The directed homotopy hypothesis

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Abstract

The homotopy hypothesis was originally stated by Grothendieck [13] : topological spaces should be "equivalent" to (weak) ∞-groupoids, which give algebraic representatives of homotopy types. Much later, several authors developed geometrizations of computational models, e.g. for rewriting, distributed systems, (homotopy) type theory etc. But an essential feature in the work set up in concurrency theory, is that time should be considered irreversible, giving rise to the field of directed algebraic topology. Following the path proposed by Porter, we state here a directed homotopy hypothesis: Grandis' directed topological spaces should be "equivalent" to a weak form of topologically enriched categories, still very close to $(\infty,1)$-categories. We develop, as in ordinary algebraic topology, a directed homotopy equivalence and a weak equivalence, and show invariance of a form of directed homology.

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1 Introduction

A central motivation in algebraic topology is to be able, through algebraic calculations, to classify topological spaces, up to homeomorphisms, or up to some shape equivalence (homotopy equivalence, or equivalence under "elastic deformations"). Similarly, computer science abounds with notions of equivalences, first and foremost, bisimulation equivalences between concurrent processes [20]. It has been observed that some of these equivalences are very geometric in nature: Pratt [23] and van Glabbeek [27], followed by many, among which [9, 12], advocated for cubical models and topological models of the execution space of concurrent systems.

Topological spaces naturally give rise to a higher dimensional category: paths, homotopies, higher homotopies provide the cells of such a structure. Moreover, by nature those cells are invertible up to higher cells: for example, paths are invertible up to homotopy, etc. Hence it is natural to ask the following question, which has become known as the homotopy hypothesis question [13]: "to what extent is the structure of spaces mirrored by that of ∞-groupoids ?" A modern answer to this question uses the language of Quillen's model structure: weak homotopy types are modeled by structures that can be interpreted as being ∞-groupoids.

The equivalences of interest in computer science are *directed*, instead: there is a direction of time, and it must be preserved by deformation. Porter [21, 22] proposed to look at a directed analogue of the homotopy hypothesis. Directed spaces also give a natural structure of higher categories (dipaths, dihomotopies, higher dihomotopies), but dipaths are not meant to be invertible up to dihomotopy. The directed homotopy hypothesis should correspond to some question of the form **"To what extent are directed spaces the same as $(\infty, 1)$-categories ?"** and should have an answer in the language of model categories as well, by comparing directed spaces and simplicial categories, a model of $(\infty, 1)$-categories.

Our goal is to give an answer to that question. We review Porter's approach in Section 3, and show some of its limitations. We then design a proposal of directed homotopy equivalence based on directed deformation retracts along inessential dipaths (Section 4). Finally, we reformulate the directed homotopy hypothesis, fixing the limitations of Porter' proposal, by designing weak equivalences for a weak version of enriched categories, based on partial enrichment and directed components, for which we prove adequacy with respect to dihomotopy equivalence (Section 5) — our main result.

2 A lexicon of equivalences

There will be many notions of equivalences in this paper and it will be important that they are not mixed up, so here is a brief lexicon:

- **weak equivalence:** the generic name for an element of the distinguished class of morphisms of a model category [24] which are meant to be turned into isomorphisms by the categorical process of localization.
- **homotopy equivalence:** invertible continuous function up to homotopy. They are the weak equivalences in the Strøm model structure on topological spaces [26].
- **weak homotopy equivalence:** continuous function that induces isomorphisms of homotopy groups in every dimensions [14]. A homotopy equivalence is a weak homotopy equivalence. They are the weak equivalences in the Quillen-Serre model structure on topological spaces. All those notions are *un*directed.
- **naive dihomotopy equivalence:** invertible dmaps (see Section 3) up to dihomotopy.
- **dihomotopy equivalence:** to be defined in Section 4.2. They will be our generalization of homotopy equivalence in the directed case, and should not be confused with other notions of (almost) the same name (e.g. [25, 12]).
- **weak equivalence of partially enriched categories:** to be defined in Section 5.3. They will be a modification of weak equivalences of enriched categories [2].
- **weak dihomotopy equivalence:** to be defined in Section 5.3. They will be our modification of Porter's proposal (see Section 2), based on weak equivalences of partially enriched categories.
- **strong equivalence of partially enriched categories:** to be defined in Section 5.4. They will be a generalization of equivalence of categories (but not of equivalence of enriched categories).

3 From homotopy hypothesis to Porter's directed homotopy hypothesis

Topological spaces naturally yield a structure of ∞-groupoids i.e. a structure with 0-cells or objects, 1-cells or morphisms between 0-cells, 2-cells or morphisms between 1-cells, and so on, such that every n-cell is invertible up to $n+1$-cells for $n \geq 1$. From a topological space X, we can construct an ∞-groupoid by taking as 0-cells, the points of X, 1-cells are the paths, i.e., continuous functions from the unit segment $I = [0, 1]$ to X, 2-cells are the homotopies between paths, i.e. continuous functions $H : I \times I \longrightarrow X$ such that $H(\_, 0)$ and $H(\_, 1)$ are constant maps, ..., higher cells are the higher homotopies. This is a ∞-groupoid since n-homotopies are invertible up to $n+1$-homotopies for all n. For example, a path γ is invertible up to homotopy since, if we note γ^{-1} the path $t \mapsto \gamma(1-t)$ and $\star$ the concatenation of paths, $\gamma \star \gamma^{-1}$ and $\gamma^{-1} \star \gamma$ are homotopic to constant paths, i.e., there is a homotopy H such that $H(0,\_) = \gamma \star \gamma^{-1}$ and $H(1,\_)$ is constant (equal to $\gamma(0)$), and similarly for $\gamma^{-1} \star \gamma$.

There are many ways to model (in the sense of model categories) ∞-groupoids. One of the simplest ones is Kan complexes, i.e. simplicial sets K for which "every horn has a filler". A horn is simply a simplicial map from $\Delta_i[n]$, the union of the faces of the standard n-simplex $\Delta[n]$, except the i-th one, to K. Having a filler means that this map extends to a simplicial map from $\Delta[n]$ to K. In the language of model categories, they are precisely the fibrant objects of the Kan-Quillen model structure on simplicial sets.

The singular simplicial complex functor provides a Kan complex from a topological space X, where the n-cells are the continuous maps from the geometric standard n-simplex to X. This functor has a left adjoint, the geometric realization that build a topological space from a simplicial set by glueing simplices together. The homotopy hypothesis can then be formulated as follows [24]: this adjunction is a Quillen-equivalence between the Quillen-Serre model structure on topological spaces (whose weak equivalences are the weak homotopy equivalences) and the Kan-Quillen model structure on simplicial sets (whose weak equivalences are simplicial maps that induce weak homotopy equivalences on the geometric realization). This formulation has many consequences: first, weak homotopy types are modeled by ∞-groupoids; secondly, one can compare topological spaces up to weak homotopy equivalence by comparing Kan complexes.

Based on this, Porter [21, 22] proposed a directed homotopy hypothesis for directed spaces. Let us first recall a few basic notions from directed topology (as in e.g. [12, 9]). A **directed space** (or **dspace** for short), is a topological space X, together with a subset of paths P_X, called the **directed paths** (or **dipaths**), satisfying the following :

- every constant path is in P_X;
- P_X is closed under concatenation;
- P_X is closed under non-decreasing reparametrization, i.e., if $\gamma \in P_X$ and $r : I \longrightarrow I$ is a continuous non-decreasing function, then $\gamma \circ r \in P_X$.

A **dmap** $f : X \longrightarrow Y$ is a continuous function such that for every $\gamma \in P_X$, $f \circ \gamma \in P_Y$. We note $dTop$, the category of dspaces and dmaps. A **dihomotopy of dipaths of** X is a homotopy between paths $H : I \times I \longrightarrow X$ such that for every $t \in I$, $H(t,\_)$ is a dipath. More generally, one can define n-dihomotopies. Contrary to topological spaces, dipaths need not be invertible up to dihomotopy: define $\overrightarrow{I}$ as the dspace I whose dipaths are the non-decreasing paths. The identity function of I is a dipath going from 0 to 1, but there is no dipath from 1 to 0, and so it cannot have an inverse modulo homotopy. Hence if a topological space is to be thought of as being "the same as" an ∞-groupoid, then a dspace, whose dipaths are not invertible up to dihomotopies, should be the same as an $(\infty,1)$-category, namely, a ∞-groupoid whose 1-cells are not required to be invertible up to 2-cells.

Much as ∞-groupoids, there are many ways to model $(\infty,1)$-categories. Two are really close to Kan complexes: quasi-categories [16] (weak Kan complexes in the sense that only "inner horns", i.e. i-horns for $i \neq 0$ and $i \neq n$ are required to have fillers) and enriched categories over Kan complexes [2]. Porter proposed to follow Quillen's program, by using the latter. Given a dspace, one can construct [22] the following simplicial category $\mathbb{T}(X)$ (which is actually enriched in Kan complexes), called the **trace category**:

- its objects are the points of X;
- for every pair of points (x,y), the simplicial set $\mathbb{T}(X)(x,y)$ is the singular simplicial

complex of the trace space $\overrightarrow{T}(X)(x,y)$, which is the space of dipaths up to non-decreasing reparametrizations with the quotient topology of the compact-open topology [8].

One can then compare dspaces through the lens of the model structure of simplicial categories [22]. The weak equivalences are those simplicially enriched functors $F : C \longrightarrow D$ such that:

- for every pair of objects (c,c') of C, the simplicial map $F_{c,c'} : C(c,c') \longrightarrow D(F(c),F(c'))$ induces a weak homotopy equivalence between geometric realizations (i.e., is a weak equivalence in the Kan-Quillen model structure);
- F induces an equivalence of categories $\pi_0(F) : \pi_0(C) \longrightarrow \pi_0(D)$ where $\pi_0(C)$, the category of components of C, is obtained from C by replacing $C(c,c')$ by the set of 0-cells of $C(c,c')$ quotient by the 1-cells (or equivalently, the set of path-connected components of the geometric realization of $C(c,c')$).

Even if this program seems natural and foreshadows the existence of a model structure for dspaces (which is a very enticing perspective), this method has its own limitations. Let us consider again the directed segment $\overrightarrow{I}$. In every obvious (weak) directed homotopy equivalence (see also later), $\overrightarrow{I}$ should be equivalent to a point space $*$. So, we expect that they will have the same trace category up to weak equivalence, which is not the case:

- $\mathbb{T}(\overrightarrow{I})(1,0)$ is empty while $\mathbb{T}(*)(*,*)$ is not and so cannot be weakly equivalent (in the Kan-Quillen model structure);
- $\pi_0(\mathbb{T}(\overrightarrow{I}))$ is isomorphic to the poset $(I,\leq)$ and so cannot be equivalent to $\pi_0(\mathbb{T}(*))$.

We must therefore better understand the meaning of (weak) directed homotopy type to make the whole dihomotopy hypothesis programme work fine.

4 Directed homotopy equivalences

4.1 A naive directed homotopy equivalence

There are numerous proposals for directed homotopy equivalences [12, 25, 18, 10, 5], but none has yet gained unequivocal approval in the community. The simplest is the textual generalization of the classical definition of homotopy equivalence in algebraic topology. A **homotopy** is a continuous function $H : I \times X \longrightarrow Y$. We then say that two continuous functions $f,g : X \longrightarrow Y$ are **homotopic** [14], if there is a homotopy H such that $H(0,\_) = f$ and $H(1,\_) = g$. This is an equivalence relation, compatible with composition. We then write $HoTop$ for the category of topological spaces and homotopy classes of continuous functions. We call **homotopy equivalence** a continuous function $f : X \longrightarrow Y$ whose homotopy class is an isomorphism in $HoTop$, i.e., such that there is a continuous function $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are homotopic to identities. We say that two spaces are **homotopy equivalent** if there is a homotopy equivalence between them.

Similarly, a ***di*homotopy** [9] is a continuous function $H : I \times X \longrightarrow Y$, *such that for every $t \in I$, $H(t,\_)$ is a dmap.* We say that two *dmaps* $f,g : X \longrightarrow Y$ are ***di*homotopic** if there is a *di*homotopy H such that $H(0,\_) = f$ and $H(1,\_) = g$. This is an equivalence relation, compatible with composition. We call ***naive di*homotopy equivalence** a *dmap* $f : X \longrightarrow Y$ which is invertible up to *di*homotopy, i.e., such that there is a *dmap* $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are *di*homotopic to identities. We say that two *d*spaces are ***naive di*homotopy equivalent** if there is a *naive di*homotopy equivalence between them.

