

Model Checking MDPs with a Unique Compact Invariant Set of Distributions

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Abstract—The semantics of Markov Decision Processes (MDPs), when viewed as transformers of probability distributions, can be described as a labeled transition system over the probability distributions over the states of the MDP. The MDP can be seen as defining a set of executions, where each execution is a sequence of probability distributions. Reasoning about sequences of distributions allows one to express properties not expressible in logics like PCTL; examples include expressing bounds on transient rewards and expected values of random variables, as well as comparing the probability of being in one set of states at a given time with another set of states. With respect to such a semantics, the problem of checking that the MDP never reaches a bad distribution is undecidable [1]. In this paper, we identify a special class of MDPs called *semi-regular* MDPs that have a unique non-empty, compact, invariant set of distributions, for which we show that checking any ω -regular property is decidable. Our decidability result also implies that for semi-regular probabilistic finite automata with isolated cut-points, the emptiness problem is decidable.

Index Terms—Markov Decision Processes; Probability Distributions; Semantics; Model Checking;

I. INTRODUCTION

Discrete Time Markov Chains (DTMCs) are a convenient model to describe systems with probabilistic transitions [2]. A DTMC has finitely many states, and a transition from a state is probabilistic. Traditionally, DTMCs are viewed as defining a probability space on sequences of states, and logics that can reason about the measure of executions satisfying some modal property can be used to verify such systems [2]–[4]. When considering systems with both probabilistic and nondeterministic behavior, Markov Decision Processes (MDPs) are used [2], [3], [5]–[7]. The difference from a DTMC is that in an MDP, each state has a set of probabilistic transitions enabled. The semantics of such an MDP is then defined with respect to a scheduler that resolves the nondeterminism by picking an enabled transition for a state based on the execution thus far. Thus, with respect to a scheduler, an MDP can be seen as a potentially infinite state DTMC which defines a measure on sequences of states. Logics for MDPs allow one to reason about schedulers and measures of certain sequences of states [2].

In this paper we view the semantics of DTMCs and MDPs differently — instead of viewing them as defining measures on

executions, we view them as transformers of probability distributions. In this approach, a DTMC defines a single execution, which is a sequence of probability distributions on states; starting from the initial distribution, each subsequent distribution is applied by multiplying the transition matrix of the DTMC with the current distribution. Thus, the i th distribution in the sequence represents the probability distribution over the state space at step i . This is the approach taken in [8], where a logic for reasoning about sequences of probability distributions was also presented. This approach can be easily generalized to MDPs. The semantics of an MDP from this viewpoint is then defined using a labeled transition system, where the states are probability distributions on the states of the MDP. Labeling the states of such a transition system by propositions (defined over distributions), one can define properties about sequences of distributions using standard modal logics. Reasoning about MDPs as transformers of distributions allows one to express properties that cannot be expressed by typical specification logics that reason about the probability space of sequences of DTMC/MDP states. The propositions over distributions allow one to express bounds on transient rewards, expected values of random variables over the state space (like expected queue length or energy consumed), software performability [9] as well as compare the probability of being in one set of states after a given number of steps with that of another set of states after the same number of steps.

Model checking MDPs with respect to such a semantics is in general undecidable [1]. This is observed by reducing the emptiness problem of Probabilistic Finite Automata (PFA), which is known to be undecidable [10], to the problem of checking whether a set of distributions is reachable from the initial distribution of an MDP. Thus, one needs to restrict attention to special MDPs. This program was initiated in [1], where it was shown that certain classes of MDPs can be verified with respect to ω -regular properties, when considering special Markovian schedulers. In this paper, we consider a different class of MDPs and prove the decidability of checking certain properties; for a detailed comparison with [1] see the related work section.

An MDP \mathcal{M} is a tuple (S, μ_0, \mathcal{P}) , where S is a finite set of states, μ_0 is the initial distribution and \mathcal{P} is a finite set

of stochastic matrices $\{P_1, P_2, \dots, P_k\}$; each matrix P_i corresponds to a particular resolution of nondeterminism from each state. \mathcal{M} defines a function from sets of distributions to sets of distributions as follows: $\mathcal{M}(C) = \{\mu P_i \mid \mu \in C, 1 \leq i \leq k\}$. An *invariant set of distributions* of \mathcal{M} is a set of distributions U such that $\mathcal{M}(U) = U$. We identify a special class of MDPs, which we call *semi-regular*. A semi-regular has a *unique* nonempty, compact, invariant set of distributions: any two nonempty invariant sets of MDP which are also compact must be identical. Semi-regular MDPs define a contracting function in a complete metric space, and the uniqueness of the invariant set follows from Banach’s theorem.

Our decidability results pertain to what we call robust, semi-regular MDPs. Observe that a function λ labeling distributions by propositions defines an equivalence relation, namely one that equates all distributions that get the same label. A semi-regular MDP \mathcal{M} is said to be robust with respect to λ if every distribution in the unique compact invariant set U of \mathcal{M} , lies in the *interior* of an equivalence class defined by λ . Our main result shows that given an ω -regular specification \mathcal{A} , and a semi-regular MDP \mathcal{M} that is robust, the problem of verifying \mathcal{M} against \mathcal{A} is decidable. Our proof proceeds in two steps. First, we show that the language of labeled executions generated by any semi-regular, robust MDP is regular. The ideas used in establishing this result crucially exploit the fact that U is compact and invariant, and that the matrices of a semi-regular MDP define contracting maps. Then we show that if we make some natural assumptions about the effectiveness of the labeling function λ , an automaton recognizing this language can be constructed. These, together with classical results about Büchi automata, give us our desired decidability result.

One important consequence of our observations applies to the decidability of the emptiness problem for probabilistic finite automata (PFAs). Recall that PFAs are finite state machines that process finite words by tossing coins at each transition. The language of such a machine is defined with respect to a threshold θ . It is the collection of all words whose probability of reaching a final state (called the acceptance probability) is above θ . As mentioned before, the emptiness problem for such machines is known to be undecidable. Our results imply that the emptiness problem is decidable for semi-regular PFAs with isolated thresholds — a threshold θ is isolated if there is an ϵ such that the acceptance probability of any word is bounded away from θ by ϵ .

We motivate our work on verifying MDPs by describing a general model in drug administration, called the compartment model. We model the absorption of insulin as an MDP, semantically viewed as a transformer of distributions, and show how our decidability results can be used in drug administration.

The rest of the paper is organized as follows. We begin by comparing our work with previous literature. In Section III, we introduce our motivating example of the drug absorption of insulin. We then (in Section IV) recall basic concepts and introduce the notation that we will use in the paper. Next, in Section V, we recall Markov decision processes and give its

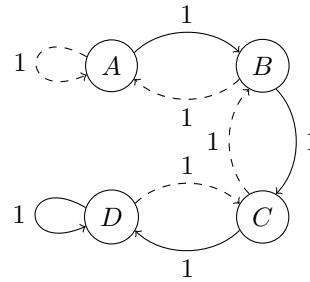


Fig. 1. MDP shown is contracting (as defined in [1]) but not semi-regular. There are two transitions from each state: a transitions are shown as solid edges, while b transitions are shown as dashed edges.

semantics as a transformer of distributions. Our special class of MDPs with a unique invariant set of distributions, namely semi-regular MDPs, are introduced in Section VI. The main decidability results are presented in Section VII. We revisit the insulin example in Section VIII, and present our conclusions in Section IX.

II. RELATED WORK

Most of the work on model checking MDPs [2], [11]–[13] has focussed on properties specified in logics such as PCTL and PCTL* that reason about the probability measure (on runs) induced by the MDP under some scheduler. Properties specified in logics like PCTL and PCTL* are incomparable to the kinds of properties considered here. On the one hand there is no way in PCTL or PCTL* to compare the probability of being in different states after the same number of steps, and on the other hand, in this paper we consider only linear time properties that do not take the branching structure of the MDP into account. For a detailed comparison see [1].

