 \vdash v : S(C \Rightarrow D)$ and $\Gamma_2 \vdash w : S(C)$, with $S(D) \preceq B$.

In any case, with $(\Gamma_1 \cup \Gamma_2) \subseteq \Gamma \cup \{x : A\}$ and $(|\Gamma| \cup \{A\}) \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$. We consider the two cases:

- $A \notin \mathcal{B}$ and $u \in \mathcal{V}^c$. Then, by Lemma 5.6, x appears free exactly once in vw. Cases:
  - Let x appear in v and not in w. Then $x : A \in \Gamma_1$ and so, by the induction hypothesis, in case 1, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D$, in case 2, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D)$. Hence, one of the following derivations is valid:

$$\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D \quad \Gamma_2 \vdash w : C}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \vdash (u/x)vw : D}\ \Rightarrow_E}{\Gamma \vdash (u/x)vw : D}\ W$$

or

$$\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D) \quad \Gamma_2 \vdash w : S(C)}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \vdash (u/x)vw : S(D)}\ \Rightarrow_{ES}}{\Gamma \vdash (u/x)vw : S(D)}\ W$$

    In any case, by Lemma 5.4, $\Gamma \vdash (u/x)vw : B$. Notice that $(u/x)vw = (u/x)(vw)$.
  - Let x appear in w and not in v. Then $x : A \in \Gamma_2$ and so, by the induction hypothesis, in case 1, $\Gamma_2 \setminus \{x : A\} \vdash (u/x)w : C$, in case 2, $\Gamma_2 \setminus \{x : A\} \vdash (u/x)w : S(C)$. Hence, one of the following derivations is valid:

$$\dfrac{\dfrac{\Gamma_1 \vdash v : C \Rightarrow D \quad \Gamma_2 \setminus \{x : A\} \vdash (u/x)w : C}{\Gamma_1, \Gamma_2 \setminus \{x : A\} \vdash v(u/x)w : D}\ \Rightarrow_E}{\Gamma \vdash v(u/x)w : D}\ W$$

or

$$\dfrac{\dfrac{\Gamma_1 \vdash v : S(C \Rightarrow D) \quad \Gamma_2 \setminus \{x : A\} \vdash (u/x)w : S(C)}{\Gamma_1, \Gamma_2 \setminus \{x : A\} \vdash v(u/x)w : S(D)}\ \Rightarrow_{ES}}{\Gamma \vdash v(u/x)w : S(D)}\ W$$

    In any case, by Lemma 5.4, $\Gamma \vdash v(u/x)w : B$. Notice that $v(u/x)w = (u/x)(vw)$.

- $A \in \mathcal{B}$ and $u \in \mathcal{B}^c$. Then if x appears free exactly once in vw, this case is analogous to the previous item. Let $x \in FV(v) \cap FV(w)$. Then, by the induction hypothesis, in case 1, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D$ and $\Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : C$, in case 2, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D)$ and $\Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : S(C)$. Therefore, one of the following derivations are valid:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \Rightarrow D \quad \Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : C}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u'/x)w : D} \Rightarrow_E}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u/x)w : D} C}{\Gamma \vdash (u/x)v(u/x)w : D} W$$

or

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C \Rightarrow D) \quad \Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : S(C)}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u'/x)w : S(D)} \Rightarrow_{ES}}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v(u/x)w : S(D)} C}{\Gamma \vdash (u/x)v(u/x)w : S(D)} W$$

  In any case, by Lemma 5.4, $\Gamma \vdash (u/x)v(u/x)w : B$. Notice that $(u/x)v(u/x)w = (u/x)(vw)$.