▶ **Example 1.**

1) The following two dspaces come from the geometric semantics of programs with semaphores [9], one process is taking locks on shared objects a and b (P actions) before relinquishing them (V actions), the second process is doing the same, in reverse order on objects :

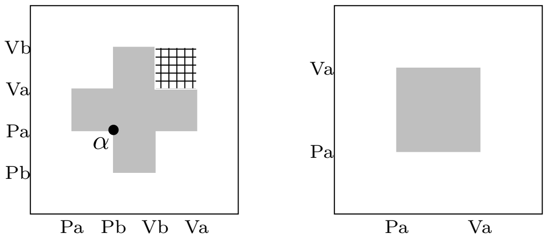

Figure 1 dspaces SF and HS

Those dspaces are subspaces of $\mathbb{R}^2$ whose points are within the white part in the square (the grey part represents the forbidden states of the program) and whose dipaths are non decreasing paths for the componentwise ordering on $\mathbb{R}^2$. They are naive dihomotopy equivalent since there are two maps, depicted in Figure 2, that form a naive dihomotopy equivalence. The points in light grey are the points which do not belong to the image

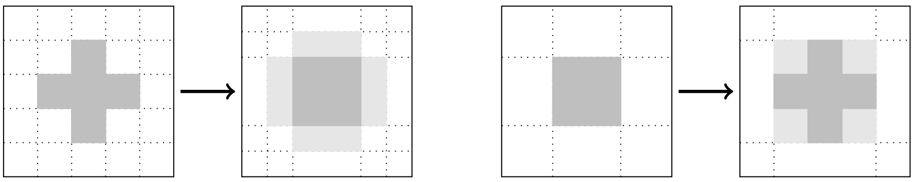

Figure 2 dmap from SF to HS (left) and from HS to SF (right)

of those maps. The problem is that those two programs are quite different: SF has a dead-lock in α and inaccessible states (depicted in Figure 1, as the hatched upper right concavity), while HC does not. Topologically, they do not have the same (directed) components in the sense of [11].

2) Next, let us consider the dspace in Figure 3, which we call the Fahrenberg matchbox [7].

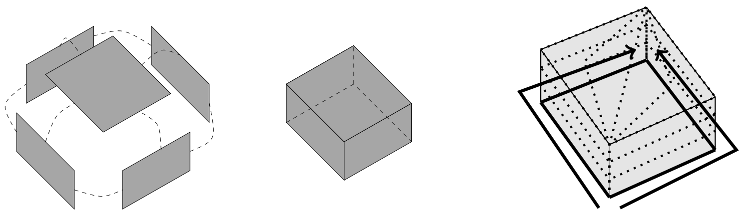

Figure 3 Fahrenberg's matchbox M and two non-dihomotopic dipaths

Geometrically, this is an empty cube without bottom face. Its dipaths are the paths that only go from bottom to top and from front to back. We expect it to be non-dihomotopically equivalent to a point because it has a non-trivial dihomotopy type (although it is contractible in the usual sense). Indeed, consider the two dipaths depicted in Figure 3. They are not dihomotopic because the only way to deform continuously one into the other is to go through the upper face, and one of the intermediate paths (namely, any such path that goes through the topmost face) will fail to be a dipath.

However, M is naive dihomotopy equivalent to its upper face (so to a point), a dihomotopy is depicted in Figure 4. More precisely, the dmap f, which maps any point of M to the point of T just above of it, is a naive dihomotopy equivalence, whose inverse modulo

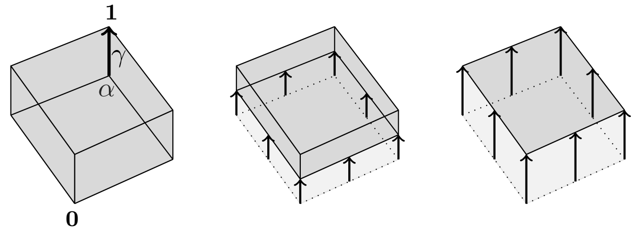

Figure 4 Naive equivalence between the Fahrenberg's matchbox M and its upper face T

dihomotopy is the embedding g of T into M. Hence, $f \circ g = id_T$ and a dihomotopy from id_M to $g \circ f$ is depicted in Figure 4.

4.2 Deformation retracts and dihomotopy equivalence

There is another way to define homotopy equivalence in classical algebraic topology, which will prove better for getting the right definition in directed algebraic topology. A homotopy equivalence can be formalized by the notion of deformation retract [14]. Let X be a space and A be a subspace of X. We say that A is a **deformation retract** of X if and only if there is a homotopy $H : I \times X \longrightarrow X$ such that $H(0, \_) = id_X$, for every $t \in I$ and every $a \in A$, $H(t, a) = a$ and for all $x \in X$, $H(1, x) \in A$. This definition of deformation retract states that the embedding of A into X is a homotopy equivalence with inverse modulo homotopy $H(1, \_)$. In fact, deformation retracts characterize homotopy equivalence in the following sense [14]: two spaces X and Y are homotopy equivalent iff there is a space Z such that X and Y are deformation retracts of Z.

A homotopy being the same as a continuous function from X to $Top(I, Y)$, where $Top(I, Y)$ is the set of paths in Y equipped with the compact-open topology, one can define "directed" deformation retracts as continuous functions from X to P_X, (equipped with the subspace topology) satisfying the same kind of axioms as deformation retracts satisfy. But one must be careful: the dihomotopy depicted in Figure 4 will be a directed deformation retract in this sense. The main problem is that the dipaths along which we deform (i.e., the dipaths in the image of the deformation retracts) will not preserve the fact that two dipaths are not dihomotopic (for example, dipath γ), and more generally the (classical) homotopy type of space of dipaths, while it is the case in the non-directed setting. Hence, some form of "components" as in [11] should underly the definition of directed deformation retract. As a side effect, we will also naturally get to define two notions of deformation retracts, one in the future, one in the past.

In the following, we write $\overrightarrow{\mathbb{P}}(X)(x, y)$ for the set of **dipaths of** X **from** x **to** y, namely dipaths γ of X such that $\gamma(0) = x$ and $\gamma(1) = y$, equipped with the compact-open topology. Imitating [11], we call a **Yoneda system of dipaths of** X any subset Λ of P_X such that:

- Λ is closed under concatenation and dihomotopy;
- for every $\gamma : x \longrightarrow y \in \Lambda$, for every $z \in X$ such that $\overrightarrow{\mathbb{P}}(X)(y, z) \neq \varnothing$, the function $\gamma \star \_ : \overrightarrow{\mathbb{P}}(X)(y, z) \longrightarrow \overrightarrow{\mathbb{P}}(X)(x, z)$, $\delta \longmapsto \gamma \star \delta$ is a homotopy equivalence;
- for every $\gamma : x \longrightarrow y \in \Lambda$, for every $w \in X$ such that $\overrightarrow{\mathbb{P}}(X)(w, x) \neq \varnothing$, the function $\_ \star \gamma : \overrightarrow{\mathbb{P}}(X)(w, x) \longrightarrow \overrightarrow{\mathbb{P}}(X)(w, y)$, $\delta \longmapsto \delta \star \gamma$ is a homotopy equivalence;
- Λ has the right Ore condition modulo dihomotopy, i.e., for every $f : x \longrightarrow y \in \Lambda$ and every dipath $g : z \longrightarrow y$ in X there are $f' : w \longrightarrow z \in \Lambda$ and a dipath $g' : w \longrightarrow x$ in X for some w such that $g' \star f$ and $f' \star g$ are dihomotopic;

$$\begin{tikzcd} w \arrow[r, dotted, "g'"] \arrow[d, dotted, "f' \in \Lambda"'] \arrow[dr, phantom, "\text{mod. dihomot.}"] & x \arrow[d, "f \in \Lambda"] \\ z \arrow[r, "g"'] & y \end{tikzcd}$$

– Λ has the left Ore condition modulo dihomotopy, i.e., for every $f : x \longrightarrow y \in \Lambda$ and every dipath $g : x \longrightarrow z$ in X there are $f' : z \longrightarrow w \in \Lambda$ and a dipath $g' : x \longrightarrow w$ in X for some w such that $g \star f'$ and $f \star g'$ are dihomotopic.

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d, "f \in \Lambda"'] \arrow[dr, phantom, "\text{mod. dihomot.}"] & y \arrow[d, dotted, "f' \in \Lambda"] \\ x \arrow[r, dotted, "g'"'] & w \end{tikzcd}$$

▶ **Lemma 2.** *The set of Yoneda systems of dipaths of X is a complete lattice for inclusion. We note $\mathfrak{I}(X)$ the largest such system and call its elements **inessential dipaths**.*

Let X be a dspace and A be a sub-dspace of X, i.e., a sub-topological space $A \subseteq X$ whose dipaths are the dipaths of X with image in A. We say that A is a **future deformation retract** of X if there is a continuous function $H : X \longrightarrow \mathfrak{I}(X)$ ($\mathfrak{I}(X)$ is equipped with the subspace topology of $\mathbf{Top}(I, X)$) such that:

– for every $x \in X$, $H(x)(0) = x$;
– for every $a \in A$ and $t \in I$, $H(a)(t) = a$;
– for every $x \in X$, $H(x)(1) \in A$;
– for every $t \in I$, the map $H_t : X \longrightarrow X$, $x \longmapsto H(x)(t)$ is a dmap;
– for every dipath δ of A from z to $H_1(x)$ there is a dipath γ of X from y to x with $H_1(y) = z$ and $H_1 \circ \gamma$ and δ are dihomotopic.

We stress here the fact that H must be with values in the inessential dipaths $\mathfrak{I}(X)$. Similarly, we define **past deformation retracts** by switching the role of 1 and 0 in the previous definition. We then say that two dspaces are **directed homotopy equivalent** if there is a zigzag of future and past deformation retracts between them.

▶ **Example 3.**

1) Observe that past deformation retracts (resp. future deformation retracts) between topological spaces (i.e. dpsaces whose set of dipaths contains all paths) coincide with non-directed deformation retracts. In particular, if two topological spaces are homotopically equivalent then they are dihomotopically equivalent. The converse also holds.
2) $\{1\}$ is a future deformation retract of $\overrightarrow{I}$. Indeed, the function $H : \overrightarrow{I} \longrightarrow \mathfrak{I}(\overrightarrow{I})$, $s \longmapsto (t \longmapsto (1-t)s + t)$ satisfies the conditions above. Similarly, $\{0\}$ is a past deformation retract of $\overrightarrow{I}$. More generally, every past face $\overrightarrow{I}^k \times \{0\} \times \overrightarrow{I}^l$ (resp. future face $\overrightarrow{I}^k \times \{1\} \times \overrightarrow{I}^l$) is a past (resp. future) deformation retract of the directed cube $\overrightarrow{I}^{k+l+1}$.
3) Is the deformation depicted in Figure 4 a future deformation retract from M to its upper face T ? The answer is no because it is not with values in $\mathfrak{I}(M)$. Indeed, the dipath γ from α to $\mathbf{1}$ does not induce a homotopy equivalence between spaces of dipaths. The space $\overrightarrow{\mathbb{P}}(M)(\mathbf{0}, \alpha)$ of dipaths from $\mathbf{0}$ to α is homotopically equivalent to a two point space, because there are two such dipaths that are not dihomotopic, while the space $\overrightarrow{\mathbb{P}}(M)(\mathbf{0}, \mathbf{1})$ of dipaths from $\mathbf{0}$ to $\mathbf{1}$ is contractible, so they cannot be homotopically equivalent.

4.3 A homological invariant of dihomotopy equivalence : natural homology

Let us recall the notion of natural homology of [6, 25]. Given a dspace X we form as previously the fundamental category $\overrightarrow{\pi_1}(X)$ whose objects are points of X and whose morphisms from x to y are classes modulo dihomotopy of dipaths from x to y. Then, we take its **category of factorizations**, noted $\mathcal{F}_X$ whose objects are classes modulo dihomotopy of dipaths and whose morphisms from the class $[\gamma]$ with γ a dipath from x to y to the class $[\gamma']$ with γ' a dipath from x' to y' are pairs of classes $([\alpha],[\beta])$ with α from x' to x and β from y to y' such that $[\alpha \star \gamma \star \beta] = [\gamma']$. Composition is concatenation and identities are pairs of classes of constant dipaths. We then define the **natural dipath functor of** X, the functor $\overrightarrow{\mathcal{P}}(X) : \mathcal{F}_X \longrightarrow HoTop$ which maps:

- every class $[\gamma]$ with γ from x to y to $\overrightarrow{\mathbb{P}}(X)(x,y)$;
- every extension $([\alpha],[\beta])$ to the class modulo homotopy of the map $\delta \longmapsto (\alpha \star \delta) \star \beta$.

We can then form the **natural homology of** X by composing with the singular homology functor. The definition of [6] is based on taking the category of factorizations of the trace category, instead of the fundamental category. This gives a "bisimilar" notion, when X is a pospace (the setting of [6]) and will be more convenient to work with here. This notion of bisimilarity of diagrams with values in Abelian groups (or more generally in any fixed category) is fully defined in [6], based on the framework of open maps [17], and is designed for comparing directed homology of pospaces. It goes as follows. The context is that of **small diagrams with values in a category** $\mathcal{M}$ which are functors from any small category $\mathcal{C}$ to the category $\mathcal{M}$. A **morphism of diagrams** from $F : \mathcal{C} \longrightarrow \mathcal{M}$ to $G : \mathcal{D} \longrightarrow \mathcal{M}$ is a pair (Φ, σ) of a functor $\Phi : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\sigma : F \longrightarrow G \circ \Phi$. We note **Diag(**$\mathcal{M}$**)** the category of small diagrams with values in Abelian groups and morphisms of diagrams.

A morphism of diagrams (Φ, σ) from $F : \mathcal{C} \longrightarrow \mathcal{M}$ to $G : \mathcal{D} \longrightarrow \mathcal{M}$ is an **open map** [6] if and only if:

- σ is a natural isomorphism;
- Φ is surjective-on-objects;
- for every morphism $j : F(c) \longrightarrow d$ of $\mathcal{D}$ there is a morphism $i : c \longrightarrow c'$ of $\mathcal{C}$ such that $F(i) = j$.

