Bounded Parikh Automata

Michaël Cadilhac[1] Alain Finkel[2] Pierre McKenzie[1]

[1] DIRO, Université de Montréal [2] LSV, ENS Cachan & CNRS

{cadilhac, mckenzie}@iro.umontreal.ca finkel@lsv.ens-cachan.fr

The Parikh finite word automaton model (PA) was introduced and studied by Klaedtke and Rueß [18]. Here, by means of related models, it is shown that the bounded languages recognized by PA are the same as those recognized by deterministic PA. Moreover, this class of languages is the class of bounded languages whose set of iterations is semilinear.

1 Introduction

Motivation. Adding features to finite automata in order to capture situations beyond regularity has been fruitful to many areas of research. Such features include making the state sets infinite, adding power to the logical characterizations, having the automata operate on infinite domains rather than finite alphabets, adding stack-like mechanisms, etc. (See, e.g., [11, 3, 17, 1].) Model checking and complexity theory below NC^2 are areas that have benefited from an approach of this type (e.g., [19, 22]). In such areas, *determinism* plays a key role and is usually synonymous with a clear understanding of the situation at hand, yet often comes at the expense of other properties, such as expressiveness. Thus, cases where determinism can be achieved without sacrificing other properties are of particular interest.

Context. Klaedtke and Rueß introduced the Parikh automaton (PA) as an extension of the finite automaton [18]. A PA is a pair (A,C) where C is a semilinear subset of $\mathbb{N}^d$ and A is a finite automaton over $(\Sigma \times D)$ for Σ a finite alphabet and D a finite subset of $\mathbb{N}^d$. The PA accepts the word $w_1 \cdots w_n \in \Sigma^*$ if A accepts a word $(w_1,\overline{v_1})\cdots(w_n,\overline{v_n})$ such that $\sum \overline{v_i} \in C$. Klaedtke and Rueß used the PA to characterize an extension of (existential) monadic second-order logic in which the cardinality of sets expressed by second-order variables is available. To use PA as symbolic representations for model checking, the closure under the Boolean operations is needed; unfortunately, PA are not closed under complement. Moreover, although they allow for great expressiveness, they are not determinizable.

Bounded and semilinear languages. Bounded languages were defined by Ginsburg and Spanier in 1964 [13] and intensively studied in the sixties. Recently, they played a new role in the theory of acceleration in regular model checking [10, 4]. A language $L \subseteq \Sigma^*$ is *bounded* if there exist words $w_1, w_2, \ldots, w_n \in \Sigma^*$ such that $L \subseteq w_1^* w_2^* \cdots w_n^*$. Bounded context-free languages received much attention thanks to better decidability properties than context-free languages [12] (e.g., inclusion between two context-free languages is decidable if one of them is bounded, while it is undecidable in the general case). Moreover, given a context-free language it is possible to decide whether it is bounded [13]. Context-free languages have a semilinear Parikh image [21]. Connecting semilinearity and boundedness, the class BSL of bounded languages $L \subseteq w_1^* \cdots w_n^*$, for which $\{(i_1,\ldots,i_n) \mid w_1^{i_1} \cdots w_n^{i_n} \in L\}$ is a semilinear set, has been studied intensively (e.g., [13, 12, 16, 6]).

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Our contribution. We study PA whose language is bounded. Our main result is that bounded PA languages are also accepted by deterministic PA, and that they correspond exactly to BSL. Moreover, we give a precise deterministic PA form into which every such language can be cast, thus relating our findings to another model in the literature, CQDD [4]. To the best of our knowledge, this is the first characterization of BSL by means of a deterministic model of one-way automata.