$t = (v + w)$. By Lemma 5.5, $\Gamma_1 \vdash v : S(C)$ and $\Gamma_2 \vdash w : S(C)$ with $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma \cup \{x : A\}$, $(|\Gamma| \cup \{A\}) \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$ and $S(C) \preceq B$. We consider two cases:

- $A \notin \mathcal{B}$, then, by Lemma 5.6, x appears free exactly once in $(v + w)$. Cases:
  - Let x appear in v and not in w. Then $x : A \in \Gamma_1$ and so, by the induction hypothesis, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C)$. Hence,

$$\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C) \quad \Gamma_2 \vdash w : S(C)}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \vdash ((u/x)v + w) : S(C)} \wedge_I}{\Gamma \vdash ((u/x)v + w) : S(C)} W$$

    Then, by Lemma 5.4, $\Gamma \vdash ((u/x)v+w) : B$. Notice that $((u/x)v+w) = (u/x)(v+w)$.
  - Let x appear in w and not in v. Analogous to previous item.
- $A \in \mathcal{B}$, then if x appears free exactly once in $(v + w)$, this case is analogous to the previous item. Let $x \in FV(v) \cap FV(w)$. Then, by the induction hypothesis, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C)$ and $\Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : S(C)$. Therefore,

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : S(C) \quad \Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : S(C)}{\Gamma_1 \setminus \{x : A\}\Gamma_2 \setminus \{x : A\} \vdash ((u/x)v + (u'/x)w) : S(C)} \wedge_I}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash ((u/x)v + (u/x)w) : S(C)} C}{\Gamma \vdash ((u/x)v + (u/x)w) : S(C)} W$$

  Then, by Lemma 5.4, $\Gamma \vdash ((u/x)v + (u/x)w) : B$. Notice that $((u/x)v + (u/x)w) = (u/x)(v + w)$.

$t = \pi_C v$. By Lemma 5.5 $\Gamma' \vdash v : S(C)$ with $\Gamma' \subseteq \Gamma \cup \{x : A\}$, $(|\Gamma| \cup \{A\}) \setminus |\Gamma'| \subseteq \mathcal{B}$ and $C \preceq B$. Notice that if $x \in FV(v)$, then $x : A \in \Gamma'$, but if $x \notin FV(v)$, then $A \subseteq \mathcal{B}$, hence by rule W we can derive $\Gamma', x : A \vdash v : S(C)$. Hence, without lost of generality, consider $x \in FV(v)$. Then, by the induction hypothesis, $\Gamma' \setminus \{x : A\} \vdash (u/x)v : S(C)$. Hence,

$$\dfrac{\dfrac{\Gamma' \setminus \{x : A\} \vdash (u/x)v : S(C)}{\Gamma' \setminus \{x : A\} \vdash \pi_C((u/x)v) : C} \wedge_E}{\Gamma \vdash \pi_C((u/x)v) : C} W$$

By Lemma 5.4, $\Gamma \vdash \pi_C((u/x)v) : B$. Notice that $\pi_C((u/x)v) = (u/x)(\pi_C v)$.

$t = |0\rangle$. By Lemma 5.5, $\mathbb{Q} \preceq B$ and $|\Gamma, x : A| \subseteq \mathcal{B}$. Therefore,

$$\dfrac{\dfrac{}{\vdash |0\rangle : \mathbb{Q}}\ Ax_{|0\rangle}}{\Gamma \vdash |0\rangle : \mathbb{Q}}\ W$$

Then, by Lemma 5.4, $\Gamma \vdash |0\rangle : B$. Notice that $|0\rangle = (u/x)\,|0\rangle$.

$t = |1\rangle$. Analogous to previous item.

$t = ?\cdot$. By Lemma 5.5, $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq B$ and $|\Gamma, x : A| \subseteq \mathcal{B}$. Therefore,

$$\dfrac{\dfrac{}{\vdash ?\cdot : \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q}}\ \mathit{If}}{\Gamma \vdash ?\cdot : \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q}}\ W$$

Then, by Lemma 5.4, $\Gamma \vdash ?\cdot : B$. Notice that $?\cdot = (u/x)(?\cdot)$.

$t = 0$. By Lemma 5.5, there exists $C \notin \mathcal{B}$ such that $C \preceq B$, and $|\Gamma, x : A| \subseteq \mathcal{B}$. Therefore,

$$\dfrac{\dfrac{}{\vdash 0 : C}\ Ax_0}{\Gamma \vdash 0 : C}\ W$$

Then, by Lemma 5.4, $\Gamma \vdash 0 : B$. Notice that $0 = (u/x)0$.