Two diagrams $F : \mathcal{C} \longrightarrow \mathcal{M}$ and $G : \mathcal{D} \longrightarrow \mathcal{M}$ are **bisimilar** if there is span of open maps between them, i.e., there are a diagram $H : \mathcal{E} \longrightarrow \mathcal{M}$ and two open maps $(\Phi, \sigma) : H \longrightarrow F$ and $(\Psi, \tau) : H \longrightarrow G$.

We can then prove that natural dipath functors are invariant modulo dihomotopy equivalence when we compare them up to bisimilarity:

▶ **Theorem 4.** *If two dspaces are dihomotopically equivalent then their natural dipath functors (and so their natural homology) are bisimilar (in* ***Diag(****$HoTop$****)****).*

Since the Fahrenberg matchbox and a point have non-bisimilar natural homology [6], they cannot be dihomotopically equivalent.

5 Weak directed homotopy equivalence

In Section 3, we have seen the limitation of too simple an implementation of Porter's programme on the example of the directed segment: empty path spaces are not well handled because we are requiring a weak homotopy equivalence for each pair of points and because the (ordinary) component category is somehow too rigid. We fix those two problems in this section. First, in Section 5.1, we introduce a notion of partially enriched categories, i.e., enriched categories where only some morphism objects between two objects are defined (intuitively, the non-empty ones). Secondly, we replace components by directed components in the style of [11]. Altogether, this defines a weak dihomotopy equivalence which is an invariant of dihomotopy equivalence (see Section 5.3).

5.1 Partially enriched categories and the dipath category

In the following, $\mathcal{V}$ is a monoidal category with $\otimes$ as tensor product, U as unit, $\alpha_{A,B,C}$: $(A\otimes B)\otimes C \longrightarrow A\otimes(B\otimes C)$ as associator, $\lambda_A : U\otimes A \longrightarrow A$ as left unit and $\rho_A : A\otimes U \longrightarrow A$ as right unit. We will mainly consider the case of Top (the category of topological spaces and continuous functions), $HoTop$ (category of topological spaces and continuous functions modulo homotopy) and Ab (category of Abelian groups and morphisms of groups) with their Cartesian structure.

A (small $\mathcal{V}$-)**partially enriched category** $\mathcal{C}$ consists of the following data:

- a set $Ob(\mathcal{C})$ of objects;
- a preorder $\leq$ on $Ob(\mathcal{C})$ called **domain**;
- for every pair $c \leq c'$ of objects of $\mathcal{C}$, an object $\mathcal{C}(c,c')$ of $\mathcal{V}$;
- for every triple $c \leq c' \leq c''$ of objects of $\mathcal{C}$, a composition morphism in $\mathcal{V}$ $\circ_{c,c',c''}$: $\mathcal{C}(c,c')\otimes\mathcal{C}(c',c'') \longrightarrow \mathcal{C}(c,c'')$;
- for every object c of $\mathcal{C}$, a unit morphism in $\mathcal{V}$ $u_c : U \longrightarrow \mathcal{C}(c,c)$.

satisfying :

- (associativity): for every quadruple $c \leq c' \leq c'' \leq c'''$ of objects of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{C}(c,c')\otimes\mathcal{C}(c',c''))\otimes\mathcal{C}(c'',c''') \arrow[rr, "\circ_{c,c',c''}\otimes id"] \arrow[d, "\alpha_{\mathcal{C}(c,c'),\mathcal{C}(c',c''),\mathcal{C}(c,c'')}"'] & & \mathcal{C}(c,c'')\otimes\mathcal{C}(c'',c''') \arrow[dd, "\circ_{c,c'',c'''}"] \\
\mathcal{C}(c,c')\otimes(\mathcal{C}(c',c'')\otimes\mathcal{C}(c'',c''')) \arrow[d, "id\otimes\circ_{c',c'',c'''}"'] & & \\
\mathcal{C}(c,c')\otimes\mathcal{C}(c',c''') \arrow[rr, "\circ_{c,c',c'''}"] & & \mathcal{C}(c,c''')
\end{tikzcd}$$

- (unit): for every pair $c \leq c'$ of objects of $\mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
U\otimes\mathcal{C}(c,c') \arrow[r, "\lambda_{\mathcal{C}(c,c')}"] \arrow[d, "u_c\otimes id"'] & \mathcal{C}(c,c') \arrow[d, "id"] & \mathcal{C}(c,c')\otimes U \arrow[l, "\rho_{\mathcal{C}(c,c')}"'] \arrow[d, "id\otimes u_{c'}"] \\
\mathcal{C}(c,c)\otimes\mathcal{C}(c,c') \arrow[r, "\circ_{c,c,c'}"] & \mathcal{C}(c,c') & \mathcal{C}(c,c')\otimes\mathcal{C}(c',c') \arrow[l, "\circ_{c,c',c'}"']
\end{tikzcd}$$

The axioms are the same as for $\mathcal{V}$-enriched categories [4], except for the fundamental role played by the domain $\leq$ in every clause. Trivially, an enriched category is a partially enriched category whose domain is $Ob(\mathcal{C})\times Ob(\mathcal{C})$. One should note that partially enriched categories in Top, in $HoTop$ and in $Simp$ (category of simplicial sets and simplicial maps) are still very close to $(\infty,1)$-categories but also to Gaucher's flows [10], which were introduced for similar motivations.

Our main use of partially enriched categories will be the following. Given a dspace (X,P_X), we construct a partially enriched category over $HoTop$ called the **dipath category** and written $\overrightarrow{\mathbb{P}}(X)$:

- its objects are points of X;
- its domain, called the **accessibility preordering** is $x \leq y$ iff there a dipath from x to y;
- for $x \leq y$, $\overrightarrow{\mathbb{P}}(X)(x,y)$ is the set of dipaths from x to y equipped with the compact-open topology, as already defined in Section 4.2;
- composition is the class modulo homotopy of the concatenation $(\gamma, \delta) \mapsto \gamma \star \delta$;
- for $x \in X$, the unit morphism u_x is the class modulo homotopy of the continuous function $\{*\} \longrightarrow \overrightarrow{\mathbb{P}}(X)(x,x)$, $* \longmapsto c_x$.

A **partially enriched functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ between partially enriched categories is the following data:
- a monotonic function $F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$;
- for every pair $c \leq c'$ of objects, a morphism $F_{c,c'} : \mathcal{C}(c,c') \longrightarrow \mathcal{D}(F(c), F(c'))$ in $\mathcal{V}$;

satisfying that:
- for every triple $c \leq c' \leq c''$ of objects of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(c,c') \otimes \mathcal{C}(c',c'') \arrow[r, "\circ_{c,c',c''}"] \arrow[d, "F_{c,c'} \otimes F_{c',c''}"'] & \mathcal{C}(c,c'') \arrow[d, "F_{c,c''}"] \\
\mathcal{D}(F(c),F(c')) \otimes \mathcal{D}(F(c'),F(c'')) \arrow[r, "\circ_{F(c),F(c'),F(c'')}"] & \mathcal{D}(F(c),F(c''))
\end{tikzcd}$$

- for every object c of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
U \arrow[r, "u_c"] \arrow[dr, "u_{F(c)}"'] & \mathcal{C}(c,c) \arrow[d, "F_{c,c}"] \\
& \mathcal{D}(F(c),F(c))
\end{tikzcd}$$

The partially enriched functors between enriched categories seen as partially enriched categories are exactly the enriched functors. We note $\mathbf{PeCat}(\mathcal{V})$ the category of small $\mathcal{V}$-partially enriched categories and partially enriched functors. Note that $\overrightarrow{\mathbb{P}}$ extends to a functor from $dTop$ to $\mathbf{PeCat}(HoTop)$.

5.2 Category of components

We recover the **fundamental category** $\overrightarrow{\pi_1}(X)$ of a dspace X [12] by applying the ordinary path-connected components functor π_0 to its category of dipaths $\overrightarrow{\mathbb{P}}(X)$:
- its objects are points of X;
- the morphisms from x to y are the elements of $\pi_0(\overrightarrow{\mathbb{P}}(X)(x,y))$, i.e., the path-connected components of $\overrightarrow{\mathbb{P}}(X)(x,y)$ if $x \leq y$ and $\varnothing$ otherwise;
- the composition is the function : $\circ : \pi_0(\overrightarrow{\mathbb{P}}(X)(y,z)) \times \pi_0(\overrightarrow{\mathbb{P}}(X)(x,y)) \longrightarrow \pi_0(\overrightarrow{\mathbb{P}}(X)(x,z))$ such that $[f] \circ [g] = [g \star f]$;
- the identity of x is the path-connected components of the constant dipath, i.e., $[c_x]$.

Morphisms of $\overrightarrow{\pi_1}(X)$ are exactly dipaths modulo dihomotopy. More generally, we can define similarly the **fundamental category** $\overrightarrow{\pi_1}(\mathcal{C})$ **of a partially enriched category** $\mathcal{C}$ **on** $HoTop$ which extends to a functor from $\mathbf{PeCat}(HoTop)$ to Cat.

From this fundamental category, we can define a category of components as in [11]. The idea is to define a class of inessential morphisms, which is the largest class of morphisms whose compositions to the left and to the right are bijections, that has a left and right calculi of fractions (exactly like in the definition of inessential dipaths in Section 4.2). Then those morphisms can be inverted to give the **category of components** $\overrightarrow{\pi_0}(\mathcal{C})$ **of a category** $\mathcal{C}$ and, by extension, of a partially enriched category $\mathcal{C}$ on **HoTop** (resp. a dspace X) by $\overrightarrow{\pi_0}(\mathcal{C}) = \overrightarrow{\pi_0}(\overrightarrow{\pi_1}(\mathcal{C}))$ (resp. $\overrightarrow{\pi_0}(X) = \overrightarrow{\pi_0}(\overrightarrow{\pi_1}(X))$).

Explicitly, given a small category $\mathcal{C}$, we define a **Yoneda system** Λ **of morphisms of** $\mathcal{C}$ as a subset of morphisms of C such that:

- Λ is closed under composition;
- for every $f : c \longrightarrow c' \in \Lambda$, for every object c'' of $\mathcal{C}$ such that $\mathcal{C}(c', c'') \neq \varnothing$, the function $\_ \circ f : \mathcal{C}(c', c'') \longrightarrow \mathcal{C}(c, c'')$ $\;g \longmapsto g \circ f$ is a bijection;
- for every $f : c \longrightarrow c' \in \Lambda$, for every object c'' of $\mathcal{C}$ such that $\mathcal{C}(c'', c) \neq \varnothing$, the function $f \circ \_ : \mathcal{C}(c'', c) \longrightarrow \mathcal{C}(c'', c')$ $\;g \longmapsto f \circ g$ is a bijection;
- Λ has the right Ore condition, i.e., for every $f : x \longrightarrow y \in \Lambda$ and every $g : z \longrightarrow y \in \mathcal{C}$ there are $f' : w \longrightarrow z \in \Lambda$ and $g' : w \longrightarrow x \in \mathcal{C}$ for some w such that $f \circ g' = g \circ f'$
- Λ has the left Ore condition, i.e., for every $f : x \longrightarrow y \in \Lambda$ and every $g : x \longrightarrow z \in \mathcal{C}$ there are $f' : z \longrightarrow w \in \Lambda$ and $g' : x \longrightarrow w \in \mathcal{C}$ for some w such that $f' \circ g = g' \circ f$.

In particular, $(\mathcal{C}, \Lambda)$ has left and right calculi of fractions [3]. This is related to the definition of [11], which additionally requires closure by pullbacks/pushouts in order to obtain a Van-Kampen theorem.

▶ **Lemma 5.** *The set of Yoneda systems of morphisms of $\mathcal{C}$ is a complete lattice for inclusion. We note $\mathfrak{I}(\mathcal{C})$ the largest such system and call its elements **inessential morphisms**. $\mathfrak{I}(\mathcal{C})$ contains all isomorphisms and has the 2-out-3 property, i.e., if two of the three morphisms f, g and $g \circ f$ are in $\mathfrak{I}(\mathcal{C})$, then so is the third. Moreover, $\{[\gamma] \in \overrightarrow{\pi_1}(X) \mid \gamma \in \mathfrak{I}(X)\} \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$.*

Now, we define the **category of components** $\overrightarrow{\pi_0}(\mathcal{C})$ **of** $\mathcal{C}$, as the localization $\mathcal{C}[\mathfrak{I}(\mathcal{C})^{-1}]$ [3]. One interesting property of this localization is that it is equivalent to the generalized quotient (in the sense of [1]) $\mathcal{C}/\mathfrak{I}(\mathcal{C})$ when $\mathcal{C}$ is loop-free [11]. We stress that this is a remarkable property, for a localization, of being equivalent to a quotient, and this will be particularly useful for examples. In fact, this can be generalized when $\mathfrak{I}(\mathcal{C})$ has a "selection". A **partial selection of a category** $\mathcal{C}$ is a subcategory of $\mathcal{C}$ which is a preorder. A **(total) selection** is a partial selection Σ satisfying moreover that for every pair (c, c') of objects of $\mathcal{C}$, if $\mathcal{C}(c, c')$ is non-empty then so is $\Sigma(c, c')$.