A predicate logic of probability, introduced by Beauquier et. al. [14], allows one to compare the probability of being in different states after the same number of steps. However, this logic of probability is also incomparable to the properties considered in this paper; for a detailed comparison see [1]. For this predicate logic of probability, the model checking problem has only been considered for DTMCs.

Kwon and Agha [8], [15] initiated the study of verifying sequences of probability distributions. They proposed a logic called iLTL, to express temporal properties of sequences of distributions, and presented an algorithm to model check DTMCs with respect to iLTL properties. In a companion paper [1], we began the study of model checking MDPs viewed as transformers of distributions. We showed that checking safety properties of general MDPs is undecidable, and we considered a special class of MDPs, called *contracting MDPs*, and proved the decidability of checking ω -regular properties with respect to special Markovian schedulers, called *almost acyclic*. The decidability results presented in this paper are incomparable to those in [1]. The MDPs we consider here, namely semi-regular MDPs, are more restricted than the class we considered in [1], namely contracting MDPs. To see that

semi-regular MDPs are contracting, consider a semi-regular MDP \mathcal{M} . Then, by definition, there is a k such that for any $P \in \mathcal{P}$, there is an inevitable state q such that $P^k(q', q) > 0$ for every q' . This means that P has only one closed class (namely the one containing q) and that this class is aperiodic, implying that P is contracting. Contracting MDPs, however, are a strict super-class; this is illustrated by the example shown in Figure 1 (that is, in fact, a deterministic transition system) which is contracting but not semi-regular, as on any sequence of alternating a 's and b 's of odd or even length, the states B and C end up in different states. On the other hand, in this paper we consider all schedulers as opposed to only certain Markovian schedulers. It is easy to see that any almost acyclic set of schedulers will not accept at least one scheduler. Recall that an almost acyclic set of scheduler is a set of schedulers (infinite sequence of matrices) accepted by a Büchi automaton \mathcal{A} such that there is a total order $>$ on the states of \mathcal{A} such that (a) if $q_1 \xrightarrow{P} q_2$ with $q_1 \neq q_2$ then $q_2 < q_1$, and (b) if for any q , $q \xrightarrow{P_1} q$ and $q \xrightarrow{P_2} q$ then $P_1 = P_2$. Suppose the MDP has at least two choices from each state; let us call these choices A and B . It can be easily shown that the sequence $(AB)^\omega$ is not accepted by any almost acyclic set of schedulers by induction on the total order $<$ on the states of the automaton. Thus, the collection of all schedulers is not almost acyclic. Apart from the decidability results being incomparable, the proof techniques used in the two papers are also very different. In particular, our extensive reliance on topological properties to prove decidability, is unique.

The semantics of MDPs considered here, is closely related to the model of probabilistic finite automata [16], [17]. However, there are a couple of differences between this work and PFAs — we consider labels on distributions, and infinite executions, rather than finite words over the transition labels. The undecidability of the emptiness problem of PFAs [10] implies that the verification problem for arbitrary MDPs under our semantics is undecidable (see [1]). The decidability results described in this paper imply that the emptiness problem for semi-regular PFAs with isolated cut-points is decidable. Finally, Probabilistic Büchi automata (PBA) is an extension of the PFA model to infinite words [18]. However, because of the way acceptance is defined for infinite words, their semantics is closer to the traditional semantics of MDPs, than ours.

Finally, our definition of semi-regular MDPs is inspired by the definition of regular interval Markov Chains in [19]. The model checking problem is however not considered in that paper.

III. MOTIVATION

We recall a pharmacokinetic model first presented in [1]. In Pharmacokinetics, compartment models have been commonly used as a mathematical model to describe the drug disposition changes in our body, where a compartment is a group of tissues with similar blood flow and drug affinity. The drug Absorption, Distribution, Metabolism, and Elimination (ADME) processes are explained in the compartment models as the drug concentration level changes [20].

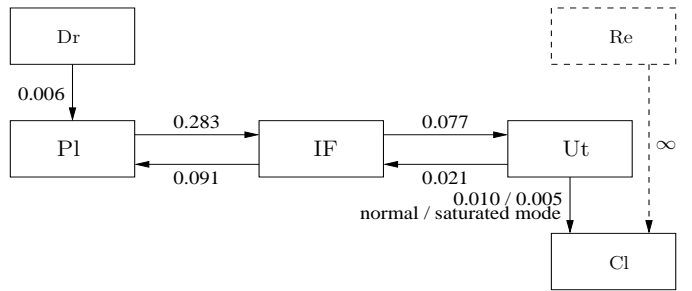


Fig. 2. A three compartment model. The unit of the rate constants is $1/\text{min}$.

In the compartment model, the amount of drug leaving from a compartment is proportional to the amount of drug in the compartment, and thus has the *memoryless property*. Thus, we can model them as Markov chains: the states are the compartments and the transition probability rates are the drug transition rates between the compartments.

Figure 2 shows a state transition diagram embedding a three compartment model of Insulin- ^{131}I [21], [22]¹. This diagram represents a Continuous Time Markov Decision Process, where the boxes are the states and the labels at the directed edges are the transition rate constants. The states Pl , IF , and Ut are the three compartments, representing plasma, interstitial fluid, and the site of utilization and degradation. Dr and Cl are the states representing the unabsorbed and the cleared drug respectively. Re state is introduced to use physical units in the specification: if α (g) of drug is initially taken, then we can put α in Dr state and $1 - \alpha$ in Re state so that the elements of the initial probability distribution adds up to one. Observe that the differential equations describing the drug disposition changes are identical to the equations for the transitions of probability distributions (Kolmogorov forward equation). Hence we can interchange the amount in probability and the amount of drug in a physical unit. Note also that, we can always make $0 \leq \alpha \leq 1$ by choosing a large unit. The interpretation of the rate from Re to Cl being infinite is that the remaining drug in this state is instantly cleared without interacting with the rest of the system. In the corresponding MDP model this transition probability is set to 1.

In the pharmaceutical analysis, one needs to account for the presence of multimodal behavior. For example, if there are more drugs than the enzymes can process, the drug elimination process shows a saturated behavior. In the saturated mode, the drug elimination rate becomes slow, and thus the drug concentration level can reach its toxic limit if this mode is not considered. The saturated mode has often been modeled as a nonlinear kinetics called *Michaelie-Menten* kinetics. However, this model can be simplified to a linear model when the drug concentration level is large compared to the Michaelie constant [20]. Thus, this multimodal behavior can be described by an MDP. In Figure 2, the rates from Ut state to Cl state

¹Although we explain this three compartment model of Insulin in this paper, our technique can generically be applied to other models as well.

$$N = \begin{bmatrix} 0.94 & 0.02434 & 0.02567 & 0.00798 & 0.00024 & 0 \\ 0.00 & 0.20724 & 0.48298 & 0.29624 & 0.01353 & 0 \\ 0.00 & 0.15531 & 0.42549 & 0.39520 & 0.02400 & 0 \\ 0.00 & 0.02598 & 0.10778 & 0.77854 & 0.08770 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.94 & 0.02435 & 0.02568 & 0.00809 & 0.00012 & 0 \\ 0.00 & 0.20728 & 0.48329 & 0.30257 & 0.00686 & 0 \\ 0.00 & 0.15540 & 0.42612 & 0.40626 & 0.01221 & 0 \\ 0.00 & 0.02653 & 0.11080 & 0.81776 & 0.04491 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \end{bmatrix}$$

Fig. 3. Stochastic matrices for the normal mode ($N = e^{10 \cdot R_n}$) and the saturated mode ($S = e^{10 \cdot R_s}$), when the states are ordered as $Dr, Pl, IF, Ut, Cl,$ and Re and R_n and R_s are the infinitesimal generator matrices of the compartment model described in Figure 2.

show the drug elimination rates in the normal mode and in the saturated mode.