$t = -v$. By Lemma 5.5, $\Gamma' \vdash v : C$, with $\Gamma' \subseteq \Gamma, x : A$, $(|\Gamma| \cup \{A\}) \setminus |\Gamma'| \subseteq \mathcal{B}$ and $S(C) \preceq B$. Notice that if $x \in FV(v)$, then $x : A \in \Gamma'$, but if $x \notin FV(v)$, then $A \subseteq \mathcal{B}$, hence, by rule W, we can derive $\Gamma', x : A \vdash v : C$. Hence, without lost of generality, consider $x \in FV(v)$. Then, by the induction hypothesis, $\Gamma' \setminus \{x : A\} \vdash (u/x)v : C$. Hence,

$$\dfrac{\dfrac{\Gamma' \setminus \{x : A\} \vdash (u/x)v : C}{\Gamma' \setminus \{x : A\} \vdash -(u/x)v : S(C)}\ N}{\Gamma \vdash -(u/x)v : S(C)}\ W$$

The, by Lemma 5.4, $\Gamma \vdash -(u/x)v : B$. Notice that $-(u/x)v = (u/x)(-v)$.

$t = v \otimes w$. By Lemma 5.5, $\Gamma_1 \vdash v : C$ and $\Gamma_2 \vdash w : D$ with $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma \cup \{x : A\}$, $(|\Gamma| \cup \{A\}) \setminus |\Gamma_1 \cup \Gamma_2| \subseteq \mathcal{B}$ and $C \otimes D \preceq B$. We consider two cases:

- $A \notin \mathcal{B}$, then, by Lemma 5.6, x appears free exactly once in $v \otimes w$. Cases:
  - Let x appear in v and not in w. Then $x : A \in \Gamma_1$ and so, by the induction hypothesis, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C$. Hence,

    $$\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \quad \Gamma_2 \vdash w : D}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \vdash (u/x)v \otimes w : C \otimes D}\ \otimes_I}{\Gamma \vdash (u/x)v \otimes w : C \otimes D}\ W$$

    Then, by Lemma 5.4, $\Gamma \vdash (u/x)v \otimes w : B$. Notice that $(u/x)v \otimes w = (u/x)(v \otimes w)$.
  - Let x appear in w and not in v. Analogous to previous item.
- $A \in \mathcal{B}$, then if x appears free exactly once in $v \otimes w$, this case is analogous to the previous item. Let $x \in FV(v) \cap FV(w)$. Then, by the induction hypothesis, $\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C$ and $\Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : D$. Therefore,

  $$\dfrac{\dfrac{\dfrac{\Gamma_1 \setminus \{x : A\} \vdash (u/x)v : C \quad \Gamma_2 \setminus \{x : A\} \vdash (u'/x)w : D}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v \otimes (u'/x)w : C \otimes D}\ \otimes_I}{\Gamma_1 \setminus \{x : A\}, \Gamma_2 \setminus \{x : A\} \vdash (u/x)v \otimes (u/x)w : C \otimes D}\ C}{\Gamma \vdash (u/x)v \otimes (u/x)w : C \otimes D}\ W$$

  Then, by Lemma 5.4, $\Gamma \vdash (u/x)v \otimes (u/x)w : B$. Notice that $(u/x)v \otimes (u/x)w = (u/x)(v \otimes w)$.

$t = \mathit{fst}\ v$. By Lemma 5.5 $\Gamma' \vdash v : C \otimes D$ with $\Gamma' \subseteq \Gamma \cup \{x : A\}$, $(|\Gamma| \cup \{A\}) \setminus |\Gamma'| \subseteq \mathcal{B}$ and $C \preceq B$. Notice that if $x \in FV(v)$, then $x : A \in \Gamma'$, but if $x \notin FV(v)$, then $A \subseteq \mathcal{B}$, hence by rule W we can derive $\Gamma', x : A \vdash v : C \otimes D$. Hence, without lost of generality, consider $x \in FV(v)$. Then, by the induction hypothesis, $\Gamma' \setminus \{x : A\} \vdash (u/x)v : C \otimes D$. Hence,

$$\dfrac{\dfrac{\Gamma' \setminus \{x : A\} \vdash (u/x)v : C \otimes D}{\Gamma' \setminus \{x : A\} \vdash \mathit{fst}\ ((u/x)v) : C}\ \otimes_{E_r}}{\Gamma \vdash \mathit{fst}\ ((u/x)v) : C}\ W$$

By Lemma 5.4, $\Gamma \vdash \mathit{fst}\ ((u/x)v) : B$. Notice that $\mathit{fst}\ ((u/x)v) = (u/x)(\mathit{fst}\ v)$.