▶ **Theorem 6.** *Let Σ be a total selection of $\mathfrak{I}(\mathcal{C})$. $\overrightarrow{\pi_0}(\mathcal{C})$ is equivalent to the generalized quotient $\mathcal{C}/\Sigma$.*

▶ **Example 7.** We show here a few examples of categories of components of dspaces which are equivalent to quotient of fundamental categories.

1) If X is a pospace (i.e. a topological space equipped with a closed partial-ordering), then the fundamental category $\overrightarrow{\pi_1}(X)$ is loop-free in the sense of [11], and so $\mathfrak{I}(\overrightarrow{\pi_1}(X))$ is itself a selection. We then recover the case of [11]. In particular, SF and HS are pospaces and their category of components are equivalent to the categories depicted in Figure 5 (more precisely, to the categories generated by those graphs with the relation $\circlearrowleft$ representing commutativity). In particular, this shows that SF and HS do not have the same category of components, as claimed in Example 1.
2) Let $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi[\}$ be the topological circle. We call non-directed circle the dspace S^1 whose dipaths are all paths. As all paths are invertible modulo homotopy, every morphism of $\overrightarrow{\pi_1}(S^1)$ is an isomorphism and so belongs to $\mathfrak{I}(\overrightarrow{\pi_1}(S^1))$. But, as they are all isomorphisms, $\overrightarrow{\pi_0}(S^1) = \overrightarrow{\pi_1}(S^1)$ which is the fundamental groupoid of the circle. Moreover, $\Sigma = \{[t \mapsto e^{i((1-t)\theta + t\theta')}] \mid \theta, \theta' \in [0, 2\pi[\}$ where $[.]$ is the class modulo homotopy, is a selection of $\mathfrak{I}(S^1)$ and $\overrightarrow{\pi_1}(S^1)/\Sigma$ is the category with one object and $\mathbb{Z}$ as set of morphisms. That category is equivalent to $\overrightarrow{\pi_1}(S^1)$ itself.

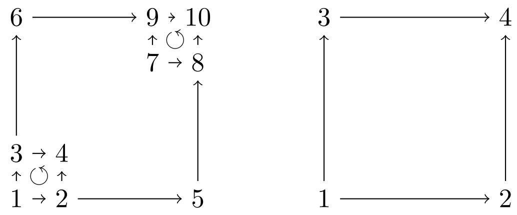

Figure 5 Category of components of SF and HS

3) Let $\overrightarrow{S^1}$ be the dspace whose underlying topological space is S^1 and whose dipaths are paths of the form $t \mapsto e^{i\Phi(t)}$ for some non-decreasing function $\Phi : I \longrightarrow \mathbb{R}$, i.e., paths that only turn anti-clockwise. In this case, the only Yoneda morphisms of $\overrightarrow{S^1}$ are the identities, i.e., dihomotopy class of constant paths. Indeed, they are the only ones that induces bijections between Hom-sets by composition: if you take any non-constant dipath γ, say from $e^{i\theta}$ to $e^{i\theta'}$ then $[\gamma] \circ \_ : \overrightarrow{\pi_1}(\overrightarrow{S^1})(e^{i\theta'}, e^{i\theta}) \longrightarrow \overrightarrow{\pi_1}(\overrightarrow{S^1})(e^{i\theta}, e^{i\theta})$ is not surjective since it never reaches the class of the constant path. Hence $\overrightarrow{\pi_0}(\overrightarrow{S^1}) = \overrightarrow{\pi_1}(\overrightarrow{S^1})$.

In this section, we could have defined the category of components in different ways. We have chosen here the classical way, from the fundamental category (just like [2, 11]). But we could have followed the path initiated in the definition of future and past deformation retracts, requiring inessential morphisms to induce homotopy equivalences by composition instead of isomorphisms of path-connected components (cf. conditions 2 and 3). It would have defined a finer notion of components in the sense that, there would have been less inessential morphisms and so less dspaces with the same components. The theorem 9 would have also hold with this definition, but there would have been some redundancy between this notion of components and the requirement of homotopy equivalences in the definition of weak equivalences.

5.3 Weak equivalences

Imitating [22], we will study dihomotopy types of dspaces using a similar notion of weak equivalences of partially enriched categories. A **weak equivalence between two partially enriched categories $\mathcal{C}$ and $\mathcal{D}$ in $HoTop$** is a partially enriched functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which induces an equivalence of categories between $\overrightarrow{\pi_0}(\mathcal{C})$ and $\overrightarrow{\pi_0}(\mathcal{D})$, and such that for every pair $c \leq c'$ in $\mathcal{C}$, $F_{c,c'}$ is an isomorphism, i.e. the homotopy class of a homotopy equivalence. We stress the fact that F induces a functor between the categories of components is not automatic since $\overrightarrow{\pi_0}$ is not a functor. We say that a dmap $f : X \longrightarrow Y$ is a **weak dihomotopy equivalence** if $\overrightarrow{\mathbb{P}}(f)$ is a weak equivalence, and we say that X and Y are **weakly dihomotopy equivalent** if there is a zig-zag of weak equivalences between X and Y.

▶ **Example 8.**

1) As we have said earlier, the directed segment is dihomotopically equivalent to a point and so is weakly equivalent to a point. We have a continuous constant map $c : \overrightarrow{I} \longrightarrow \{1\}$, which is a dmap. Let us prove that this is a weak equivalence. First, for all $x \leq y$, $\overrightarrow{\mathbb{P}}(c)_{x,y}$ is a homotopy equivalence because it is a constant map and $\overrightarrow{\mathbb{P}}(X)(x,y)$ is contractible. Now, it is easy to check that $\mathfrak{I}(\overrightarrow{\pi_1}(\overrightarrow{I})) = \overrightarrow{\pi_1}(\overrightarrow{I})$ and is itself a selection. Therefore $\overrightarrow{\pi_0}(\overrightarrow{I})$ is equivalent to the category with one object and one morphism, namely $\overrightarrow{\pi_0}(\{1\})$. In fact, c induces an equivalence between those two categories.

2) As we have said earlier, the Fahrenberg matchbox M is not dihomotopically equivalent to a point. In fact, they also are non-weakly equivalent. We have seen that there are two dipaths that are not dihomotopic. This implies that $\overrightarrow{\mathbb{P}}(M)(\mathbf{0}, \alpha)$ (with the notation of Figure 4) is homotopically equivalent to a two point spaces. But if two dspaces are weak-equivalent then they have the same homotopy types of non-empty spaces of dipaths. Since the spaces of dipaths of a point are all contractible, the matchbox cannot be weakly equivalent to a point.

▶ **Theorem 9.** *If two dspaces are dihomotopically equivalent then they are weakly equivalent.*

That is, weak dihomotopy equivalence is well-suited to prove that two dspaces are not dihomotopy equivalent (like in the case of T and M). In particular, if two dspaces do not have the same homotopy types of spaces of dipaths then they are not weakly dihomotopy equivalent and therefore not dihomotopy equivalent. The previous examples show in particular that our notion of dihomotopy equivalence is different from the one of [12] (because that one does not distinguish the matchbox from a point) and that the notion of weak equivalence used in [22] is strictly stronger than that introduced above as it distinguishes the directed segment from the point.

5.4 Natural homology and weak equivalences

We have proved that natural homology is an invariant of dihomotopy equivalence (Theorem 4). Is it also an invariant of weak equivalence ? At the time of this article, this is still a conjecture. But as a step in that direction, we observe that bisimilarity is strongly tied to (strong) equivalence of partially enriched categories (Theorem 11 below).

Equivalence of enriched categories is usually defined as in the non-enriched case using (enriched) natural isomorphisms. Using the axiom of choice, this definition is equivalent to the existence of a fully-faithful essentially surjective functor [19]. Nevertheless, we will not use these definitions in the partially enriched case : one problem is that there is no clear non-trivial notion of partially enriched natural transformations. We will rather use the following:

▶ **Lemma 10.** *Two categories are equivalent iff there is a span of fully-faithful surjective-on-objects functors*

We say that a partially enriched functor $F : \mathcal{E} \longrightarrow \mathcal{C}$ is:

- **fully-faithful** if for every pair $e \leq e'$ in $\mathcal{E}$, $F_{e,e'} : \mathcal{E}(e, e') \longrightarrow \mathcal{C}(F(e), F(e'))$ is an isomorphism;
- **surjective** if $F : Ob(\mathcal{E}) \longrightarrow Ob(\mathcal{C})$ is surjective;
- **fibrational** if for every $e \in Ob(\mathcal{E})$ and $c \in Ob(\mathcal{C})$ such that $F(e) \leq c$ there is e' such that $e \leq e'$ and $F(e') = c$.

We call **strong equivalence** any partially enriched functor which is fully-faithful, surjective and fibrational.

We say that two partially enriched categories $\mathcal{C}$ and $\mathcal{D}$ are **strongly equivalent** if there are a partially enriched category $\mathcal{E}$ and a span $F : \mathcal{E} \longrightarrow \mathcal{C}$ and $G : \mathcal{E} \longrightarrow \mathcal{D}$ of strong equivalences.

Without the fibrational condition, this equivalence would be a bit trivial: taking a suitable $\mathcal{E}$ whose domain is equality would make equivalent two partially enriched categories which have the same endomorphisms. Moreover, the strong equivalences between two enriched categories are exactly the fully-faithful surjective enriched functors between them.

Let us look at the case $\mathcal{M} = \mathbf{Ab}$, the category of Abelian groups. In [6] we were only considering diagrams whose domains were pre-orders. Let us call them **po-diagrams** and denote by **PoDiag(Ab)** the full subcategory of those po-diagrams.

Given a diagram $F : \mathcal{C} \longrightarrow \mathbf{Ab}$, define its **unfolding** as the diagram $Unf(F) : Unf(\mathcal{C}) \longrightarrow \mathbf{Ab}$ such that:

- the objects of $Unf(\mathcal{C})$ are non-empty finite sequences $(f_1, ..., f_n)$ of composable morphisms of $\mathcal{C}$, i.e. domain of f_i = codomain of f_{i-1};
- the set of morphisms of $Unf(\mathcal{C})$ from $(f_1, ..., f_n)$ to $(g_1, ..., g_p)$ is $\{(g_{n+1}, ..., g_p)\}$ if $n \leq p$ and for all $i \leq n$, $f_i = g_i$, and is empty otherwise;
- composition of $Unf(\mathcal{C})$ is concatenation;
- identities of $Unf(\mathcal{C})$ are empty sequences;
- $Unf(F)(f_1, ..., f_n) = F(c)$ where c is the codomain of f_n;
- $Unf(F)(g_{n+1}, ..., g_p) = F(g_p \circ \ldots \circ g_{n+1})$.

Given a po-diagram $F : \mathcal{C} \longrightarrow \mathbf{Ab}$, we extend the Grothendieck construction [15] to partially enriched categories in $\mathbf{Ab}$ as follows:

- the objects are objects of $\mathcal{C}$;
- the domain is $\mathcal{C}$, which we recall is a preorder;
- for $c \leq c'$, $\mathcal{G}(F)(c, c') = F(c')$;
- for $c \leq c' \leq c''$, the composition $\circ_{c,c',c''} : F(c') \times F(c'') \longrightarrow F(c'')$ is the morphism of groups that maps (g', g'') to $g'' + F(c' \leq c'')(g')$, where $c \leq c'$ is the unique morphism from c to c' in $\mathcal{C}$;
- the unit $u_c : \{0\} \longrightarrow F(c)$ is the null morphism.

▶ **Theorem 11.** *Two po-diagrams are bisimilar iff their Grothendieck constructions are equivalent. Two diagrams are bisimilar iff the Grothendieck construction of their unfoldings are equivalent.*

We see Theorem 11 as a tool that we may use later to prove that if the dipath categories of X and Y are weakly equivalent, then their natural homologies are bisimilar.

6 Conclusion and future work

Using partially enriched categories allows us to compare dspaces modulo dihomotopy equivalence. Are partially enriched categories the right models for dspaces modulo dihomotopy equivalence ? We hope to have at least conveyed the idea that this should be the case. Are topologically partially enriched categories a nice model of $(\infty, 1)$-categories ? Can we conversely understand $(\infty, 1)$-categories using (weak) dihomotopy types of dspaces ? Those questions are left to future work.

Also, a recurrent and associated question concerns the algebraic structure of directed homotopy. Ordinary homotopy theories can be described in the framework of Quillen model categories, giving axioms linking classes of morphisms (in the category of topological spaces, or of simplicial sets, for instance) called weak equivalences, fibrations and cofibrations.

In this paper, we have developed a class of weak equivalences, which could be part of such an axiomatics. More precisely, most of what has been described in Section 5.3 can be parameterized by the class of weak equivalences on the category of topological spaces that is consistent with the standard model category theoretic framework. We used the "stronger" one, Strøm model category where "weak equivalences" are in fact (strong) homotopy equivalences, but indeed, one would be tempted to use weak homotopy equivalences instead. We only "lifted' homotopy equivalences onto the hom-sets of our (partially-enriched) category of dspaces, and of course, we could as well think of lifting fibrations and cofibrations of any closed model category structure on topological spaces, to give a reasonable notion of directed homotopy structure on directed spaces. The resulting classes of morphisms, lifts of weak equivalences, fibrations and cofibrations should verify some of Quillen axioms at least, maybe others. The directed topological community is currently undecided with respect to whether there is a model category of directed spaces which accounts, faithfully, for directed algebraic topological phenomena.