Now, we build an MDP model from the state transition diagram of Figure 2 by setting the sampling time to 10 (min). Let R_n and R_s be the infinitesimal generator matrices for the normal mode and the saturated mode respectively. Then, their corresponding probability transition matrices $N = e^{10 \cdot R_n}$ and $S = e^{10 \cdot R_s}$ are shown in Figure 3.

The amount of the drug in the compartments satisfies the following difference equation:

$$\mu_{t+1} = \mu_t \cdot P_{t+1},$$

where μ_t is a probability distribution at time t and P_t is either N or S . Given an initial dose μ_0 , the distribution of the drug in the compartments at time t is:

$$\mu_t = \mu_0 \cdot P_1 \cdot P_2 \cdots P_t.$$

Observe that given an initial dose the drug disposition changes depend on the choices of the stochastic matrices.

There are several conditions that must be considered when administering a drug. These conditions include:

- 1) The drug concentration level should never exceed its *Minimum Toxic Concentration* (MTC).
- 2) To be effective, the drug concentration level should eventually be above its *Minimum Effective Concentration* (MEC).
- 3) All the administered drug should be cleared from the body eventually.

Specifically, if we assume that the MTC is 2.1 ($\mu g/ml$), the MEC is 1.4 ($\mu g/ml$), the body weight is 60 (kg), and the volume of Ut compartment is 15.8% of the body weight, then the amount of drug in the Ut compartment at MTC and MEC are $mtc = 0.020$ and $mec = 0.013(g)$ respectively.

To describe these conditions, we begin by defining propositions *effective*, *nontoxic*, and *cleared* over the space of probability distributions. A probability distribution μ is labeled

effective if $\mu(Ut) > mec$ or labeled *nontoxic* if $\mu(Ut) < mtc$ or labeled *cleared* if $\mu(Ut) < \epsilon$, where ϵ is a small value.

Using \square operator (*always*) the first condition can be simply written as $\square nontoxic$. Regarding the second condition, let the required active duration be at least two sampling periods which can be expressed as $(effective \wedge X effective \wedge XX effective)$. Since this condition is only required to occur eventually and not immediately, we can write the second condition as $\diamond (effective \wedge X effective \wedge XX effective)$, where \diamond stands for “eventually”. Finally, the third condition can be expressed as $\diamond \square cleared$.

Combining these three conditions together, the entire specification can be written as:

$$\begin{aligned}
\psi &= \square(nontoxic) \wedge \diamond \square(cleared) \\
&\wedge \diamond(effective \wedge X effective \wedge XX effective)).
\end{aligned}$$

IV. PRELIMINARIES

We introduce the basic notations and definitions that we will use in this paper.

A. Languages

Sequences and Words. Let Σ be a finite set. $|\Sigma|$ will denote the cardinality of Σ . Let $\eta = s_0, s_1, \dots$ be a possibly infinite sequence/word over Σ . The length of η , denoted as $|\eta|$, is defined to be the number of elements in η , if η is finite, and ω otherwise. Σ^* denotes the set of finite sequences/words, Σ^+ the set of finite sequences/words of length ≥ 1 and Σ^ω denotes the set of infinite sequences/words. If β is a finite sequence, and η is either a finite or an infinite sequence then $\beta\eta$ denotes the concatenation of the two sequences in that order. For integers i and j such that $0 \leq i \leq j < |\eta|$, $\eta[i, j]$ denotes the (finite) sequence/word s_i, \dots, s_j and the element $\eta[i]$ denotes the element s_i . A finite prefix of η is any $\eta[0, j]$ for $j < |\eta|$.

Languages and Automata. A Büchi automaton over Σ is $\mathcal{A} = (Q, \delta, q_0, F)$, where Q is a finite set of states, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation. A run of \mathcal{A} on an infinite input string β is a sequence $\rho = r_0, r_1, \dots$ over Q such that $r_0 = q_0$ and for each i , $(r_i, \beta[i], r_{i+1}) \in \delta$. We say a run ρ is accepting if there is some $q \in F$ that appears infinitely often in ρ . A word β is accepted by \mathcal{A} if β has an accepting run, and $L(\mathcal{A})$ is the collection of all infinite words over Σ that are accepted by \mathcal{A} . We say that a language $L \subseteq \Sigma^\omega$ is ω -regular if there is a Büchi automaton \mathcal{A} such that $L = L(\mathcal{A})$.

B. Distributions and Stochastic Matrices

Distributions. A probability distribution over S is $\mu : S \rightarrow [0, 1]$ such that $\sum_{q \in S} \mu(q) = 1$. We will denote $\text{Dist}(S)$ to be the set of all distributions over S .

Stochastic Matrices. A stochastic matrix over a set of states S is a matrix $P : S \times S \rightarrow [0, 1]$ such that $\forall s \in S. \sum_{s' \in S} P(s, s') = 1$. $\text{Mat}_{=1}(S)$ will denote the set of all stochastic matrices over the set S . For $P \in \text{stocmat}S$ and $\mu \in \text{Dist}(S)$, μP denotes the distribution, given by $\mu P(q) = \sum_{q' \in S} \mu(q') P(q', q)$.

C. Metric Spaces

We recall some standard definitions from topology [23]. Recall that a metric space $\mathcal{M} = (M, d)$ is a nonempty set M equipped with a distance metric $d : M \times M \rightarrow \mathbb{R}_{\geq 0}$ with the following properties: for every $x, y, z \in M$, $d(x, y) = 0$ iff $x = y$; $d(x, y) = d(y, x)$; and $d(x, z) \leq d(x, y) + d(y, z)$.

Open and Closed Sets. An *open ball* of radius $\epsilon > 0$ around an element x in M is defined as the set $\mathbb{B}(x; \epsilon) = \{y \in M \mid d(x, y) < \epsilon\}$. Similarly, an open ball of radius $\epsilon > 0$ around a set $U \subseteq M$ is defined as the set $\mathbb{B}(U; \epsilon) = \cup_{x \in U} \mathbb{B}(x; \epsilon)$. A subset O of M is called *open* if for every $x \in O$ there exists an $\epsilon > 0$ such that $\mathbb{B}(x; \epsilon)$ is contained in O . The complement of an open set is said to be *closed*. The *interior* of a set U , $\text{int}(U)$ is the largest open set contained in U ; the existence of such a set follows from the fact that open sets are closed under arbitrary unions. An infinite sequence $\sigma = x_1, x_2, \dots$ of elements from M *converges* to x if for every ϵ , there is a N_ϵ such that for every $n \geq N_\epsilon$, $d(x_n, x) < \epsilon$; x is said to be the *limit* of the sequence σ . A function $f : M \rightarrow M$ is said to be *continuous* if for every open set O , $f^{-1}(O) = \{x \mid f(x) \in O\}$ is open. Given a function $f : M \rightarrow M$, $f^n : M \rightarrow M$ denotes the n -fold composition of f with itself.

Cauchy Sequences, limit points, complete metric spaces. Given a metric space $\mathcal{M} = (M, d)$, an infinite sequence x_1, x_2, \dots of elements from M is said to be *Cauchy* if for every $\epsilon > 0$ there is an N_ϵ such that for all $m, n > N_\epsilon$, $d(x_m, x_n) < \epsilon$. An element x is said to be the limit of the Cauchy sequence x_1, x_2, \dots if for every $\epsilon > 0$ there is an N_ϵ such that for all $m > N_\epsilon$, $d(x_m, x) < \epsilon$. A metric space $\mathcal{M} = (M, d)$ is said to be *complete* if every Cauchy sequence has a limit in M .