$t = \mathit{snd}\ v$. Analogous to previous item. ◀

Proof of Theorem 5.9 (Subject reduction on closed terms). We proceed by induction on the rewrite relation.

(β) Let $\vdash (\lambda x : A\ t)u : B$, with $u \in \mathcal{V}^c$. Then by Lemma 5.5, $\vdash \lambda x : A\ t : C \Rightarrow B$ and $\vdash u : C$. So, by Lemma 5.5 again, $C = A$ and $x : A \vdash t : B$. Therefore, by Lemma 5.8, $\vdash (u/x)t : B$.

($\mathsf{lin_r}$) Let $\vdash t(u + v) : A$, with $\vdash t : B_0 \Rightarrow B$. Then, by Lemma 5.5, one of the following cases happens:

1. $\vdash t : C \Rightarrow D$ and $\vdash (u + v) : C$, with $D \preceq A$.

2. $\vdash t : S(C \Rightarrow D)$ and $\vdash (u + v) : S(C)$, with $S(D) \preceq A$.

Then, by Lemma 5.5 again, $\vdash u : S(E)$ and $\vdash v : S(E)$, with $S(E) \preceq C$, in case 1, or $S(E) \preceq S(C)$, in case 2. However, by Lemma 5.2, case 1 is impossible and so only case 2 remains. By Lemma 5.4, $\vdash u : S(C)$ and $\vdash v : S(C)$. Hence,

$$\dfrac{\dfrac{\vdash t : S(C \Rightarrow D) \quad \vdash u : S(C)}{\vdash tu : S(D)} \qquad \dfrac{\vdash t : S(C \Rightarrow D) \quad \vdash v : S(C)}{\vdash tv : S(D)}}{\vdash (tu + tv) \vdash S(D)}$$

We conclude by Lemma 5.4.

($\mathsf{lin_l}$) Let $\vdash (t + u)v : A$. Then by Lemma 5.5, one of the following cases happens:

1. $\vdash (t + u) : B \Rightarrow C$ and $\vdash v : B$, with $C \preceq A$.

2. $\vdash (t + u) : S(B \Rightarrow C)$ and $\vdash v : S(B)$, with $S(C) \preceq A$.

Then, by Lemma 5.5 again, $\vdash t : S(D)$ and $\vdash u : S(D)$, with $S(D) \preceq B \Rightarrow C$, in case 1, or $S(D) \preceq S(B \Rightarrow C)$, in case 2. However, by Lemma 5.2, case 1 is imposible and so only case 2 remains. By Lemma 5.4, $\vdash t : S(B \Rightarrow C)$ and $\vdash u : S(B \Rightarrow C)$. Hence,

$$\dfrac{\dfrac{\overline{\vdash t : S(B \Rightarrow C)} \quad \overline{\vdash v : S(B)}}{\vdash tv : S(C)} \qquad \dfrac{\overline{\vdash u : S(B \Rightarrow C)} \quad \overline{\vdash v : S(B)}}{\vdash uv : S(C)}}{\vdash (tv + uv) : S(C)}$$

We conclude by Lemma 5.4.

($\mathsf{proj_n}$) Let $\vdash \pi_B(t + u) : A$, where t has type B. Then t is closed, and so $\vdash t : B$. By Lemma 5.5, $B \preceq A$, so, by Lemma 5.4, $\vdash t : A$.

($\mathsf{proj_1}$) Let $\vdash \pi_B t : A$, where t has type B. Then t is closed, and so $\vdash t : B$. By Lemma 5.5, $B \preceq A$, so by Lemma 5.4, $\vdash t : A$.