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A Proof of Lemma 2

Proof. Let Γ be the set of Yoneda systems of X. Γ has suprema for inclusion given by: if $(Y_i)_{i\in I}$ is a family of Γ then $sup(Y_i)_{i\in I} = \{\gamma \mid \exists n \geq 1, \exists \gamma_1, \ldots, \gamma_n \in \gamma_1 \star \ldots \star \gamma_n \in \bigcup_{i\in I} Y_i$ such that γ is dihomotopic to $\gamma_1 \star \ldots \star \gamma_n\}$. We must prove that $sup(Y_i)_{i\in I}$ is in Γ:

- it is closed under concatenation and dihomotopy by definition;
- $\gamma \star \_$ is homotopic to $(\gamma_1 \star \ldots \star \gamma_n) \star \_$ which is homotopic to $(\gamma_n \star \_) \circ \ldots \circ (\gamma_1 \star \_)$ and so is a homotopy equivalence;
- the right Ore condition is proved by induction on n using the right Ore condition in the Y_j.

It is the supremum because:

- every Y_i is included in it (case $n = 1$);
- if Z is a Yoneda system that contains all the Y_i, then by closure under concatenation and dihomotopy, it must contains $sup(Y_i)_{i\in I}$.

◀

B Proof of Theorem 4

Let us begin with the following lemma:

▶ **Lemma 12.** *If H is a future (or past) deformation retract and if $\overrightarrow{\mathbb{P}}(X)(x,x') \neq \varnothing$ then the map $H_1 \circ \_ : \overrightarrow{\mathbb{P}}(X)(x,x') \longrightarrow \overrightarrow{\mathbb{P}}(A)(H_1(x), H_1(x'))$, $\delta \longmapsto H_1 \circ \delta$ is a homotopy equivalence.*

Proof of Lemma 12. By definition of a future deformation retract, $H(x')$ is an inessential dipath and since $\overrightarrow{\mathbb{P}}(X)(x,x') \neq \varnothing$, then the map $\_ \star H(x') : \overrightarrow{\mathbb{P}}(X)(x,x') \longrightarrow \overrightarrow{\mathbb{P}}(X)(x, H_1(x'))$ is a homotopy equivalence.

Let $\gamma \in \overrightarrow{\mathbb{P}}(X)(x,x')$. Since H_1 is a dmap, $H_1(\gamma) \in \overrightarrow{\mathbb{P}}(X)(H_1(x), H_1(x'))$ and so $\overrightarrow{\mathbb{P}}(X)(H_1(x), H_1(x')) \neq \varnothing$. By definition of a future deformation retract, $H(x)$ is an inessential dipath and therefore, the map $H(x) \star \_ : \overrightarrow{\mathbb{P}}(X)(H_1(x), H_1(x')) \longrightarrow \overrightarrow{\mathbb{P}}(X)(x, H_1(x'))$.

To conclude, we prove that $(H(x) \star \_) \circ (H_1 \circ \_)$ is homotopic to $\_ \star H(x')$ and so $H_1 \circ \_$ is a homotopy equivalence by the 2-out-of-3 property. Let $K : I \times \overrightarrow{\mathbb{P}}(X)(x,x') \longrightarrow \overrightarrow{\mathbb{P}}(X)(x, H_1(x'))$ which maps (t, π) to the dipath sending $s \in I$ to:

- $H(x)(2s)$ if $s \leq \frac{t}{2}$
- $H_t(\pi(2s-t))$ if $\frac{t}{2} \leq s \leq \frac{t+1}{2}$
- $H(x')(2s-1)$ if $s \geq \frac{t+1}{2}$

K is continuous and satisfies that $K(1,\pi) = H(x) \star (H_1 \circ \pi) = (H(x) \star \_) \circ (H_1 \circ \_)(\pi)$ and $K(0,\pi) = \pi \star H(x') = (\_ \star H(x'))(\pi)$ and so is a homotopy. ◀

Proof of Theorem 4. We prove that when A is a future deformation retract of X then the map H_1 induces a bisimulation between the natural dipath functors.

First, it induces a functor $\Phi_{H_1} : \mathcal{F}_X \longrightarrow \mathcal{F}_A$ by sending every class $[\gamma]$ to $[H_1 \circ \gamma]$. This functor is surjective since H_1 is the identity on A. It satisfies the lifting property: given a class $[H_1 \circ \gamma]$ of dipath modulo dihomotopy in A with γ from x to y and let $([\alpha], [\beta])$ be an extension in A, i.e., α and β are dipaths in A with α from x' to $H_1(x)$ and β from $H_1(y)$ to y'.

By the last condition of a future deformation retract, there is a dipath α' in X from w to x such that $H_1 \circ \alpha'$ and α are dihomotopic.

$\overrightarrow{\mathbb{P}}(X)(y,y')$ is non-empty since it contains $H(y) \star \beta$. By the previous lemma, $H_1 \circ \_ : \overrightarrow{\mathbb{P}}(X)(y,y') \longrightarrow \overrightarrow{\mathbb{P}}(A)(H_1(y), y')$, $\delta \longmapsto H_1 \circ \delta$ is a homotopy equivalence and so

$H_1 \circ \_ : \overrightarrow{\pi_1}(X)(y, y') \longrightarrow \overrightarrow{\pi_1}(A)(H_1(y), y')$, $[\delta] \longmapsto [H_1 \circ \delta]$ is a bijection. There is, thus, a dipath β' from y to y' such that $H_1 \circ \beta'$ is dihomotopic to β. Then $([\alpha'], [\beta'])$ is the lifting we were looking for.

Now, H_1 induces a natural transformation, $\sigma_{H_1} : \overrightarrow{\mathcal{P}}(X) \longrightarrow \overrightarrow{\mathcal{P}}(A) \circ \Phi_{H_1}$ by $\sigma_{H_1,[\gamma]} : \overrightarrow{\mathbb{P}}(X)(x, y) \longrightarrow \overrightarrow{\mathbb{P}}(A)(H_1(x), H_1(y))$, $\delta \longmapsto H_1 \circ \delta$ which is a homotopy equivalence by the previous lemma. ◀

C Proof of Lemma 5

Let us begin with the following lemma:

▶ **Lemma 13.** *Let W be a subset of morphisms of $\mathcal{C}$ containing all the identities and satisfying all the conditions a Yoneda system except the closure under composition then:*
i) $\langle W \rangle$, the subcategory of $\mathcal{C}$ generated by W satisfies those conditions too
ii) $TOOT(W)$, the 2-out-of-3 completion of W, i.e., the set

$$\{f \in \mathcal{C} \mid \exists u, v \in W,\ u \circ f = v\} \cup \{f \in \mathcal{C} \mid \exists u, v \in W,\ f \circ u = v\}$$

also satisfies these conditions.

Proof of Lemma 13.

i) The verification is long but strait-forward.

ii) First, remark that since W contains identities, $W \subseteq TOOT(W)$. Now, let us prove the conditions:

- **isomorphisms induced by composition:** Let $f \in TOOT(W)$ and let $u \in W$ such that $f \circ u \in W$ (the other case will be symmetric). Assume that $f : x \longrightarrow y$ and $u : w \longrightarrow x$.
  - ⋆ First, let z such that $\mathcal{C}(y, z) \neq \varnothing$. Then $\_ \circ f = (\_ \circ u)^{-1} \circ (\_ \circ (f \circ u))$ and is an isomorphism.
  - ⋆ Now, let z such that $\mathcal{C}(z, x) \neq \varnothing$. Let $h : z \longrightarrow x$. By the right Ore condition, there exist $u' : \alpha \longrightarrow z \in W$ and $h' : \alpha \longrightarrow w \in \mathcal{C}$ such that $h \circ u' = u \circ h'$. So, let us prove that $f \circ \_ : \mathcal{C}(z, x) \longrightarrow \mathcal{C}(z, y)$ is injective and surjective.
    injective: let $g, g' : z \longrightarrow x$ such that $f \circ g = f \circ g'$. Then $f \circ g \circ u' = f \circ g' \circ u'$. As $u \circ \_ : \mathcal{C}(\alpha, w) \longrightarrow \mathcal{C}(\alpha, x)$ is an isomorphism, there are $h, h' : \alpha \longrightarrow w$ such that $g \circ u' = u \circ h$ and $g' \circ u' = u \circ h$ and $f \circ u \circ h = f \circ u \circ h'$. Since $(f \circ u) \circ \_ : \mathcal{C}(\alpha, w) \longrightarrow \mathcal{C}(\alpha, y)$ is an isomorphism then $h = h'$ and so $g \circ u' = g' \circ u'$. Since $\_ \circ u' : \mathcal{C}(z, x) \longrightarrow \mathcal{C}(\alpha, x)$ is an isomorphism, $g = g'$.
    surjective: Let $h : z \longrightarrow y$. Then $h \circ u' : \alpha \longrightarrow y$. Since $(f \circ u) \circ \_ : \mathcal{C}(\alpha, w) \longrightarrow \mathcal{C}(\alpha, y)$ is an isomorphism then there exists $h' : \alpha \longrightarrow w$ such that $h \circ u' = f \circ u \circ h'$. Since $\_ \circ u' : \mathcal{C}(z, x) \longrightarrow \mathcal{C}(\alpha, x)$ is an isomorphism, there exists $g : z \longrightarrow x$ such that $u \circ h' = g \circ u'$ and so $h \circ u' = f \circ g \circ u'$. Since $\_ \circ u' : \mathcal{C}(z, y) \longrightarrow \mathcal{C}(\alpha, y)$ is an isomorphism, $h = f \circ g$.
- **right and left Ore conditions:** Let $f \in TOOT(W)$ and let $u \in W$ such that $f \circ u \in W$ (the other case will be symmetric). Assume that $f : x \longrightarrow y$ and $u : w \longrightarrow x$.
  - ⋆ **right:** Let $g : z \longrightarrow y \in \mathcal{C}$. Since $f \circ u \in W$ then by the right Ore condition on W, there is $u' : \alpha \longrightarrow z \in W \subseteq TOOT(W)$ and $g' : \alpha \longrightarrow w$ such that $g \circ u' = f \circ (u \circ g')$.
  - ⋆ **left:** Let $g : x \longrightarrow z \in \mathcal{C}$. By the left Ore condition on W with $f \circ u \in W$ and $g \circ u \in \mathcal{C}$, there are $u' : z \longrightarrow \alpha \in W \subseteq TOOT(W)$ and $g' : y \longrightarrow \alpha$ such that $g' \circ f \circ u = u' \circ g \circ u$. Since $\_ \circ u$ is an isomorphism, $g' \circ f = u' \circ g$.

◀

Proof of Lemma 5. Let Γ be the set of Yoneda systems of morphisms of $\mathcal{C}$.
- **complete lattice:** similar to lemma 2.
- **isomorphisms:** the set of isomorphisms belongs to Γ so isomorphisms belong to $\mathfrak{I}(\mathcal{C})$.
- **2-out-of-3 property:** We complete $\mathfrak{I}(\mathcal{C})$ inductively this way:
  - ⋆ $W_0 = W'_0 = \mathfrak{I}(\mathcal{C})$
  - ⋆ $W_{i+1} = TOOT(W'_i)$
  - ⋆ $W'_{i+1} = \langle W_{i+1}\rangle$

  Clearly, $W_i \subseteq W'_i$ and $W'_i \subseteq W_{i+1}$ (because W'_i contains identities). So define $W_\infty = \bigcup\limits_{i\in\mathbb{N}} W_i = \bigcup\limits_{i\in\mathbb{N}} W'_i$. It satisfies that:
  - ⋆ $\mathfrak{I}(\mathcal{C}) = W_0 \subseteq W_\infty$
  - ⋆ it is a wide subcategory of $\mathcal{C}$ which has the 2-out-of-3 property
  - ⋆ it satisfies the other conditions of the definition of $\mathfrak{I}(\mathcal{C})$ by the previous lemma.

  Thus, by maximality of $\mathfrak{I}(\mathcal{C})$, $\mathfrak{I}(\mathcal{C}) = W_\infty$ and so has the 2-out-of-3 property.
- **link between $\mathfrak{I}(X)$ and $\mathfrak{I}(\overrightarrow{\pi_1}(X))$:** the set $\{[\gamma] \in \overrightarrow{\pi_1}(X) \mid \gamma \in \mathfrak{I}(X)\}$ is a Yoneda system of morphisms of $\overrightarrow{\pi_1}(X)$.

◀

D Proof of Theorem 6

Before proving Theorem 6, let us recall some definitions from [1] on generalized quotients. First, a **generalized congruence** on a small category $\mathcal{C}$ is the following data:
- an equivalence relation $\simeq_0$ on objects of $\mathcal{C}$;
- a partial equivalence relation (i.e., symmetric and transitive relation) $\simeq_m$ on $Mor_+(\mathcal{C})$ (i.e., the set of non-empty finite sequences of morphisms of $\mathcal{C}$). We call the **support of** $\simeq_m$, the set $\{\gamma \in Mor_+(\mathcal{C}) \mid \gamma \simeq_m \gamma\}$.