Compact Sets. An *open cover* of $U \subseteq M$ is a collection of open sets $\mathcal{O} = \{O_i \mid i \in I\}$ such that $U \subseteq \cup_{i \in I} O_i$. A set $C \subseteq M$ is said to be *compact* if every open cover $\mathcal{O} = \{O_i \mid i \in I\}$ of C has a *finite sub-cover*, i.e., there is a finite subset $J \subseteq I$ such that $C \subseteq \cup_{i \in J} O_i$. Observe that every finite set is compact trivially. The collection of nonempty compact spaces of \mathcal{M} will be denoted by $\mathcal{C}(\mathcal{M})$.

We recall the following properties of compact sets. Given a continuous function f and a compact set C , the set $f(C) = \{f(x) \mid x \in C\}$ is also compact. The collection of compact sets is closed under finite union and intersection with closed sets. The intersection of a nonempty collection of compact sets is also compact. A collection $\mathcal{X} = \{X_i \mid i \in I, X_i \subseteq S\}$ of sets (non necessarily compact) is said to have the *finite intersection property* if every finite sub-collection $\{X_1, X_2, \dots, X_n\}$ of \mathcal{X} has a nonempty intersection. A set C is compact iff every nonempty collection of closed subsets of C having the finite intersection property has a nonempty intersection. Every sequence of elements of a compact set C has a Cauchy subsequence which converges to an element in C . Thus, every compact metric space is complete. Finally, we have the following property.

Lemma IV.1. *If C is a compact set that is contained in an open set O then there is an $\epsilon > 0$ such that $\mathbb{B}(C; \epsilon) \subseteq O$.*

Proof: We first show that if C is compact and A is a closed set such that $C \cap A = \emptyset$, then there is a $\epsilon > 0$ such that $d(x, y) > \epsilon$ for each $x \in C, y \in A$.

Suppose (for contradiction) that for each $\epsilon > 0$ there are $x_\epsilon \in C$ and $y_\epsilon \in A$ such that $d(x_\epsilon, y_\epsilon) \leq \epsilon$. Then, there are $\{c_n\}_{n \in \mathbb{N}}$ in C and $\{a_n\}_{n \in \mathbb{N}}$ in A such that $d(c_n, a_n) < 1/n$. Now since C is compact, there is a convergent subsequence c_{n_k} that converges to $c \in C$. It is easy to show that the subsequence a_{n_k} also converges to c . As A is a closed set, this implies that $c \in A$. Therefore, $c \in C \cap A$ contradicting the emptiness of $C \cap A$.

Now, the result follows by taking A to be the complement of the set O . ■

Metric Space on Compact Sets. Given nonempty sets $X, Y \subseteq M$ the *Hausdorff distance* between X and Y is defined as follows

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}.$$

Let $\mathcal{C}(\mathcal{M})$ be the collection of all nonempty compact sets in the metric space $\mathcal{M} = (M, d)$. Then $(\mathcal{C}(\mathcal{M}), d_H)$ forms a metric space, i.e., the Hausdorff distance defines a metric space over the nonempty compact sets.

Contracting maps and fixpoints. A mapping $f : M \rightarrow M$ is said to *contracting* if there is an α , with $0 \leq \alpha < 1$, such that for all $x, y \in M$, $d(f(x), f(y)) < \alpha d(x, y)$. The fixed point of a function $f : M \rightarrow M$ is an element x such that $f(x) = x$. Banach's fixed point theorem is an important result on the existence of unique fixed points.

Theorem IV.2 (Banach's fixed point theorem). *Let (M, d) be a nonempty, complete metric space. Let $f : M \rightarrow M$ and natural number $k > 0$ be such that f^k a contracting map. Then f has a unique fixed point $x^* \in M$. Furthermore, x^* is the limit of the (Cauchy) sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_{i+1} = f(x_i)$ and x_0 is an arbitrary element of M .*

Metric space on Distributions. We will consider the standard L^1 metric on distributions. For a finite set S , let $\mathbb{D}(S) = (\text{Dist}(S), d)$ where

$$d(\mu, \nu) = \frac{1}{2} \sum_{x \in S} |\mu(x) - \nu(x)| = \max_{A \subseteq S} |\mu(A) - \nu(A)|.$$

In this space, the set $\text{Dist}(S)$ is a compact set. Furthermore every stochastic matrix defines a nonexpanding map.

Proposition IV.3. *Every stochastic matrix $P \in \text{Mat}_{=1}(S)$ is nonexpanding, i.e., for all $\mu, \nu \in S$, $d(\mu P, \nu P) \leq d(\mu, \nu)$.*

We will also consider the space $\mathbb{C}(S) = (\mathcal{C}(\mathbb{D}(S)), d_H)$, the Hausdorff metric space on the nonempty compact sets of $\mathbb{D}(S)$. Since $\mathbb{C}(S)$ is also compact, it is a complete metric space.

Proposition IV.4. *The space $\mathbb{C}(S)$ is a complete metric space.*

V. MDPs AS TRANSFORMERS OF DISTRIBUTIONS

Markov decision processes (MDPs) are a natural representation for modeling and analysis of systems with both probabilistic and nondeterministic behavior.

Definition An Markov decision process (MDP) is a tuple $\mathcal{M} = (S, \mu_0, \mathcal{P})$, where S is a (finite) set of states, $\mu_0 \in \text{Dist}(S)$ is the initial distribution, $\mathcal{P} \subseteq \text{Mat}_{=1}(S)$ is a finite nonempty set of stochastic matrices (also call *transition matrices*). \mathcal{M} is said to be a *Markov chain* (MC) if \mathcal{P} contains exactly one element.

Remark Our definition of an MDP differs from the more commonly used definition of an MDP. An MDP \mathcal{M} is defined as a tuple (S, μ_0, Steps) , where S and μ_0 have the same interpretation as ours and *Steps* maps each state in S to a finite subset of $\text{Dist}(S)$. In this definition, a transition from state s involves (nondeterministically) picking a distribution $\mu \in \text{Steps}(s)$ and then transitions to state s' with probability $\mu(s')$. It is easy to see that our definition of MDP is more general when compared to the standard definition.

We will define the semantics of an MDP as a transition system. This semantics differs from the traditional semantics of an MDP. Traditionally, the informal semantics of an MDP is taken as follows. The semantics is defined using *schedulers* which resolve nondeterminism: given a sequence of states (the *history* of the state sequence visited) ending in s , a scheduler picks a matrix $P \in \mathcal{P}$, and then transitions to state s' with probability $P(s, s')$. This yields a probability measure on executions, once the nondeterministic choices are resolved through an adversary.

Semantics. Instead of the usual semantics of an MDP as a probability measure on distributions once the nondeterminism is resolved, we consider MDPs as transformers of probability distributions. We give this semantics by defining a labeled transition system. Given $\mathcal{M} = (S, \mu_0, \mathcal{P})$, the transition system associated with \mathcal{M} is $\mathcal{T}(\mathcal{M}) = (Q, \rightarrow, \mu_0)$, where $Q = \text{Dist}(S)$, and $\rightarrow \subseteq Q \times \mathcal{P} \times Q$ is defined as $(\mu, P, \nu) \in \rightarrow$ iff $\nu = \mu P$. From now on, we will say $\mu \xrightarrow{P} \nu$ instead of $(\mu, P, \nu) \in \rightarrow$.

An execution of \mathcal{M} starting from a distribution μ is a sequence ν_0, ν_1, \dots such that $\nu_0 = \mu$ and for all i , $\nu_i \xrightarrow{P} \nu_{i+1}$, for some $P \in \mathcal{P}$. The collection of all executions starting from μ will be denoted by $\mathcal{E}(\mathcal{M}, \mu)$. An execution of \mathcal{M} is an execution starting from the initial distribution μ_0 , and the set of executions of \mathcal{M} is denoted as $\mathcal{E}(\mathcal{M})$.