($\mathsf{if_1}$) Let $\vdash |1\rangle?u{\cdot}v : A$. Then, by Lemma 5.5,

1. a. $(\vdash |1\rangle?u{\cdot} : B_0 \Rightarrow C$ and $\vdash v : B_0)$
with $C \preceq A$, or

b. ($\vdash |1\rangle?u\cdot : S(B \Rightarrow C)$ and $\vdash v : S(B)$)
with $S(C) \preceq A$.

2. a. ($\vdash |1\rangle?\cdot : D_0 \Rightarrow E$ and $\vdash u : D_0$) with
$E \preceq B_0 \Rightarrow C$, in case 1a, or
$E \preceq S(B \Rightarrow C)$, in case 1b, or

b. ($\vdash |1\rangle?\cdot : S(D \Rightarrow E)$ and $\vdash u : S(D)$) with
$S(E) \preceq S(B \Rightarrow C)$, in case 1b.
(Notice that case 1a would imply $S(E) \preceq B_0 \Rightarrow C$, which is impossible by Lemma 5.2).

3. a. ($\vdash ?\cdot : F_0 \Rightarrow G$ and $\vdash |1\rangle : F_0$) with
$G \preceq D_0 \Rightarrow E$, in case 2a, or
$G \preceq S(D \Rightarrow E)$, in case 2b, or

b. ($\vdash ?\cdot : S(F \Rightarrow G)$ and $\vdash |1\rangle : S(F)$) with
$S(G) \preceq S(D \Rightarrow E)$, in case 2b.
(Notice that case 2a would imply $S(G) \preceq D_0 \Rightarrow E$, which is impossible by Lemma 5.2).

4. a. $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq F_0 \Rightarrow G$ and $\mathbb{Q} \preceq F_0$, in case 3a, or

b. $\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \preceq S(F \Rightarrow G)$ and $\mathbb{Q} \preceq S(F)$, in case 3b.

There are 6 possible paths to follow. We analize only two paths, any other path is analogous.

- Following the path 4a – 3a – 2a – 1a, we have

$$\begin{aligned}\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} &\preceq F_0 \Rightarrow G\\ &\preceq F_0 \Rightarrow D_0 \Rightarrow E\\ &\preceq F_0 \Rightarrow D_0 \Rightarrow B_0 \Rightarrow C\\ &\preceq F_0 \Rightarrow D_0 \Rightarrow B_0 \Rightarrow A\end{aligned}$$

  Hence, $\mathbb{Q} \preceq D_0$ and $\mathbb{Q} \preceq A$. So, by Lemma 5.2, $\mathbb{Q} = D_0$ and so $D_0 \preceq A$. Therefore, from case 2a and Lemma 5.4, we have $\vdash u : A$.

- Following the path 4a – 3a – 2b – 1b, we have

$$\begin{aligned}\mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} \Rightarrow \mathbb{Q} &\preceq F_0 \Rightarrow G\\ &\preceq F_0 \Rightarrow S(D \Rightarrow E)\\ \text{(by Lemma 5.3)} &\preceq F_0 \Rightarrow S(D) \Rightarrow S(E)\\ &\preceq F_0 \Rightarrow S(D) \Rightarrow S(B \Rightarrow C)\\ \text{(by Lemma 5.3)} &\preceq F_0 \Rightarrow S(D) \Rightarrow S(B) \Rightarrow S(C)\\ &\preceq F_0 \Rightarrow S(D) \Rightarrow S(B) \Rightarrow A\end{aligned}$$

  Since $S(C) \preceq A$, by Lemma 5.2, either $A \notin \mathcal{B}$, or $A = S(C)$. In any case, $A = S(\mathbb{Q})$, same as $S(D) = S(\mathbb{Q})$. Therefore, from case 2b, we have $\vdash u : A$.

(if_0) Analogous to case (if_1).

($\mathsf{lineg}_\mathsf{r}$) Let $\vdash t(-u) : A$, with $\vdash t : B_0 \Rightarrow B$. Then, by Lemma 5.5, one of the following cases happens:

1. $\vdash t : C \Rightarrow D$ and $\vdash -u : C$, with $D \preceq A$.

2. $\vdash t : S(C \Rightarrow D)$ and $\vdash -u : S(C)$, with $S(D) \preceq A$.