These are to satisfy:
- if $(\beta_m, \ldots, \beta_0, \alpha_n, \ldots, \alpha_0)$ is in the support of $\simeq_m$, then source of $\beta_0 \simeq_0$ target of α_n;
- if $(\beta_m, \ldots, \beta_0) \simeq_m (\alpha_n, \ldots, \alpha_0)$, then target of $\beta_m \simeq_0$ target of α_n;
- if $c \simeq_0 d$, then $(id_c) \simeq_m (id_d)$;
- if $(\beta_m, \ldots, \beta_0) \simeq_m (\alpha_n, \ldots, \alpha_0)$, $(\gamma_p, \ldots, \gamma_0) \simeq_m (\delta_q, \ldots, \delta_0)$ and source of $\beta_0 \simeq_0$ target of α_n, then
$$(\beta_m, \ldots, \beta_0, \alpha_n, \ldots, \alpha_0) \simeq_m (\gamma_p, \ldots, \gamma_0, \delta_q, \ldots, \delta_0);$$
- if α and β are composable (i.e., source of β = target of α), then $(\beta, \alpha) \simeq_m (\beta \circ \alpha)$.

Given a relation R_0 on objects of $\mathcal{C}$ and a relation R_m on $Mor_+(\mathcal{C})$, there is a least generalized congruence that contains (R_0, R_m) [1].

Given a generalized congruence $(\simeq_0, \simeq_m)$ on $\mathcal{C}$, we define the **generalized quotient** $\mathcal{C}/(\simeq_0, \simeq_m)$ as the category whose:
- objects are equivalence classes $[x]_0$ of objects of $\mathcal{C}$ modulo $\simeq_0$
- morphisms from $[x]_0$ to $[y]_0$ are equivalence classes
$$[(\alpha_n, \ldots, \alpha_0)]_m$$
of elements of the domain of $\simeq_m$ modulo $\simeq_m$ such that the target of $\alpha_n \simeq_0 y$ and the source of $\alpha_0 \simeq_0 x$;
- composition is $[(\beta_m, \ldots, \beta_0)]_m \circ [(\alpha_n, \ldots, \alpha_0)]_m = [(\beta_m, \ldots, \beta_0, \alpha_n, \ldots, \alpha_0)]_m$;
- identity on $[x]_0$ is $[(id_x)]_m$.

Proof. Let Σ be a selection of $\mathfrak{I}(C)$. Let $(\sim_0, \sim_m)$ be the generalized congruence on $\mathcal{C}$ generated by $R_0 = \varnothing$ and $R_m = \{((f),(id_x)) \mid f \in \Sigma, x \in \{\text{source of } f, \text{ target of } f\}\}$. Then the generalized quotient $\mathcal{C}/\Sigma$ is $\mathcal{C}/(\sim_0, \sim_m)$. We note $\sigma_{x,y}$ the unique morphism of Σ from x to y when it exists.

We start by defining a functor $Q : \overrightarrow{\pi_0}(\mathcal{C}) \longrightarrow \mathcal{C}/\Sigma$. It maps every object x to the class $[x]_0$ modulo $\sim_0$ and every class of span $[x \xleftarrow{f} y \xrightarrow{g} z]$ in the localization with $f \in \mathfrak{I}(\mathcal{C})$ to $[g \circ \tilde{f}^{-1}]_m$ the class of $g \circ \tilde{f}$ modulo $\sim_m$ where $\tilde{f}$ is the unique morphism from y to y such that $\sigma_{y,x} \circ \tilde{f} = f$. $\tilde{f}$ is an isomorphism because it is an endomorphism which belongs to $\mathfrak{I}(\mathcal{C})$ by the 2-out-of-3 property.

This is well defined: indeed, it does not depend on the span representing $[x \xleftarrow{f} y \xrightarrow{g} z]$. If we take another span representing it $x \xleftarrow{f'} y' \xrightarrow{g'} z$, there exists a span $y \xleftarrow{u} w \xrightarrow{v} y'$ such that $f \circ u = f' \circ v$, $g \circ u = g' \circ v$ and $g \circ u = g' \circ v$ belongs to $\mathfrak{I}(\mathcal{C})$. By the 2-out-of-3 property, u and v belongs to $\mathfrak{I}(\mathcal{C})$. Since $(u, f) \sim_m (v, f') \sim_m (u, \tilde{f}) \sim_m (v, \tilde{f}')$. Moreover $(u, \tilde{f}, \tilde{f}^-1, g) \sim_m (u, g) \sim_m (v, g') \sim_m (v, \tilde{f}', \tilde{f'}^{-1}, g') \sim_m (u, \tilde{f}, \tilde{f'}^{-1}, g')$. But, since $\tilde{f} \circ u\mathfrak{I}(\mathcal{C})$, there is a h such that $\tilde{f} \circ u \circ h = \sigma_{w,y}$ and so $(\tilde{f}^{-1}, g) \sim_m (\tilde{f'}^{-1}, g')$. It is a functor because $Q([x \xleftarrow{id_x} x \xrightarrow{id_x} x]) = [id_x \circ \tilde{id_x}]_m = [id_x \circ id_x]_m = [id_x]_m$ and if $x \xleftarrow{f} y \xrightarrow{g} z \xleftarrow{h} y' \xrightarrow{k} x'$, let $y \xleftarrow{h'} w \xrightarrow{g'} y'$ coming from the right Ore condition on h and g with $h' \in \mathfrak{I}(\mathcal{C})$. Then $Q([z \xleftarrow{h} y' \xrightarrow{k} x'] \circ [x \xleftarrow{f} y \xrightarrow{g} z]) = Q([x \xleftarrow{f \circ h'} w \xrightarrow{k \circ g'} x']) = [f \tilde{\circ} h'^{-1}, k \circ g']_m$ and $Q([z \xleftarrow{h} y' \xrightarrow{k} x']) \circ Q([x \xleftarrow{f} y \xrightarrow{g} z]) = [\tilde{f}^{-1}, g, \tilde{h}^{-1}, k]_m$. Since $g \circ h' = g' \circ h$, $(g) \sim_m (\tilde{h'}^{-1}, h', g) \sim_m (\tilde{h'}^{-1}, g', h)$. So $(\tilde{f}^{-1}, g, \tilde{h}^{-1}, k) \sim_m (\tilde{f}^{-1}, \tilde{h'}^{-1}, g', k)$. But $(f \tilde{\circ} h'^{-1}, f \circ h') \sim_m (id) \sim_m (\tilde{f}^{-1}, \tilde{h'}^{-1}, f \circ h')$. Since $f \circ h'\mathfrak{I}(\mathcal{C})$, $(f \tilde{\circ} h'^{-1}) \sim_m (\tilde{f}^{-1}, \tilde{h'}^{-1})$ and $(\tilde{f}^{-1}, g, \tilde{h}^{-1}, k) \sim_m (f \tilde{\circ} h'^{-1}, f \circ h')$.

Now, we define a functor $R : \mathcal{C}/\Sigma \longrightarrow \overrightarrow{\pi_0}(\mathcal{C})$. For every class α modulo $\sim_0$, make a choice $R(\alpha) \in \alpha$. Note that by the Ore conditions, if $c \sim_0 d$, then there is a span $c \xleftarrow{f} e \xrightarrow{g} d$ of morphisms of $\mathfrak{I}(\mathcal{C})$ and so of Σ. Every $[f_1, \ldots, f_n]_m$ with $f_i : d_i \longrightarrow c_i$, $c_{i-1} \sim_0 d_i$ and with $c_0 = R([d_1]_0)$ and $d_{n+1} = R([c_n]_0)$ will be mapped to $[c_n \xleftarrow{\sigma e_n, c_n} e_n \xrightarrow{\sigma e_n, d_{n+1}} d_{n+1}] \circ [c_{n-1} \xleftarrow{\sigma e_{n-1}, c_{n-1}} e_{n-1} \xrightarrow{f_n \circ \sigma e_{n-1}, d_n} c_n] \circ \ldots \circ [c_0 \xleftarrow{\sigma e_0, c_0} e_0 \xrightarrow{f_1 \circ \sigma e_0, d_1} c_1]$. Notice that this depends on neither the choice of the e_i, nor the choice of the element representing $[f_1, \ldots, f_n]_m$.

We now prove that $Q \circ R = id$. But first, let us prove by induction on n that for every $(f_1, \ldots, f_n)$ with $f_i : d_i \longrightarrow c_i$ and $c_i \sim_0 d_{i+1}$ and for every $x \sim_0 d_1$ and $y \sim_0 c_n$, there is a morphism $h : z \longrightarrow y$ such that there is a morphism in $\mathfrak{I}(\mathcal{C})$ from z to x and $(f_1, ..., f_n) \sim_m (h)$:

- **if $n = 1$:** we have $f : c \longrightarrow d$, $x \sim_0 c$ and $y \sim_0 d$. We have then e such that there are morphisms in $\mathfrak{I}(\mathcal{C})$ from e to d and from e to y. By the right Ore condition on $\sigma_{e,d} \in \mathfrak{I}(\mathcal{C})$ and f there are $g : w \longrightarrow e$ and $\gamma : w \longrightarrow c \in \mathfrak{I}(\mathcal{C})$ with $f \circ \gamma = \sigma_{e,d} \circ g$. So $\sigma_{w,c}$ exists and $\mathcal{C}(w, e)$ is non-empty. Since, $\sigma_{e,d} \circ \_ : \mathcal{C}(w, e) \longrightarrow \mathcal{C}(w, d)$ is a bijection, there is $h : w \longrightarrow e$ such that $\sigma_{e,d} \circ h = f \circ \sigma_{w,c}$. Since $x \sim_0 c \sim_0 w$, there is u such that there are morphisms in $\mathfrak{I}(\mathcal{C})$ from u to x and from u to w. Then $\sigma_{e,y} \circ h \circ \sigma_{u,x}$ and u are what we were looking for.
- **inductive case:** by the induction hypothesis on $(f_2, ..., f_n)$, c_1 and y, there is a morphism

$f : w \longrightarrow y$ such that $(f_2, ..., f_n) \sim_m (f)$ and there is a morphism in $\mathfrak{I}(\mathcal{C})$ from w to c_1. By the right Ore condition on $\sigma_{w,c_1} \in \mathfrak{I}(\mathcal{C})$ and f_1 there are $g : u \longrightarrow w$ and $\gamma : u \longrightarrow d_1\mathfrak{I}(\mathcal{C})$ such that $f \circ \gamma = \sigma_{w,c_1} \circ g$. As previously, we can assume that $\gamma = \sigma_{u,c_1}$. Then, the rest is similar as the previous case.

Now, it is clear that for every class α modulo $\sim_0$, $Q(R(\alpha)) = [R(\alpha)]_0 = \alpha$. Then for every class γ modulo $\sim_m$, by what we just proved, there is $f : x \longrightarrow y$ with $R([y]) = y$ and there is a morphism in $\mathfrak{I}(\mathcal{C})$ from x to $R([x])$ and $\gamma = [f]_m$. So $R(\gamma) = [R([x]) \xleftarrow{\sigma_{x,R([x])}} x \xrightarrow{f} y]$ and $Q(R(\gamma)) = [f]_m = \gamma$.

It remains to define a natural isomorphism $\tau : id \longrightarrow R \circ Q$ which will be defined by $\tau_c : c \longrightarrow R([c])$ is equal to $[c \xleftarrow{\sigma_{\alpha,c}} \alpha \xrightarrow{\sigma_{\alpha,R([c])}} R([c])]$ (such an α exists since $x \sim_0 R([c])$). This does not depend on the choice of α and is an isomorphism with inverse $[R([c]) \xleftarrow{\sigma_{\alpha,R([c])}} \alpha \xrightarrow{\sigma_{\alpha,c}} c]$. ◀

E Proof of Theorem 9

Proof. We will prove the case where A is a future deformation retract of X. So we have a continuous function $H : X \longrightarrow \mathfrak{I}(X)$ satisfying all the conditions of the definition. We will prove that $H_1 : X \longrightarrow A$ is a weak equivalence.

First, notice that the conditions imply that H_1 is a dmap with values in A. First, for $x \leq x'$, $\overrightarrow{\mathbb{P}}(H_1)_{x,x'}$ is a homotopy equivalence by lemma 12.

Now, we have to prove that $\overrightarrow{\mathbb{P}}(H_1)$ induces a functor $\overrightarrow{\pi_0}(H_1)$ from $\overrightarrow{\pi_0}(X)$ to $\overrightarrow{\pi_0}(A)$, i.e., if $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$ then $[H_1 \circ \gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$.

First, let us note some things about dihomotopies in A and X:

- two dipaths in A are dihomotopic in A iff they are dihomotopic in X: the only if is trivial because a dihomotopy in A is also a dihomotopy in X. The converse holds because a dihomotopy K in X between two dipaths γ, γ' in X induces a dihomotopy $H_1 \circ K$ in A between $H_1 \circ \gamma = \gamma$ and $H_1 \circ \gamma' = \gamma'$.
- for any pair (x, y) of points of A, $\overrightarrow{\pi_1}(A)(x, y)$ is isomorphic to $\overrightarrow{\pi_1}(X)(x, y)$: by the previous point, $\overrightarrow{\pi_1}(A)(x, y)$ embeds in $\overrightarrow{\pi_1}(X)(x, y)$. This injection is also surjective because any dipath γ in X between x and y is dihomotopic to $H_1 \circ \gamma$.