2 Preliminaries

We write $\mathbb{N}$ for the nonnegative integers and $\mathbb{N}^+$ for $\mathbb{N} \setminus \{0\}$. Let $d > 0$ be an integer. Vectors in $\mathbb{N}^d$ are noted with a bar on top, e.g., $\overline{v}$ whose elements are $v_1, \ldots, v_d$. We write $\overline{e_i} \in \{0,1\}^d$ for the vector having a 1 only in position i. We view $\mathbb{N}^d$ as the additive monoid $(\mathbb{N}^d, +)$. For a monoid $(M, \cdot)$ and $S \subseteq M$, we write S^* for the monoid generated by S, i.e., the smallest submonoid of $(M, \cdot)$ containing S. A subset C of $\mathbb{N}^d$ is *linear* if there exist $\overline{c} \in \mathbb{N}^d$ and a finite $P \subseteq \mathbb{N}^d$ such that $C = \overline{c} + P^*$. The subset C is said to be *semilinear* if it is a finite union of linear sets. Semilinear sets are the sets expressible by a quantifier-free first-order formula ϕ which uses the function symbol $+$, the congruence relations $\equiv_i$, for $i \geq 2$, and the order relation $<$ (see, e.g., [9]). More precisely, a subset C of $\mathbb{N}^d$ is semilinear iff there is such a formula with d free variables, with $(x_1, \ldots, x_d) \in C \Leftrightarrow \mathbb{N} \models \phi(x_1, \ldots, x_d)$, where $\mathbb{N}$ is the standard model of arithmetic.

Let $\Sigma = \{a_1, \ldots, a_n\}$ be an (ordered) alphabet, and write ε for the empty word. The *Parikh image* is the morphism $\Phi \colon \Sigma^* \to \mathbb{N}^n$ defined by $\Phi(a_i) = \overline{e_i}$, for $1 \leq i \leq n$. A language $L \subseteq \Sigma^*$ is said to be *semilinear* if $\Phi(L) = \{\Phi(w) \mid w \in L\}$ is semilinear.

A language $L \subseteq \Sigma^*$ is *bounded* [13] if there exist $n > 0$ and a sequence of words $w_1, \ldots, w_n \in \Sigma^+$, which we call *a socle of* L, such that $L \subseteq w_1^* \cdots w_n^*$. The *iteration set* of L w.r.t. this socle is defined as $\mathsf{Iter}_{(w_1,\ldots,w_n)}(L) = \{(i_1, \ldots, i_n) \in \mathbb{N}^n \mid w_1^{i_1} \cdots w_n^{i_n} \in L\}$. BOUNDED stands for the class of bounded languages. We denote by BSL the class of *bounded semilinear languages*, defined as the class of languages L for which there exists a socle $w_1, \ldots, w_n$ such that $\mathsf{Iter}_{(w_1,\ldots,w_n)}(L)$ is semilinear; in particular, the Parikh image of a language in BSL is semilinear.

We then fix our notation about automata. An automaton is a quintuple $A = (Q, \Sigma, \delta, q_0, F)$ where Q is the finite set of states, Σ is an alphabet, $\delta \subseteq Q \times \Sigma \times Q$ is the set of transitions, $q_0 \in Q$ is the initial state and $F \subseteq Q$ are the final states. For a transition $t \in \delta$, where $t = (q, a, q')$, we define $\mathsf{From}(t) = q$ and $\mathsf{To}(t) = q'$. Moreover, we define $\mu_A \colon \delta^* \to \Sigma^*$ to be the morphism given by $\mu_A(t) = a$, and we write μ when A is clear from the context. A *path* on A is a word $\pi = t_1 \cdots t_n \in \delta^*$ such that $\mathsf{To}(t_i) = \mathsf{From}(t_{i+1})$ for $1 \leq i < n$; we extend From and To to paths, letting $\mathsf{From}(\pi) = \mathsf{From}(t_1)$ and $\mathsf{To}(\pi) = \mathsf{To}(t_n)$. We say that $\mu(\pi)$ is the *label* of π. A path π is said to be *accepting* if $\mathsf{From}(\pi) = q_0$ and $\mathsf{To}(\pi) \in F$; we write $\mathsf{Run}(A)$ for the language over δ of accepting paths on A. We write $L(A)$ for the language of A, i.e., the labels of the accepting paths. The automaton A is said to be *deterministic* if $(p, a, q), (p, a, q') \in \delta$ implies $q = q'$. An ε-automaton is an automaton $A = (Q, \Sigma, \delta, q_0, F)$ as above, except with $\delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ so that in particular μ_A becomes an erasing morphism.