Labeling. Given a finite set of *labels* Σ , a Σ -labeling function for distributions over S is $\lambda : \text{Dist}(S) \rightarrow 2^\Sigma$. A labeling function λ defines a partition $\pi_\lambda = \{U_K \mid K \subseteq \Sigma\}$ on $\text{Dist}(S)$, where $\mu \in U_K$ iff $\lambda(\mu) = K$. The *interior* of partition π_λ is $\text{int}(\pi_\lambda) = \cup_{K \subseteq \Sigma} \text{int}(U_K)$.

The labeling function λ can be extended to an execution as follows. Given $\rho = \nu_0, \nu_1, \dots \in \text{Dist}(S)^\omega$, $\lambda(\rho) = \lambda(\nu_0), \lambda(\nu_1), \dots \in (2^\Sigma)^\omega$. The language $L_\lambda(\mathcal{M}, \mu)$ of labeled sequences of MDP \mathcal{M} starting from μ with respect to λ , is

defined as

$$L_\lambda(\mathcal{M}, \mu) = \{\lambda(\rho) \mid \rho \in \mathcal{E}(\mathcal{M}, \mu)\}.$$

Finally the language of \mathcal{M} with respect to λ is given by $L_\lambda(\mathcal{M}) = L_\lambda(\mathcal{M}, \mu_0)$.

Model Checking Problem. The verification question that we will consider in this paper is as follows. Given an MDP \mathcal{M} , a Σ -labeling function λ and a language $L \subseteq (2^\Sigma)^\omega$, decide if $L_\lambda(\mathcal{M}) \subseteq L$. This problem is undecidable for general MDPs [1]. In this paper we identify sufficient conditions on the MDP and the labeling function for which this question becomes decidable for any ω -regular language L .

VI. MDPs WITH A UNIQUE COMPACT INVARIANT SET OF DISTRIBUTIONS

As mentioned in the previous section, the problem of verifying MDPs (viewed as transformers of distributions) against regular specifications was shown to be undecidable in [1]. In this section, we will identify a special subclass of MDPs, called *semi-regular MDPs*, for which the model checking problem is shown to be decidable in the next section.

We start by defining some notations. Given a set $\mathcal{P} \subseteq \text{Mat}_{=1}(S)$, we can define an operator $\mathcal{P} : 2^{\text{Dist}(S)} \rightarrow 2^{\text{Dist}(S)}$ where $\mathcal{P}(W) = \{\mu P \mid \mu \in W \text{ and } P \in \mathcal{P}\}$.² We will often write $W\mathcal{P}$ to mean $\mathcal{P}(W)$.

Given a natural number $n > 0$, the n -th power of \mathcal{P} is given by $\mathcal{P}^n = \{P_1 P_2 \dots P_n \mid P_i \in \mathcal{P}\}$. Thus, \mathcal{P}^n contains all matrices obtained by taking the product of n matrices (possibly same) from \mathcal{P} . Observe that the resulting operator $\mathcal{P}^n : 2^{\text{Dist}(S)} \rightarrow 2^{\text{Dist}(S)}$ is the n -fold composition of the operator \mathcal{P} .

We begin by defining what we mean by an invariant set of distributions, before defining semi-regular MDPs.

A. Invariant Set of Distributions

One of the most important concepts in the study of Markov chains is the notion of an invariant/stationary distribution. Given a Markov chain with transition matrix P , an invariant/stationary distribution is a probability distribution μ such that $\mu = \mu P$. When considering MDPs, since we have a set of transition matrices, we need to consider the notion of a *set of invariant distributions*.

Formally, a set $U \subseteq \text{Dist}(S)$ is an invariant set of MDP $\mathcal{M} = (S, \mu_0, \mathcal{P})$ iff U is a fixed point of the operator \mathcal{P} , i.e., iff $U\mathcal{P} = U$. Thus an invariant set is closed under multiplication with the set of possible transition matrices. Note, it is not the case that each distribution in an invariant set is invariant for some matrix in \mathcal{P} . Please observe that the emptyset is trivially an invariant set. Tarski's fixed point theorem implies that the set of fixpoints of \mathcal{P} ordered by set inclusion forms a complete lattice [24]. We show that the greatest fixed point of the operator \mathcal{P} is a nonempty compact set.

²The overloading of \mathcal{P} as both a set of stochastic matrices and as an operator on powerset of distributions is deliberate.

Proposition VI.1. *Given an MDP $\mathcal{M} = (S, \mu_0, \mathcal{P})$, there is a nonempty compact set C^∞ such that $C^\infty \mathcal{P} = C^\infty$. Furthermore, for every set U such that $U\mathcal{P} = U$, $U \subseteq C^\infty$.*

Proof: Let $C^\infty = \bigcap_{i \in \mathbb{N}} (\text{Dist}(S)\mathcal{P}^i)$. We have:

- 1) If U is such that $U\mathcal{P} = U$, then $U \subseteq C^\infty$.
- 2) For each i , $\text{Dist}(S)\mathcal{P}^{i+1} = \bigcup_{P \in \mathcal{P}} P(\text{Dist}(S)\mathcal{P}^i)$. Since compact sets are closed under finite union, images of continuous maps and arbitrary nonempty intersection, C^∞ is compact.
- 3) The sequence $\text{Dist}(S)\mathcal{P}^i$ is a sequence monotonically decreasing of nonempty closed sets. By the finite intersection property, we have that C^∞ is nonempty.
- 4) $C^\infty \mathcal{P} \subseteq C^\infty$.

Hence, we will be done if we can show that $C^\infty \subseteq C^\infty \mathcal{P}$.

Pick $x \in C^\infty$ and fix it. Consider the sequence of sets $W_i = (\text{Dist}(S)\mathcal{P}^i) \cap \mathcal{P}^{-1}(\{x\})$ where $\mathcal{P}^{-1}(\{x\}) = \bigcup_{P \in \mathcal{P}} P^{-1}(\{x\})$. Observe that W_i is a monotonically decreasing sequence of closed sets. As $x \in C^\infty$, W_i is nonempty for every i , and so satisfies the finite intersection property. Since the metric space $\mathbb{D}(S)$ is compact, we get $\bigcap_{i \in \mathbb{N}} W_i$ is nonempty. Pick $y \in \bigcap_{i \in \mathbb{N}} W_i$. The proposition follows by observing that $y \in \bigcap_{i \in \mathbb{N}} W_i$ implies that $y \in \mathcal{P}^{-1}(\{x\})$ and $y \in C^\infty$. ■

The number of different compact invariant sets, and their structure, plays an important role in solving the model checking problem.

B. Semi-regular MDPs

We will define and study the properties of a class of MDPs which will have a unique (nonempty) compact invariant set of distributions.

Definition Given an MDP $\mathcal{M} = (S, \mu_0, \mathcal{P})$, a set of states $\text{Inevt} \subseteq S$ is set to be *inevitable in \mathcal{M}* if there exists a natural number $\ell > 0$ such that for each $P \in \mathcal{P}^\ell$ there is a $s_P \in \text{Inevt}$ such that $P(s, s_P) > 0$ for all $s \in Q$.

An MDP $\mathcal{M} = (S, \mu_0, \mathcal{P})$ is said to be *semi-regular* if there is a set $\text{Inevt} \subseteq S$ inevitable in \mathcal{M} .

Semi-regular MDPs generalize the notion of regular Markov Chains [25]. An important property of regular Markov Chains is that they have unique stationary distributions. One method of proving this fact is to show that Markov chains define contracting maps on the set of distributions under the L^1 metric. We extend this observation to semi-regular MDPs. In particular, by using a proof similar to the one used for regular Markov Chains, we show that there is a number k such that each matrix $P \in \mathcal{P}^k$ is a contracting map on the space of distributions.