Let us prove that $\mathfrak{I}(H_1) = \{[H_1 \circ \gamma] \mid [\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))\}$ satisfies the conditions in the definition of $\mathfrak{I}(\overrightarrow{\pi_1}(A))$ and so that $\mathfrak{I}(H_1) \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(A))$:

- **$\mathfrak{I}(H_1)$ contains all identities of $\overrightarrow{\pi_1}(A)$:** those identities are the classes modulo dihomotopy of the constant paths in A. Those classes belong to $\mathfrak{I}(\overrightarrow{\pi_1}(X))$ and as H_1 is the identity on A, they also belong to $\mathfrak{I}(H_1)$
- **$\mathfrak{I}(H_1)$ is closed under composition:** easy
- **$\mathfrak{I}(H_1)$ induces isomorphisms by composition:** let γ be a dipath from x to y such that $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$ and let $z \in A$ such that $\overrightarrow{\pi_1}(A)(H_1(y), z) \neq \varnothing$ (the other case will be symmetric). We must prove that $(H_1 \circ \gamma) \star \_ : \overrightarrow{\pi_1}(X)(H_1(y), z) \longrightarrow \overrightarrow{\pi_1}(X)(H_1(x), z)$, $[\delta] \longmapsto [(H_1 \circ \gamma) \star \delta]$ is an isomorphism. The following diagram is commutative:

$$\begin{tikzcd}
\overrightarrow{\pi_1}(X)(H_1(x), z) \arrow[r, "H(x) \star _"] & \overrightarrow{\pi_1}(X)(x, z) \\
\overrightarrow{\pi_1}(X)(H_1(y), z) \arrow[r, "H(y) \star _"'] \arrow[u, "(H_1 \circ \gamma) \star _"] & \overrightarrow{\pi_1}(X)(y, z) \arrow[u, "\gamma \star _"']
\end{tikzcd}$$

because $\gamma \star H(y)$ and $H(x) \star (H_1 \circ \gamma)$ are dihomotopic, and $H(x) \star \_$ and $H(y) \star \_$ are isomorphisms because $H(x)$ and $H(y)$ belongs to $\mathfrak{I}(X)$ and $\gamma \star \_$ is an isomorphism because $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$. $(H_1 \circ \gamma) \star \_$ is, thus, an isomorphism.

- **right Ore condition:** let $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$ with γ dipath from x to y and let δ be a dipath in A from z to $H_1(y)$. Since $H(y) \in \mathfrak{I}(X)$, $[\gamma \star H(y)] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$. By the right Ore condition in $\overrightarrow{\pi_1}(X)$ on $[\gamma \star H(y)]$ and $[\delta]$ there are a dipath η in X from w to x and a dipath μ in X from w to z such that $[\mu] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$ and $\mu \star \delta$ is dihomotopic to $\eta \star \gamma \star H(y)$ and so to $\eta \star H(x) \star (H_1 \circ \gamma)$. $\eta \star H(x)$ is dihomotopic to $H(w) \star (H_1 \circ \eta)$ and since $z \in A$, μ is dihomotopic to $H(w) \star (H_1 \circ \mu)$. $H(w) \star (H_1 \circ \mu) \star \delta$ is dihomotopic to $H(w) \star (H_1 \circ \eta) \star (H_1 \circ \gamma)$. Since $H(w) \in \mathfrak{I}(X)$, $(H_1 \circ \mu) \star \delta$ is dihomotopic to $(H_1 \circ \eta) \star (H_1 \circ \gamma)$ within A and $[H_1 \circ \mu] \in \mathfrak{I}(H_1)$.
- **left Ore condition:** similar.

Now, we want to prove that the injection $\iota : A \longrightarrow X$ induces a functor $\overrightarrow{\pi_0}(\iota)$ from $\overrightarrow{\pi_0}(A)$ to $\overrightarrow{\pi_0}(X)$, i.e., $\mathfrak{I}(\overrightarrow{\pi_1}(A)) \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$. Let us assume that $H_1^{-1}(\mathfrak{I}(\overrightarrow{\pi_1}(A))) = \{[f] \mid [H_1 \circ f] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))\} \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$ (we will prove this later). We prove now that $\mathfrak{I}(\overrightarrow{\pi_1}(A)) \cup \mathfrak{I}(\overrightarrow{\pi_1}(X))$ satisfies all the conditions of $\mathfrak{I}(\overrightarrow{\pi_1}(X))$ except the closure by composition and by the previous lemma we will deduce that $\langle \mathfrak{I}(\overrightarrow{\pi_1}(A)) \cup \mathfrak{I}(\overrightarrow{\pi_1}(X)) \rangle \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$ by maximality of $\mathfrak{I}(\overrightarrow{\pi_1}(X))$ and so that $\mathfrak{I}(\overrightarrow{\pi_1}(A)) \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$. So let us prove those conditions:

- $\mathfrak{I}(\overrightarrow{\pi_1}(A))$ **induces isomorphisms by composition on the left in** X**:** let $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$ with γ dipath in A from x to y. Let $z \in X$ such that $\overrightarrow{\pi_1}(X)(y, z) \neq \varnothing$. We prove that $\gamma \star \_ : \overrightarrow{\pi_1}(X)(y, z) \longrightarrow \overrightarrow{\pi_1}(X)(x, z)$, $[\delta] \longmapsto [\gamma \star \delta]$ is an isomorphism.
  - ⋆ **injective:** let two dipaths δ and δ' from y to z such that $\gamma \star \delta$ and $\gamma \star \delta'$ are dihomotopic. Then $\gamma \star \delta \star H(z)$ and $\gamma \star \delta' \star H(z)$ are dihomotopic. So $\gamma \star (H_1 \circ \delta)$ and $\gamma \star (H_1 \circ \delta')$. But since $H_1 \circ \delta$ and $H_1 \circ \delta'$ are dipaths of A then $H_1 \circ \delta$ and $H_1 \circ \delta'$ are dihomotopic so $\delta \star H(z)$ and $\delta' \star H(z)$ are dihomotopic. Since $H(z) \in \mathfrak{I}(X)$, δ and δ' are dihomotopic.
  - ⋆ **surjective:** let δ be a dipath from x to z. Then $H_1 \circ \delta$ is a dipath from x to $H_1(z)$ dihomotopic to $\delta \star H(z)$. Since $\overrightarrow{\pi_1}(X)(y, z) \neq \varnothing$, $\overrightarrow{\pi_1}(A)(y, H_1(z)) \neq \varnothing$. So since $H_1 \circ \delta$ is in A and $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$, there is a dipath γ' from y to $H_1(z)$ such that $H_1 \circ \delta$ is dihomotopic to $\gamma \star \delta'$. Since $H(z) \in \mathfrak{I}(X)$, there is a dipath δ'' from y to z such that δ' is dihomotopic to $\delta'' \star H(z)$. So $\delta \star H(z)$ is dihomotopic to $\gamma \star \delta'' \star H(z)$. Since $H(z) \in \mathfrak{I}(X)$, δ is dihomotopic to $\gamma \star \delta''$.
- $\mathfrak{I}(\overrightarrow{\pi_1}(A))$ **induces isomorphisms by composition on the right in** X**:** similar.
- **right Ore condition with respect to** X**:** let $[\gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$ with γ dipath in A from x to y. Let δ be a dipath in X from z to y. Then δ is dihomotopic to $H(z) \star (H_1 \circ \delta)$. Since $H_1 \circ \delta$ is a dipath in A from $H_1(z)$ to y, then by the right Ore condition in $\overrightarrow{\pi_1}(A)$, there are a dipath η from w to x and μ a dipath in A from w to $H_1(z)$ with $[\mu] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$ and $\eta \star \gamma$ is dihomotopic to $\mu \star (H_1 \circ \delta)$. By the last condition of a future deformation retract, there is a dipath μ' in X from α to z with $[H_1 \circ \mu'] = [\mu]$ (and so $H_1(\alpha) = w$). $[\mu']$ belongs to $H_1^{-1}(\mathfrak{I}(\overrightarrow{\pi_1}(A))) \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$ and $\mu' \star \delta$ is dihomotopic to $\mu' \star H(z) \star (H_1 \circ \delta)$ which is dihomotopic to $H(\alpha) \star \mu \star (H_1 \circ \delta)$ which is dihomotopic to $H(\alpha) \star \eta \star \gamma$.
- **left Ore condition with respect to** X**:** similar.
- the other conditions are easy.

So now, we must prove that $H_1^{-1}(\mathfrak{I}(\overrightarrow{\pi_1}(A))) \subseteq \mathfrak{I}(\overrightarrow{\pi_1}(X))$. As previously, we prove that $H_1^{-1}(\mathfrak{I}(\overrightarrow{\pi_1}(A))) \cup \mathfrak{I}(\overrightarrow{\pi_1}(X))$ has all the condition of $\mathfrak{I}(\overrightarrow{\pi_1}(X))$ except the closure by composition. The only difficult condition to prove is the right Ore condition. Let γ be a dipath from x to y such that $[H_1 \circ \gamma] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$ and let δ be a dipath from z to y. $\delta \star H(y)$ is dihomotopic to $H(z) \star (H_1 \circ \delta)$. Moreover, by the right Ore condition in $\overrightarrow{\pi_1}(A)$ between

$H_1 \circ \gamma$ and $H_1 \circ \delta$, there are a dipath η from w to x and a dipath μ from w to $H_1(z)$ such that $[\mu] \in \mathfrak{I}(\overrightarrow{\pi_1}(A))$ and $\mu \star (H_1 \circ \delta)$ is dihomotopic to $\eta \star (H_1 \circ \gamma)$. Then, by the last condition of a future deformation retract, there is a dipath μ' from α to z such that $[H_1 \circ \mu'] = [\mu]$ (and so $H_1(\alpha) = w$). Since $H(x) \in \mathfrak{I}(X)$ then by the right Ore condition in $\overrightarrow{\pi_1}(X)$ between $H(x)$ and $H(\alpha) \star \eta$, there are a dipath ϵ from β to α with $[\epsilon] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$ and a dipath κ from β to x such that $\epsilon \star H(\alpha) \star \eta$ is dihomotopic to $\kappa \star H(x)$. Then $\kappa \star \gamma$ is dihomotopic to $\epsilon \star \mu' \star \delta$ with $[\epsilon \star \mu'] \in \langle H_1^{-1}(\mathfrak{I}(\overrightarrow{\pi_1}(A))) \cup \mathfrak{I}(\overrightarrow{\pi_1}(X))\rangle$.

Now, we have two functors $\overrightarrow{\pi_0}(H_1) : \overrightarrow{\pi_0}(X) \longrightarrow \overrightarrow{\pi_0}(A)$ and $\overrightarrow{\pi_0}(\iota) : \overrightarrow{\pi_0}(A) \longrightarrow \overrightarrow{\pi_0}(X)$. Let us prove that they form an equivalence of categories.

First, $\overrightarrow{\pi_0}(H_1) \circ \overrightarrow{\pi_0}(\iota) = id_{\overrightarrow{\pi_0}(A)}$.

Secondly, we have a natural transformation $\nu : id_{\overrightarrow{\pi_0}(X)} \longrightarrow \overrightarrow{\pi_0}(\iota) \circ \overrightarrow{\pi_0}(H_1)$ defined by $\nu_x : x \longrightarrow H_1(x) = [H(x)] \in \mathfrak{I}(\overrightarrow{\pi_1}(X))$ and so is an isomorphism in $\overrightarrow{\pi_0}(X)$. ◀

F Proof of Lemma 10

We prove the following result:

▶ **Lemma 14.** *Given two categories $\mathcal{C}$ and $\mathcal{D}$, the following assertions are equivalent:*

(i) *there are two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ and two natural isomorphisms $\epsilon : G \circ F \longrightarrow id_{\mathcal{C}}$ and $\eta : F \circ G \longrightarrow id_{\mathcal{D}}$*

(ii) *there is a fully-faithful essentially surjective functor $F : \mathcal{C} \longrightarrow \mathcal{D}$*

(iii) *there are a category $\mathcal{E}$ and a span $F : \mathcal{E} \longrightarrow \mathcal{C}$ and $G : \mathcal{E} \longrightarrow \mathcal{D}$ of fully-faithful surjective-on-objects functors*

Proof.

- the equivalence between $i)$ and $ii)$ is classic.
- $iii) \Rightarrow ii)$ is obvious because surjectivity implies essential surjectivity
- $ii) \Rightarrow iii)$ Suppose that there is a fully faithful and essentially surjective functor $F : \mathcal{C} \longrightarrow \mathcal{D}$. Let $\mathcal{E}$ be the category whose:
  - ⋆ objects are tuples (c, θ, η, d) which satisfies the requirement of essential surjectivity, i.e., $\theta : F(c) \longrightarrow d$ is inverse of $\eta : d \longrightarrow F(c)$.
  - ⋆ the Hom-set $\mathcal{E}((c, \theta, \eta, d), (c', \theta', \eta', d'))$ is $\mathcal{C}(c, c')$
  - ⋆ compositions and identities are those of $\mathcal{C}$

  Let $F_{\mathcal{C}} : \mathcal{E} \longrightarrow \mathcal{C}$ be the functor such that:
  - ⋆ $F_C(c, \theta, \eta, d) = c$
  - ⋆ $F_C((c, \theta, \eta, d), (c', \theta', \eta', d'))$:
    $\mathcal{E}((c, \theta, \eta, d), (c', \theta', \eta', d')) \longrightarrow \mathcal{C}(c, c')$ is $id_{\mathcal{C}(c,c')}$

  It is fully faithful (obvious) and surjective because $(c, id_{F(c)}, id_{F(c)}, F(c))$ is an object of $\mathcal{E}$.