Following [4, 2, 8], we say that an automaton is *flat* if it consists of a set of states $q_0, \ldots, q_n$ such that q_0 is initial, q_n is final, there is one and only one transition between q_i and q_{i+1}, and there may exist, at most, a path of fresh states from q_j to q_i iff $i \leq j$ and no such path exists with one of its ends between i and j. Note that no nested loop is allowed in a flat automaton, and that the language of a flat automaton is bounded. Flat automata are called *restricted simple* in [4].

Let Σ and T be two alphabets. Let A be an automaton over the alphabet $(\Sigma \cup \{\varepsilon\}) \times (\mathrm{T} \cup \{\varepsilon\})$, where the concatenation is seen as $(u_1, v_1).(u_2, v_2) = (u_1 u_2, v_1 v_2)$. Then A defines the *rational transduction* τ_A

from languages L on Σ to languages on T given by $\tau_A(L) = \{v \in \mathrm{T}^* \mid (\exists u \in L)[(u,v) \in L(A)]\}$. Closure under rational transduction for a class $\mathscr{C}$ is the property that for any language $L \in \mathscr{C}$ and any automaton A, $\tau_A(L) \in \mathscr{C}$. We say that τ_A is a *deterministic* rational transduction if A is deterministic with respect to the first component of its labels, i.e., if $(p,(a,b),q)$ and $(p,(a,b'),q')$ are transitions of A, then $b = b'$ and $q = q'$.

3 Parikh automata and constrained automata

The following notations will be used in defining Parikh finite word automata (PA) formally. Let Σ be an alphabet, $d \in \mathbb{N}^+$, and D a finite subset of $\mathbb{N}^d$. Following [18], let $\Psi\colon (\Sigma \times D)^* \to \Sigma^*$ and $\tilde{\Phi}\colon (\Sigma \times D)^* \to \mathbb{N}^d$ be two morphisms defined, for $\ell = (a,\overline{v}) \in \Sigma \times D$, by $\Psi(\ell) = a$ and $\tilde{\Phi}(\ell) = \overline{v}$. The function Ψ is called the *projection on* Σ and the function $\tilde{\Phi}$ is called the *extended Parikh image*. As an example, for a word $\omega \in \{(a_i,\overline{e_i}) \mid 1 \le i \le n\}^*$, the value of $\tilde{\Phi}(\omega)$ is the Parikh image of $\Psi(\omega)$.

Definition 1 (Parikh automaton [18])**.** Let Σ be an alphabet, $d \in \mathbb{N}^+$, and D a finite subset of $\mathbb{N}^d$. A *Parikh automaton (PA)* of dimension d over $\Sigma \times D$ is a pair (A,C) where A is a finite automaton over $\Sigma \times D$, and $C \subseteq \mathbb{N}^d$ is a semilinear set. The PA language is $L(A,C) = \{\Psi(\omega) \mid \omega \in L(A) \wedge \tilde{\Phi}(\omega) \in C\}$.

The PA is said to be *deterministic (DetPA)* if for every state q of A and every $a \in \Sigma$, there exists at most one pair $(q',\overline{v})$ with q' a state and $\overline{v} \in D$ such that $(q,(a,\overline{v}),q')$ is a transition of A. The PA is said to be *flat* if A is flat. We write $\mathscr{L}_{\mathrm{PA}}$ (resp. $\mathscr{L}_{\mathrm{DetPA}}$) for the class of languages recognized by PA (resp. DetPA).

In [5], PA are characterized by the following simpler model:

Definition 2 (Constrained automaton [5])**.** A *constrained automaton (CA)* over an alphabet Σ is a pair (A,C) where A is a finite automaton over Σ with d transitions, and $C \subseteq \mathbb{N}^d$ is a semilinear set. Its language is $L(A,C) = \{\mu(\pi) \mid \pi \in \mathsf{Run}(A) \wedge \Phi(\pi) \in C\}$.