Proposition VI.2. *Let $\mathcal{M} = (S, \mu_0, \mathcal{P})$ be a semi-regular MDP. There is an $0 \leq \alpha < 1$ and a natural number $k > 0$ such that for all $P \in \mathcal{P}^k$,*

$$\forall \mu, \nu \in \text{Dist}(S). d(\mu P, \nu P) < \alpha d(\mu, \nu).$$

Proof: As \mathcal{M} is semi-regular, there is a set Inevt inevitable in \mathcal{M} . Pick ℓ such that

$$\forall P \in \mathcal{P}^\ell. \exists s_P \in S. \forall s \in S. P(s, s_P) > 0.$$

Fix $P \in \mathcal{P}^\ell$ and let s_P be such that $\forall s \in S. P(s, s_P) > 0$. Let

$$\epsilon_P = \min_{s \in S} P(s, s_P) \text{ and } \alpha_P = 1 - \epsilon_P.$$

Note that $0 < \epsilon_P \leq 1, 0 \leq \alpha_P < 1$ and $P(s, s_P) \geq \epsilon_P$ for each $s \in S$. We will now show that for all $\mu, \nu \in \text{Dist}(S)$, $d(\mu P, \nu P) < \alpha_P d(\mu, \nu)$. We have

$$\begin{aligned} 2d(\mu P, \nu P) &= |\sum_{s \in S} (\mu(s)P(s, s_P) - \nu(s)P(s, s_P))| + \\ &\quad \sum_{s' \in S \setminus s_P} |\sum_{s \in S} (\mu(s) - \nu(s))P(s, s')| \\ &= |\sum_{s \in S} (\mu(s)(P(s, s_P) - \epsilon_P + \epsilon_P) - \\ &\quad \nu(s)(P(s, s_P) - \epsilon_P + \epsilon_P))| + \\ &\quad \sum_{s' \in S \setminus s_P} |\sum_{s \in S} (\mu(s) - \nu(s))P(s, s')| \\ &= |\sum_{s \in S} (\mu(s)(P(s, s_P) - \epsilon_P) - \\ &\quad \nu(s)(P(s, s_P) - \epsilon_P))| + \\ &\quad \sum_{s' \in S \setminus s_P} |\sum_{s \in S} (\mu(s) - \nu(s))P(s, s')| \\ &\leq \sum_{s \in S} |\mu(s) - \nu(s)| (P(s, s_P) - \epsilon_P) + \\ &\quad \sum_{s' \in S \setminus s_P} \sum_{s \in S} |\mu(s) - \nu(s)| P(s, s') \\ &\leq \sum_{s \in S} |\mu(s) - \nu(s)| (\sum_{s' \in S} P(s, s') - \epsilon_P) \\ &\leq \sum_{s \in S} |\mu(s) - \nu(s)| (1 - \epsilon_P) \\ &\leq 2\alpha_P d(\mu, \nu) \end{aligned}$$

The result now follows by letting $\alpha = \max_{P \in \mathcal{P}} \alpha_P$. ■

For each natural number $k > 0$, observe that the operator \mathcal{P}^k maps nonempty compact sets to nonempty compact sets. In other words, for every nonempty compact set C , $C\mathcal{P}^k$ is nonempty and compact. Thus, the map $C \mapsto C\mathcal{P}^k$, for any k , is a map from the the space $\mathbb{C}(S)$ of nonempty compact sets of distributions to itself. Our next observation shows that when \mathcal{M} is semi-regular, this map is contracting.

Proposition VI.3. *Let $\mathcal{M} = (S, \mu_0, \mathcal{P})$ be a semi-regular MDP. There is a number k such that the mapping on $\mathbb{C}(S)$ given by $C \mapsto C\mathcal{P}^k$ is contracting. In other words, there is an α , such that $0 \leq \alpha < 1$, and*

$$d_H(C_1\mathcal{P}^k, C_2\mathcal{P}^k) < \alpha d_H(C_1, C_2).$$

Proof: Let $k > 0$ and $0 \leq \alpha < 1$ be such that for all $P \in \mathcal{P}^k$,

$$\forall \mu, \nu \in \text{Dist}(S). d(\mu P, \nu P) < \alpha d(\mu, \nu).$$

We have

$$\begin{aligned} d_H(C_1\mathcal{P}^k, C_2\mathcal{P}^k) &= \\ &\max \left\{ \sup_{\mu \in C_1, P_1 \in \mathcal{P}^k} \inf_{\nu \in C_2, P_2 \in \mathcal{P}^k} d(\mu P_1, \nu P_2), \right. \\ &\quad \left. \sup_{\nu \in C_2, P_2 \in \mathcal{P}^k} \inf_{\mu \in C_1, P_1 \in \mathcal{P}^k} d(\mu P_1, \nu P_2) \right\} \end{aligned}$$

Now observe that

$$\begin{aligned} & \sup_{\mu \in C_1, P_1 \in \mathcal{P}^k} \inf_{\nu \in C_2, P_2 \in \mathcal{P}^k} d(\mu P_1, \nu P_2) \\ & \leq \sup_{P_1 \in \mathcal{P}^k} \sup_{\mu \in C_1} \inf_{\nu \in C_2} d(\mu P_1, \nu P_1) \\ & \leq \sup_{\mu \in C_1} \inf_{\nu \in C_2} \alpha \cdot d(\mu, \nu) \\ & \leq \alpha \cdot d_H(C_1, C_2). \end{aligned}$$

Similarly, the other term will also be bounded by $\alpha \cdot d_H(C_1, C_2)$. Thus, $d_H(C_1 \mathcal{P}^k, C_2 \mathcal{P}^k) \leq \alpha \cdot d_H(C_1, C_2)$. ■

Recall that the space $\mathbb{C}(S)$ is complete (see Proposition IV.4). We use Banach's fixpoint theorem to conclude the uniqueness of nonempty compact invariant sets.

Theorem VI.4. *Let $\mathcal{M} = (S, \mu_0, \mathcal{P})$ be a semi-regular MDP. Then there is a unique compact, nonempty invariant set U of \mathcal{M} . Furthermore, if C is a (nonempty) compact set of distributions, then the sequence $\{C \mathcal{P}^k\}_{k \in \mathbb{N}}$ converges in space $\mathbb{C}(S)$ to U .*

Remark Please note that the above theorem only guarantees that any two nonempty compact invariant sets of an semi-regular MDP \mathcal{M} are identical. There might be other invariant sets of \mathcal{M} which are not compact.

C. Robust Semi-regular MDPs

We conclude this section by defining a notion of robustness that will play an important role in the decidability result. For a semi-regular MDP $\mathcal{M} = (S, \mu_0, \mathcal{P})$, let $U(\mathcal{M})$ denote the unique nonempty compact invariant set. Recall that a Σ -labeling function λ defines a partition π_λ on the collection $\text{Dist}(S)$. Robustness is defined as follows.

Definition Let $\mathcal{M} = (S, \mu_0, \mathcal{P})$ be a semi-regular MDP with $U(\mathcal{M})$ being the unique nonempty compact invariant set of distributions. \mathcal{M} is said to be *robust* with respect to a Σ -labeling function λ if $U(\mathcal{M}) \subseteq \text{int}(\pi_\lambda)$.

In other words, \mathcal{M} is robust if none of the distributions in the invariant set lie on the boundary of any of the partitions of π_λ . Thus, slight changes in the labeling function λ will not change the labels of the invariant set in a robust MDP.