Then, by Lemma 5.5 again, $\vdash u : E$, with $S(E) \prec C$, in case 1, or $S(E) \prec S(C)$ in case 2. However, since $\vdash t : B_0 \Rightarrow B$, by Lemma 5.2, case 1 is impossible and so only case 2 remains. Since $E \preceq S(E) \preceq S(C)$, by Lemma 5.4, $\vdash u : S(C)$. Hence,

$$\dfrac{\dfrac{\vdash t : S(C \Rightarrow D) \quad \vdash u : S(C)}{\vdash tu : S(D)}}{\vdash -tu : S(S(D))}$$

We conclude by Lemma 5.4.

(lineg$_\mathsf{l}$) Let $\vdash (-t)u : A$. Then, by Lemma 5.5, one of the following cases happens:

1. $\vdash -t : B \Rightarrow C$ and $\vdash u : B$, with $C \preceq A$.

2. $\vdash -t : S(B \Rightarrow C)$ and $\vdash u : S(B)$, with $S(C) \preceq A$.

Then, by Lemma 5.5 again, $\vdash t : D$, with $S(D) \prec B \Rightarrow C$, in case 1, or $S(D) \prec S(B \Rightarrow C)$ in case 2. However by Lemma 5.2, case 1 is impossible and so only case 2 remains. Since $D \preceq S(D) \preceq S(B \Rightarrow C)$, by Lemma 5.4, $\vdash t : S(B \Rightarrow C)$. Hence,

$$\dfrac{\dfrac{\dfrac{\vdash t : S(B \Rightarrow C)}{\vdash -t : S(S(B \Rightarrow C))}}{\vdash -t : S(B \Rightarrow C)} \quad \vdash u : S(B)}{\vdash (-t)u : S(C)}$$

We conclude by Lemma 5.4.

(neutral) Let $\vdash (0 + t) : A$. Then, by Lemma 5.5, $\vdash t : S(B)$, with $S(B) \preceq A$. Then, by Lemma 5.4, $\vdash t : A$.

(inverse) Let $\vdash t - t : A$. Then, by Lemma 5.5, $\vdash t : S(B)$ and $\vdash -t : S(B)$, with $S(B) \preceq A$. We conclude by rule Ax_0 and Lemma 5.4.

(negneg) Let $\vdash -(-t) : A$. Then, by Lemma 5.5, $\vdash -t : B$ with $S(B) \preceq A$. Then, by Lemma 5.5 again, $\vdash t : C$, with $S(C) \preceq B$. Then $C \preceq S(C) \preceq B \preceq S(B) \preceq A$, hence we conclude by Lemma 5.4.

(negzero) Let $\vdash -0 : A$. Then, by Lemma 5.5, $\vdash 0 : B$ with $S(B) \preceq A$. We conclude by Lemma 5.4.

(nosup) Let $\vdash (t + t) : A$. Then, by Lemma 5.5, $\vdash t : S(B)$, with $S(B) \preceq A$. We conclude by Lemma 5.4.

(lin0$_\mathsf{r}$) Let $\vdash t0 : A$, with $\vdash t : B_0 \Rightarrow B$. Then, by Lemma 5.5, one of the following cases happens:

1. $\vdash t : C \Rightarrow D$ and $\vdash 0 : C$, with $D \preceq A$, or

2. $\vdash t : S(C \Rightarrow D)$ and $\vdash 0 : S(C)$, with $S(D) \preceq A$.

Case 1, by Lemma 5.5, implies there exists $E \notin \mathcal{B}$ such that $E \preceq C$, however since $\vdash t : B_0 \Rightarrow B$, it is impossible as shown by Lemma 5.2. Hence, only case 2 remains. By rule Ax_0, $\vdash 0 : S(D)$, hence we conclude by Lemma 5.4.

(lin0$_\mathsf{l}$) Let $\vdash 0t : A$. Then, by Lemma 5.5, one of the following cases happens:

1. $\vdash 0 : C \Rightarrow D$ and $\vdash t : C$, with $D \preceq A$, or

2. $\vdash 0 : S(C \Rightarrow D)$ and $\vdash t : S(C)$, with $S(D) \preceq A$.