The CA is said to be *deterministic (DetCA)* if A is deterministic. An ε-CA is defined as a CA except that A is an ε-automaton. Finally, the CA is said to be *flat* if A is flat.

Theorem 1 ([5])**.** *CA and ε-CA define the same class of languages, and the following are equivalent for any $L \subseteq \Sigma^*$:*

(i) $L \in \mathscr{L}_{\mathrm{PA}}$ (resp. $\in \mathscr{L}_{\mathrm{DetPA}}$);

(ii) L is the language of an ε-CA (resp. deterministic CA).

4 Bounded Parikh automata

Let $\mathscr{L}_{\mathrm{BoundedPA}}$ be the set $\mathscr{L}_{\mathrm{PA}} \cap \mathrm{BOUNDED}$ of bounded PA languages, and similarly let $\mathscr{L}_{\mathrm{BoundedDetPA}}$ be $\mathscr{L}_{\mathrm{DetPA}} \cap \mathrm{BOUNDED}$.

Theorem 2 below characterizes $\mathscr{L}_{\mathrm{BoundedPA}}$ as the class BSL of bounded semilinear languages. In one direction of the proof, given $L \in \mathrm{BSL}$, an ε-CA (A,C) for L is constructed. We describe this simple construction here and, for future reference, we will call (A,C) the *canonical ε-CA* of L.

Let $L \in \mathrm{BSL}$, there exists a socle $w_1,\ldots,w_n$ of L (minimum under some ordering for uniqueness) such that $E = \mathsf{Iter}_{(w_1,\ldots,w_n)}(L)$ is a semilinear set. Informally, the automaton A will consist of n disjoint cycles labeled $w_1,\ldots,w_n$ and traversed at their origins by a single ε-labeled path leading to a unique final state. Then C will be defined to monitor $(\#t_1,\ldots,\#t_n)$ in accordance with E, where t_i is the first transition of the cycle for w_i and $\#t_i$ is the number of occurrences of t_i in a run of A. Formally, let $k_j = \sum_{1\le i\le j} |w_i|$, $0 \le j \le n$, and set $Q = \{1,\ldots,k_n\}$. Then A is the ε-automaton (Q,Σ,δ,q_0,F) where

$q_0 = 1$, $F = \{k_{n-1}+1\}$ and for any $1 \le i < n$, there is a transition $(k_{i-1}+1, \varepsilon, k_i+1)$ and for any $1 \le i \le n$ a cycle $t_i\rho_i$ labeled w_i on state $k_{i-1}+1$, where t_i is a transition and ρ_i a path. Then $C \subseteq \mathbb{N}^{|\delta|}$ is the semilinear set defined by $(\#t_1, \ldots, \#t_2, \ldots, \ldots, \ldots, \#t_n, \ldots) \in C$ iff $(\#t_1, \#t_2, \ldots, \#t_n) \in E$.

Theorem 2. $\mathscr{L}_{\text{BoundedPA}} = \text{BSL}$.

Proof. ($\mathscr{L}_{\text{BoundedPA}} \subseteq \text{BSL}$): Let $L \subseteq \Sigma^*$ be a bounded language of $\mathscr{L}_{\text{PA}}$, and $w_1, \ldots, w_n$ be a socle of L. Define $E = \mathsf{Iter}_{(w_1,\ldots,w_n)}(L)$. Let $T = \{a_1, \ldots, a_n\}$ be a fresh alphabet, and let $h\colon T^* \to \Sigma^*$ be the morphism defined by $h(a_i) = w_i$. Then the language $L' = h^{-1}(L) \cap (a_1^* \cdots a_n^*)$ is in $\mathscr{L}_{\text{PA}}$ by closure of $\mathscr{L}_{\text{PA}}$ under inverse morphisms and intersection [18]. But $\Phi(L') = E$, and as any language of $\mathscr{L}_{\text{PA}}$ is semilinear [18], E is semilinear. Thus the iteration set E of the bounded language L with respect to its socle $w_1, \ldots, w_n$ is semilinear, and this is the meaning of L belonging to BSL.