VII. MODEL CHECKING ROBUST, SEMI-REGULAR MDPs

In this section we present our main decidability result. We begin by observing that for a semi-regular MDP \mathcal{M} that is robust with respect to a Σ -labeling function λ , the language of labeled executions $L_\lambda(\mathcal{M})$ is ω -regular. We then show that under some effectiveness assumptions on the labeling function λ , we can in fact construct an automaton recognizing the language $L_\lambda(\mathcal{M})$. These two facts allow us to conclude that checking the correctness of such MDPs against regular specifications is decidable. We conclude the section with applications of this result to the emptiness checking problem for probabilistic finite automata with isolated cut-points.

A. Decidability Result

We start by showing that the language of executions of a semi-regular MDP is ω -regular.

Theorem VII.1. *Let $\mathcal{M} = (S, \mu_0, \mathcal{P})$ be a semi-regular MDP that is robust with respect to the Σ -labeling function λ . The language $L_\lambda(\mathcal{M})$ is ω -regular.*

Proof: Consider the equivalence relation \equiv defined on $\text{Dist}(S)$ as follows.

$$\mu \equiv \nu \quad \text{iff} \quad L_\lambda(\mathcal{M}, \mu) = L_\lambda(\mathcal{M}, \nu).$$

The crux of this proof is to show that \equiv has finitely many equivalence classes. Observe that since $L_\lambda(\mathcal{M})$ is a safety language,³ proving that \equiv has finite index establishes the regularity of $L_\lambda(\mathcal{M})$.

Let U be the unique, nonempty, compact, invariant set of \mathcal{M} . By the definition of robustness we know that $U \subseteq \text{int}(\pi_\lambda)$. Since U is compact and $\text{int}(\pi_\lambda)$ is an open set, fix $\epsilon > 0$ to be such that $\mathbb{B}(U; \epsilon) \subseteq \text{int}(\pi_\lambda)$; note, here we are taking the ball in the space $\mathbb{D}(S)$ and not $\mathbb{C}(S)$. The existence of such an ϵ is guaranteed by Lemma IV.1. Consider the following sequence of nonempty compact sets — $C_0 = \text{Dist}(S)$, and $C_{i+1} = C_i \mathcal{P}$. Since this sequence converges to U (Theorem VI.4), it follows that there is an N such that for all $i \geq N$, $d_H(C_i, U) < \epsilon$. Observe that by our choice of N and ϵ we have

- 1) $\mathbb{B}(U; \epsilon) \subseteq \text{int}(\pi_\lambda)$, and
- 2) $\mathbb{B}(U; \epsilon) \supseteq \cup_{i \geq N} C_i$

where these balls are taken in the space $\mathbb{D}(S)$.

Given a set X let us denote by \equiv_X the equivalence \equiv restricted to set X . Using this notation, observe that to show that \equiv has finite index, all we need to show is that \equiv_{C_i} has finite index, for all i . This will be accomplished by first showing that $\equiv_{\mathbb{B}(U; \epsilon)}$ has finite index, and then inductively establishing that \equiv_{C_i} , for $i < N$, has finite index.

Let us assume (to be proved later) that $\equiv_{\mathbb{B}(U; \epsilon)}$ has finite index. We will use this to show that \equiv_{C_i} , for $i < N$, has finite index. Consider $\mu, \nu \in C_{N-1}$. Observe that if $\lambda(\mu) = \lambda(\nu)$, and for every $P \in \mathcal{P}$, $\mu P \equiv_{\mathbb{B}(U; \epsilon)} \nu P$ then $\mu \equiv_{C_{N-1}} \nu$. Given that \mathcal{P} is finite, we can conclude that $\equiv_{C_{N-1}}$ has finite index. Inductively, for $\mu, \nu \in C_i$, $i < N-2$, we have if $\lambda(\mu) = \lambda(\nu)$, and for every $P \in \mathcal{P}$, $\mu P \equiv_{C_{i+1}} \nu P$ then $\mu \equiv_{C_i} \nu$. Thus, inductively, we can conclude that \equiv_{C_i} , for $i < N$ are all of finite index.

So the crux of the proof is in establishing that $\equiv_{\mathbb{B}(U; \epsilon)}$ is of finite index. In what follows, we will make use of the following claim.

Claim 1: Let $\mathcal{R} = \{R_i \mid i \in I\}$ be a collection of subsets of distributions over S such that for all $i, \mu, \nu \in R_i$ implies $\lambda(\mu) = \lambda(\nu)$. Suppose for every $i \in I$ and $P \in \mathcal{P}$, there is a $j \in I$ such that $R_i P = \{\mu P \mid \mu \in R_i\} \subseteq R_j$. Then each R_i is contained in an equivalence class of \equiv .

Claim 1 is a straightforward consequence of the fact that $\cup_{i \in I} (R_i \times R_i)$ is a bisimulation. The next important

³A language $L \subseteq \Sigma^\omega$ is a safety language iff for any $\alpha \in \Sigma^\omega$ if every prefix of α is the prefix of some string in L then $\alpha \in L$.

observation we make is the following.

Claim 2: Let $\mu \in U$ and $\nu \in \mathbb{B}(\mu; \epsilon)$. Then $\mu \equiv \nu$.

Proof of Claim 2: Observe that, since P is a stochastic matrix, every $P \in \mathcal{P}$ defines a nonexpanding map. In other words, for any μ_1, μ_2 , $d(\mu_1 P, \mu_2 P) \leq d(\mu_1, \mu_2)$. Thus, for any $\mu' \in U$, $\mathbb{B}(\mu'; \epsilon)P \subseteq \mathbb{B}(\mu'P; \epsilon)$. Now, since μ' is in the invariant set, $\mu'P$ is also in U . Moreover, since $\mathbb{B}(U; \epsilon) \subseteq \text{int}(\pi_\lambda)$, it follows that all distributions in $\mathbb{B}(\mu'; \epsilon)$, for $\mu' \in U$, have the same labels. Thus, the collection $\{\mathbb{B}(\mu'; \epsilon) \mid \mu' \in U\}$ satisfies the conditions in Claim 1, yielding the desired observation.

Let E be an equivalence class of \equiv_U . The following sequence of observations completes the proof.

- (a) \equiv_U is of finite index. This is seen as follows. The collection $\mathcal{B} = \{\mathbb{B}(\mu'; \epsilon) \mid \mu' \in U\}$ is an open cover of U , and (by Claim 2) each set in \mathcal{B} belongs to the same equivalence class of \equiv . Now, since U is compact, there is a finite sub-cover of \mathcal{B} , and the size of this sub-cover gives a bound on the number of equivalence classes of \equiv_U .
- (b) For every $\mu, \nu \in \mathbb{B}(E; \epsilon)$, $\mu \equiv \nu$. This is just an immediate consequence of Claim 2.
- (c) Now, by part (b), $\equiv_{\mathbb{B}(U; \epsilon)}$ has the same number of equivalence classes as \equiv_U , which by part (a) is finite. ■

Having established the regularity of $L_\lambda(\mathcal{M})$, a natural question to ask is if it can be effectively constructed. For this we need to make some assumptions about the effectiveness of the labeling set and the set of matrices \mathcal{P} that define \mathcal{M} . We will assume that the MDP $\mathcal{M} = (S, \mu_0, \mathcal{P})$ is such that for every $s \in S$, $\mu_0(s)$ is a rational number and every matrix $P \in \mathcal{P}$ has rational entries; we will call such an MDP as *having rational entries*.

The model checking algorithm and the labeling function all need a representation for sets of distributions. Thus, it is useful to introduce the abstract notion of a symbolic representation of sets of distributions.

Definition A *symbolic representation* for distributions is a family $\mathcal{R} = \{R_i \mid i \in \mathbb{N}\}$ of subsets $R_i \subseteq \text{Dist}(S)$ such that each R_i has a finite representation $\text{rep}(R_i)$. A symbolic representation \mathcal{R} is *effective* iff the following conditions hold.