Case 1, by Lemma 5.5, implies there exists $E \notin \mathcal{B}$ such that $E \preceq C \Rightarrow D$, however it is impossible as shown by Lemma 5.2. Hence, only case 2 remains. By rule Ax_0, $\vdash 0 : S(D)$, hence we conclude by Lemma 5.4.

(fst) Let $\vdash \mathit{fst}(t \otimes u) : A$. Hence, by Lemma 5.5, $\vdash t \otimes u : B \otimes C$, with $B \preceq A$. Then, by Lemma 5.5 again, $\vdash t : D$ and $\vdash u : E$, with $D \otimes E \preceq B \otimes C$. Hence, $D \preceq B$, and so we conclude by Lemma 5.4.

(snd) Analogous to case (fst).

(abs$_\mathsf{r}$) Let $\vdash t \otimes 0 : A$. Then, by Lemma 5.5, $\vdash t : B$ and $\vdash 0 : C$, with $B \otimes C \preceq A$. Then, by Lemma 5.5 again, there exists $D \notin \mathcal{B}$ such that $D \preceq C$. Then, by Lemma 5.2, $C \notin \mathcal{B}$, and so $B \otimes C \notin \mathcal{B}$. We conclude by rule Ax_0 and Lemma 5.4.

(abs$_\mathsf{l}$) Analogous to case (abs$_\mathsf{r}$).

(linten$_\mathsf{r}$) Let $\vdash t((r+s)\otimes u) : A$. Then, by Lemma 5.5, one of the following cases happens:

1. $\vdash t : B \Rightarrow C$ and $\vdash (r+s)\otimes u : B$, with $C \preceq A$, or

2. $\vdash t : S(B \Rightarrow C)$ and $\vdash (r+s)\otimes u : S(B)$, with $S(C) \preceq A$.

Then, by Lemma 5.5 again, we have $\vdash (r+s) : D$ and $\vdash u : E$, with $D \otimes E \preceq B_0$, in case 1, or $D \otimes E \preceq S(B)$ in case 2. Then, again using Lemma 5.5, we have $\vdash r : S(F)$ and $\vdash s : S(F)$, with $S(F) \preceq D$. Hence, by Lemma 5.2 we discard case 1. So, by Lemma 5.4, $\vdash r : D$, then

$$\frac{\vdash r : D \qquad \vdash u : E}{\vdash r \otimes u : D \otimes E}$$

So, by Lemma 5.4, $\vdash r \otimes u : S(B)$, then

$$\frac{\vdash t : S(B \Rightarrow C) \qquad \vdash r \otimes u : S(B)}{\vdash t(r \otimes u) : S(C)}$$

Analogously, $\vdash t(s \otimes u) : S(C)$, so

$$\frac{\vdash t(r \otimes u) : S(C) \qquad \vdash t(s \otimes u) : S(C)}{\vdash (t(r \otimes u) + t(s \otimes u) : S(C))}$$

We conclude with Lemma 5.4.

(linten$_\mathsf{l}$) Analogous to case (linten$_\mathsf{r}$).

(comm) Let $\vdash (u+v) : A$. Then, by Lemma 5.5, $\vdash u : S(B)$ and $\vdash v : S(B)$, with $S(B) \preceq A$. So,

$$\frac{\vdash v : S(B) \qquad \vdash u : S(B)}{\vdash (v+u) : S(B)}$$

We conclude with Lemma 5.4.

(assoc) Let $\vdash ((u+v)+w) : A$. Then, by Lemma 5.5, $\vdash (u+v) : S(B)$ and $\vdash w : S(B)$, with $S(B) \preceq A$. Then, by Lemma 5.5 again, $\vdash u : S(C)$ and $\vdash v : S(C)$, with $S(C) \preceq S(B)$. So, by Lemma 5.4, we have $\vdash u : S(B)$ and $\vdash v : S(B)$. Hence,

$$\frac{\vdash u : S(B) \qquad \dfrac{\vdash v : S(B) \quad \vdash w : S(B)}{\vdash (v+w) : S(B)}}{\vdash (u+(v+w)) : S(B)}$$

We conclude with Lemma 5.4.

Contextual rules Contextual cases are straightfoward, and hence omited. ◀