  Let $F_{\mathcal{D}} : \mathcal{E} \longrightarrow \mathcal{D}$ be the functor such that:
  - ⋆ $F_D(c, \theta, \eta, d) = d$
  - ⋆ $F_D((c, \theta, \eta, d), (c', \theta', \eta', d'))$:
    $\mathcal{E}((c, \theta, \eta, d), (c', \theta', \eta', d')) \longrightarrow \mathcal{D}(d, d'')$ is the function $f \longmapsto \theta' \circ F_{c,c'}(f) \circ \eta$ (which is a bijection with inverse $g \longmapsto F^{-1}_{c,c'}(\eta' \circ g \circ \theta)$)

It is fully faithful because F is and surjective because F is essentially surjective. ◀

When $\mathcal{C}$ and $\mathcal{D}$ satisfy one of these conditions, we say that they are **equivalent** (this is actually the classical notion of equivalence of categories).

G Proof of Theorem 11

In order to prove this theorem, we have to prove several intermediary results.

We first prove that strong equivalence of partially-enriched categories is indeed an equivalence relation.

Proof. Reflexivity and symmetry are obvious. The only interesting part is transitivity, i.e., assume that we have four strong equivalences like this:

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

then we must construct two strong equivalences like this:

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X & & Z \end{tikzcd}$$

As strong equivalences are closed by composition, it is enough to prove the following. Assume that we have two strong equivalences like this:

$$\begin{tikzcd} U \arrow[dr, "\Phi"'] & & V \arrow[dl, "\Psi"] \\ & Y & \end{tikzcd}$$

then we must construct two other strong equivalences like this:

$$\begin{tikzcd} & W \arrow[dl, "\theta"'] \arrow[dr, "\eta"] & \\ U & & V \end{tikzcd}$$

Construct the partially enriched category W whose:
- objects are pairs of (u, v) with u object of U and v object of V such that $\Phi(u) = \Psi(v)$;
- its domain has $(u, v) \leq (u', v')$ iff $u \leq u'$ and $v \leq v'$;
- Hom-object $W((u, v), (u', v')) = Y(\Phi(u), \Phi(u')) = Y(\Psi(v), \Psi(v'))$ (well defined for $u \leq u'$ and $v \leq v'$);
- compositions and identities are those of Y.

Define the partially enriched functor $\theta : W \longrightarrow U$ such that:
- $\theta(u, v) = u$;
- $\theta_{(u,v),(u',v')} : Y(\Phi(u), \Phi(u')) \longrightarrow U(u, u') = \Phi^{-1}_{u,u'}$.

It is a strong equivalence:
- **fully-faithful** obvious;
- **surjective** θ is surjective because Ψ is;
- **fibrational** given $u \leq u'$ and v such that $\Phi(u) = \Psi(v)$, then $\Psi(v) = \Phi(u) \leq \Phi(u')$ and as Ψ is fibrational, there is v' such that $v \leq v'$ and $\Psi(v') = \Phi(u')$ that is (u', v') is an object of W, $\theta(u', v') = u'$ and $(u, v) \leq (u', v')$.

We construct a strong equivalence $\eta : W \longrightarrow V$ the same way. ◀

Now we have:

▶ **Lemma 15.**

i) *For all F, $Unf(F)$ is well defined and is a po-diagram.*

ii) *$Unf(F)$ extends to a functor Unf : $\boldsymbol{Diag(Ab)} \longrightarrow \boldsymbol{PoDiag(Ab)}$.*

iii) *Unf preserves open maps, i.e., if (Φ, σ) is an open map then so is $Unf(\Phi, \sigma)$.*

iv) *If we denote the injection by ι : $\boldsymbol{PoDiag(Ab)} \longrightarrow \boldsymbol{Diag(Ab)}$, there is a natural transformation $\mu : \iota \circ Unf \longrightarrow id_{\boldsymbol{Diag(Ab)}}$ such that for all F, μ_F is an open map.*

In particular, F and $Unf(F)$ are bisimilar.

Note that Unf is not a right adjoint of ι. Now, composing with μ_H and since open maps are closed by composition:

▶ **Lemma 16.** *Two diagrams F and G are bisimilar iff there are a po-diagram H and a span of open maps $F \longleftarrow H \longrightarrow G$.*

Proof of Lemma 15.

i) Easy.

ii) Given a morphism of diagrams (Φ, σ) from $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ to $G : \mathcal{D} \longrightarrow \mathbf{Ab}$, we define $Unf(\Phi, \sigma) = (Unf(\Phi), Unf(\sigma))$ with:
   - $Unf(\Phi)(f_1, ..., f_n) = (\Phi(f_1), ..., \Phi(f_n))$
   - $Unf(\sigma)_{(f_1,...,f_n)} : F(codom(f_n)) \longrightarrow G \circ \Phi(codom(f_n))$ (the codomain being equal to $\sigma_{codom(f_n)}$)

   It is easy to check that $Unf(\Phi)$ is a functor. Naturality of $Unf(\sigma)$ comes from the naturality of σ.

iii) Now assume that (Φ, σ) is open.
   - **surjectivity on objects:** Let $(g_1, ..., g_n)$ be a non-empty finite sequence of composable morphisms of $\mathcal{D}$. By surjectivity on objects of Φ, there is c_0 object of $\mathcal{C}$ such that $\Phi(c) = dom(f_1)$. Then by the lifting property of Φ, there is a morphism $f_1 : c_0 \longrightarrow c_1$ of $\mathcal{C}$ such that $\Phi(f_1) = g_1$. Then, $dom(g_2) = codom(g_1) = c_1$, then by the lifting property of Φ there is a morphism f_2 such that... By induction, we construct a non-empty finite sequence of composable morphisms of $\mathcal{C}$ such that for all i, $g_i = \Phi(f_i)$ and so $Unf(\Phi)(f_1, ..., f_n) = (g_1, ..., g_n)$.
   - **lifting property:** the proof is the same as the previous point.
   - $Unf(\sigma)$ **is a natural isomorphism:** because σ is.

iv) Let $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ be a diagram. We construct $\mu_F = (\Phi_F, id_{Unf(F)}) : Unf(F) \longrightarrow F$ this way:
   - $\Phi_F(f_1, ..., f_n) = codom(f_n)$
   - $\Phi_F(g_{n+1}, ..., g_p) = g_p \circ ... \circ g_{n+1}$

   So, $Unf(F) = F \circ \Phi_F$. Then:
   - **surjectivity on objects:** $\Phi_F(id_c) = c$
   - **lifting property:** If we have a morphism $f : codom(f_n) \longrightarrow c$ then (f) is a morphism from $(f_1, ..., f_n)$ to $(f_1, ..., f_n, f)$ such that $\Phi_F(f) = f$.
   - $id_{Unf(F)}$ **is a natural isomorphism:** OK.

   Naturality of μ is easy.

◀

Then we prove the following :

▶ **Lemma 17.**

i) *For every po-diagram F, $\mathcal{G}(F)$ is a **Ab**-partially enriched category.*
ii) *$\mathcal{G}$ extends to a functor from **PoDiag(Ab)** to **PeCat(Ab)**.*
iii) *$\mathcal{G}$ transforms open maps into strong equivalences, i.e., if $(\Phi,\sigma) : F \longrightarrow G$ is an open map between po-diagrams then $\mathcal{G}(\Phi,\sigma) : \mathcal{G}(F) \longrightarrow \mathcal{G}(G)$ is a strong equivalence.*

Proof of Lemma 17.

i) We must prove (associativity) and (unit):
- (unit) can be expressed as: for all $g \in F(c')$,

$$g + F(c \leq c')(0) = g = F(c' \leq c')(g) + 0$$

which is true because $F(c' \leq c') = id$ since F is a functor and $F(c \leq c')(0) = 0$ since $F(c \leq c')$ is a morphism of groups.
- (associativity) can be expressed as for all $g' \in F(c')$, $g'' \in F(c'')$ and $g''' \in F(c''')$:

$$F(c'' \leq c''')(F(c' \leq c'')(g') + g'') + g''' =$$

$$F(c'' \leq c''')(F(c' \leq c'')(g')) + F(c'' \leq c''')(g') + g''' =$$

$$F(c' \leq c''')(g') + F(c'' \leq c''')(g'') + g'''$$

ii) If (Φ,σ) is a morphism of po-diagrams from $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ to $G : \mathcal{D} \longrightarrow \mathbf{Ab}$, we define a partially enriched functor $\mathcal{G}(\Phi,\sigma)$ from $\mathcal{G}(F)$ to $\mathcal{G}(G)$ this way:
- the monotonous function part is Φ
- for $c \leq c'$, $\mathcal{G}(\Phi,\sigma)_{c,c'} : F(c') \longrightarrow G(\Phi(c'))$ is $\sigma_{c'}$

The first axiom of partially enriched functor comes from the naturality of σ and the second is trivial.

iii) Assume now that (Φ,σ) is open. Let us prove that $\mathcal{G}(\Phi,\sigma)$ is a strong equivalence:
- **surjectivity:** comes from the surjectivity on Φ.
- **fibrationality:** comes from the lifting property of Φ.
- **fully-faithful:** comes from the fact that σ is a natural isomorphism.

◀

We can now prove the main theorem :

Proof of Theorem 11.

$\Rightarrow$) If F and G, two po-diagrams are bisimilar then there is span of open maps whose tip is also a po-diagram. Then by Lemma 17 iii), there is a span of strong equivalences between $\mathcal{G}(F)$ and $\mathcal{G}(G)$.

$\Leftarrow$) Let $F : \mathcal{C} \longrightarrow \mathbf{Ab}$ and $G : \mathcal{D} \longrightarrow \mathbf{Ab}$ be two po-diagrams. Assume there is a span of strong equivalences between $\mathcal{G}(F)$ and $\mathcal{G}(G)$, i.e., there are a partially enriched category $\mathcal{M}$ and two strong equivalences $K : \mathcal{M} \longrightarrow \mathcal{G}(F)$ and $L : \mathcal{M} \longrightarrow \mathcal{G}(G)$. Let us start by constructing a diagram $H : \mathcal{E} \longrightarrow \mathbf{Ab}$ this way:
- $\mathcal{E}$ is the set $\{(e,e') \mid e,e' \in Ob(\mathcal{M}) \wedge e \leq e'\}$ equipped with the preorder $(e,e') \leq (e'',e''')$ iff $e = e''$ and $e' \leq e'''$.
- $H(e,e') = \mathcal{M}(e,e')$
- $H((e,e') \leq (e,e'')) : \mathcal{M}(e,e') \longrightarrow \mathcal{M}(e,e'')$ is the morphism of groups that maps $g \in \mathcal{M}(e,e')$ to $\circ_{e,e',e''}(g, 0_{\mathcal{M}(e',e'')})$

Let us prove that this is a functor:
- $H((e,e') \leq (e,e'))(g) = \circ_{e,e',e'}(g, 0_{\mathcal{M}(e',e')})$
$= \circ_{e,e',e'}(g, u_{e'}(0)) = g$ by (unit).

- $H((e,e'') \leq (e,e''')) \circ H((e,e') \leq (e,e''))(g)$
  $= H((e,e'') \leq (e,e'''))(\circ_{e,e',e''}(g, 0_{\mathcal{M}(e',e'')}))$
  $= \circ_{e',e'',e'''}(\circ_{e,e',e''}(g, 0_{\mathcal{M}(e',e'')}), 0_{\mathcal{M}(e'',e''')})$
  $= \circ_{e,e',e'''}(g, \circ_{e',e'',e'''}(0_{\mathcal{M}(e',e'')}, 0_{\mathcal{M}(e'',e''')}))$ by (associativity)
  $= \circ_{e,e',e'''}(g, 0_{\mathcal{M}(e',e''')})$ because $\circ_{e',e'',e'''}$ is a morphism of groups
  $= H((e,e') \leq (e,e'''))(g)$.

Let us define a open map (Φ, σ) from H to F as follows:

- $\Phi(e,e') = K(e')$ (which is monotonous)
- for every $e \leq e'$, $\sigma_{(e,e')} : \mathcal{M}(e,e') \longrightarrow F(K(e')) = \mathcal{G}(F)(K(e), K(e'))$ is $K_{e,e'}$

Let us prove that this is a well defined open map:

- **naturality of σ:** it can be reformulated as for every $e \leq e' \leq e''$, for every $g \in \mathcal{M}(e,e')$,
  $K_{e,e'}(H((e,e') \leq (e,e''))(g)) = K_{e,e''}(\circ_{e,e',e''}(g, 0))$
  $= \circ_{K(e),K(e'),K(e'')}(K_{e,e'}(g), K_{e,e''}(0))$ because K is a partially enriched functor
  $= \circ_{K(e),K(e'),K(e'')}(K_{e,e'}(g), 0)$ because $K_{e,e''}$ is a morphism of groups
  $= F(e' \leq e'')(K_{e,e'}(g))$ by definition of the composition in $\mathcal{G}(F)$.
- **surjectivity:** comes from the surjectivity of K
- **lifting property:** comes from the fibrational condition of K
- σ **natural isomorphism:** comes from the fully-faithfulness of K

The same way, we can construct an open map from H to G.

◀