- 1) \mathcal{R} is closed under all Boolean operations and their representations can be effectively computed. That is, given $\text{rep}(R_i)$ and $\text{rep}(R_j)$, we can compute $\text{rep}(R_i \cap R_j)$, $\text{rep}(R_i \cup R_j)$ and $\text{rep}(\overline{R_i})$.
- 2) Given $P \in \text{Mat}_{=1}(S)$ with rational entries, and $\text{rep}(R_i)$, the set $P^{-1}R_i = \{\mu \mid \mu P \in R_i\}$ is in \mathcal{R} and $\text{rep}(P^{-1}R_i)$ can be effectively computed.
- 3) Given $\text{rep}(R_i)$ and $\text{rep}(R_j)$ the following questions are decidable: $R_i \subseteq R_j$ and $R_i = \emptyset$
- 4) Given $\mu \in \text{Dist}(S)$ such that $\mu(s)$ is rational for all $s \in S$, and $\text{rep}(R)$, determining if $\mu \in R$ is decidable.

The effectiveness requirements are the usual ones imposed on representations to carry out symbolic model checking. In this context, one symbolic representation that is effective is the first order theory of real-closed fields [26].

We are now ready to present the main decidability result of the paper. We will say that a Σ -labeling function λ is *represented using a symbolic representation* \mathcal{R} iff for every $K \subseteq \Sigma$, $\lambda^{-1}(K) = \{\mu \in \text{Dist}(S) \mid \lambda(\mu) = K\}$ belongs to \mathcal{R} .

Theorem VII.2. *Let \mathcal{M} be a semi-regular MDP having rational entries. Let λ be a Σ -labeling function represented using an effective symbolic representation \mathcal{R} . If \mathcal{M} is robust with respect to λ then one can effectively construct a Büchi automaton \mathcal{A} such that $L(\mathcal{A}) = L_\lambda(\mathcal{M})$.*

Proof: Recall the equivalence \equiv on $\text{Dist}(S)$ defined in the proof of Theorem VII.1 as $\mu \equiv \nu$ iff $L_\lambda(\mathcal{M}, \mu) = L_\lambda(\mathcal{M}, \nu)$. From the proof of Theorem VII.1, it follows that \equiv has finite index. Now we can run a partition refinement algorithm starting from the partition π_λ . The effectiveness assumptions on the symbolic representation ensures that the partition refinement algorithm can be carried out. Thus, \equiv can be computed, and the automaton \mathcal{A} constructed. ■

We use the main theorem to conclude the decidability of the model checking question.

Corollary VII.3. *Let \mathcal{A} be a Büchi automaton over alphabet 2^Σ . Let \mathcal{M} be a semi-regular MDP with rational entries that is robust with respect to a Σ -labeling function λ . Let λ be represented in an effective symbolic representation. Then, the problem of checking if $L_\lambda(\mathcal{M}) \subseteq L(\mathcal{A})$ is decidable.*

Proof: From Theorem VII.2, we can construct the automaton \mathcal{B} such that $L(\mathcal{B}) = L_\lambda(\mathcal{M})$. The result then follows from the fact that language containment between Büchi automata is decidable. ■

B. Emptiness of PFAs

Probabilistic finite automata [16] (PFA) are finite automata on finite words that can toss coins while making transitions. Formally, a PFA $\mathcal{A} = (Q, \Delta, \{P_a\}_{a \in \Delta}, q_0, F)$, where Q is a finite set of states, Δ the input alphabet, $P_a \in \text{Mat}_{=1}(Q)$ for each $a \in \Delta$, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is the set of final states. Informally, the automaton on input $w = a_1 \cdots a_n$, starts at the initial state q_0 and takes transitions according to P_{a_i} at step i . We will now define its behavior formally.

Let $\delta(q) \in \text{Dist}(Q)$ be the distribution that assigns probability 1 to q and probability 0 to all other states. The *probability of accepting* word $w = a_1 \cdots a_n$, denoted as $\text{acc}_{\mathcal{A}}(w)$, is given by $\sum_{q \in F} \mu(q)$, where $\mu = \delta(q_0)P_{a_1} \cdots P_{a_n}$. Given a threshold $0 \leq \theta \leq 1$, the language accepted by \mathcal{A} is given by $L_\theta(\mathcal{A}) = \{w \in \Delta^* \mid \text{acc}_{\mathcal{A}}(w) \geq \theta\}$. We say a threshold θ is *isolated* if there exists $\epsilon > 0$ such that for all $w \in \Delta^*$, $|\text{acc}_{\mathcal{A}}(w) - \theta| > \epsilon$. One of the most celebrated results about PFAs establishes the undecidability of the emptiness problem; this is formally stated next.

Theorem VII.4 (Paz [17], Condon-Lipton [10]). *Given a PFA \mathcal{A} and threshold θ , the problem of determining if $L_\theta(\mathcal{A})$ is empty is undecidable.*

Our results identify an important subclass of PFAs with isolated cut-points, for which the emptiness problem is decidable. This is the formal content of the next theorem.

Theorem VII.5. *Let $\mathcal{A} = (Q, \Delta, \{P_a\}_{a \in \Delta}, q_0, F)$ be semi-regular; i.e., the MDP $\mathcal{M} = (Q, \delta(q_0), \{P_a\}_{a \in \Delta})$ is semi-regular. Suppose θ is an isolated cut-point for \mathcal{A} . Then the problem of determining if $L_\theta(\mathcal{A}) = \emptyset$ is decidable.*

Proof: The result essentially follows from Corollary VII.3. Take $\Sigma = \{f\}$ and Σ -labeling function λ such that $\lambda(\mu) = \{f\}$ iff $\sum_{q \in F} \mu(q) \geq \theta$. Observe that the fact that θ is isolated ensures that the MDP \mathcal{M} is robust with respect to λ , and the result follows from the decidability of the model checking problem for robust, semi-regular MDPs. ■

VIII. ANALYZING THE COMPARTMENT MODEL OF INSULIN-¹³¹I

In this section, we show that the MDP $\mathcal{M} = (\{Dr, Pl, IF, Ut, Cl, Re\}, \mu_0, \{N, S\})$ corresponding to compartment model described in Section III is semi-regular and is robust with respect to a Σ -labeling function λ where Σ contains the labels *effective*, *nontoxic* and *cleared*, as defined earlier. Note that, both the matrices S and N have the same closed class $\{Cl\}$. It is also trivial to see that there exists some l such that from every state $s \in \{Dr, Pl, IF, Ut, Cl, Re\}$ there is nonzero probability of reaching the state Cl , no matter what sequence of matrices is chosen. Hence the set $\{Cl\}$ is inevitable in \mathcal{M} . Thus, by definition, MDP \mathcal{M} is semi-regular. We observe that the invariant/stationary set of distributions of this special semi-regular MDP contains only a single distribution μ_l which is a unit distribution at state Cl . In other words, in the long run, all the drug will end up in the compartment Cl irrespective of the mode. Consider the set of distributions \mathcal{D} such that $\mathcal{D} = \{\mu \mid \lambda(\mu) = \{nontoxic, cleared\}\}$. Since the invariant distribution μ_l of \mathcal{M} is in the interior of the set \mathcal{D} , the MDP \mathcal{M} is robust with respect to Σ -labeling function λ . Thus by Corollary VII.3 model checking \mathcal{M} against any Büchi specification \mathcal{B} is decidable. The MDP model satisfies all the the properties of interest listed in Section III.

IX. CONCLUSIONS

We identified a new class of MDPs called semi-regular MDPs that have a unique compact invariant set of distributions. We showed that for this class of MDPs, under some robustness and effectiveness assumptions, the problem of checking if all executions (as sequences of distributions) belong to some ω -regular specification is decidable. Our decidability result also establishes the decidability of the emptiness problem for semi-regular PFAs with isolated cut-points.

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