($\text{BSL} \subseteq \mathscr{L}_{\text{BoundedPA}}$): Let $L \in \text{BSL}$. Of course $L \in \text{BOUNDED}$. We leave out the simple proof that L equals the language of its "canonical ε-CA" constructed prior to Theorem 2. Since ε-CA and PA capture the same languages by Theorem 1, $L \in \mathscr{L}_{\text{PA}}$. □

Theorem 2 and the known closure properties of BOUNDED and $\mathscr{L}_{\text{PA}}$ imply:

Proposition 3. BSL *is closed under union, intersection, concatenation.*

5 Bounded Parikh automata are determinizable

Parikh automata cannot be made deterministic in general. Indeed, Klaedtke and Rueß [18] have shown that $\mathscr{L}_{\text{DetPA}}$ is closed under complement while $\mathscr{L}_{\text{PA}}$ is not, so that $\mathscr{L}_{\text{DetPA}} \subsetneq \mathscr{L}_{\text{PA}}$, and [5] further exhibits languages witnessing the separation. In this section, we show that PA can be determinized when their language is bounded:

Theorem 4. *Let $L \in \mathscr{L}_{\text{BoundedPA}}$. Then L is the union of the languages of flat DetCA (i.e., of DetCA with flat underlying automata as defined in Section 2). In particular, as $\mathscr{L}_{\text{BoundedDetPA}}$ is closed under union, $\mathscr{L}_{\text{BoundedPA}} = \mathscr{L}_{\text{BoundedDetPA}}$.*

Proof. Let $L \in \mathscr{L}_{\text{BoundedPA}}$. By Theorem 2, $L \in \text{BSL}$. Recall the canonical ε-CA of a BSL language constructed prior to Theorem 2. Inspection of the canonical ε-CA (A, C) of L reveals a crucial property which we will call "constraint-determinism," namely, the property that no two paths π_1 and π_2 in $\mathsf{Run}(A)$ for which $\mu_A(\pi_1) = \mu_A(\pi_2)$ can be distinguished by C (i.e., the property that for any two paths π_1 and π_2 in $\mathsf{Run}(A)$ for which $\mu_A(\pi_1) = \mu_A(\pi_2)$, $\Phi(\pi_1) \in C$ iff $\Phi(\pi_2) \in C$). To see this property, note that if two accepting paths π_1 and π_2 in the ε-CA have the same label w, then π_1 and π_2 describe two ways to iterate the words in the socle of L. As the semilinear set C describes *all* possible ways to iterate these words to get a specific label, $\Phi(\pi_1) \in C$ iff $\Phi(\pi_2) \in C$.

We will complete the proof in two steps. First, we will show (Lemma 5) that *any* ε-CA having the constraint-determinism property can be simulated by a suitable deterministic extension of the PA model, to be defined next. Second (Lemma 9), we will show that any bounded language accepted by such a suitable deterministic extension of the PA model is a finite union of languages of flat DetCA, thus concluding the proof. □

Our PA extension leading to Lemma 5 hinges on associating affine functions (rather than vectors) to PA transitions. In the following, we consider the vectors in $\mathbb{N}^d$ to be *column* vectors. Let $d > 0$. A function $f\colon \mathbb{N}^d \to \mathbb{N}^d$ is a (total and positive) *affine function* of dimension d if there exist a matrix

$M \in \mathbb{N}^{d\times d}$ and $\overline{v} \in \mathbb{N}^d$ such that for any $\overline{x} \in \mathbb{N}^d$, $f(\overline{x}) = M.\overline{x} + \overline{v}$. We note $f = (M, \overline{v})$ and write $\mathscr{F}_d$ for the set of such functions; we view $\mathscr{F}_d$ as the monoid $(\mathscr{F}_d, \diamond)$ with $(f \diamond g)(\overline{x}) = g(f(\overline{x}))$.

Intuitively, an affine Parikh automaton will be defined as a finite automaton that also operates on a tuple of counters. Each transition of the automaton will blindly apply an affine transformation to the tuple of counters. A word w will be deemed accepted by the affine Parikh automaton iff some run of the finite automaton accepting w also has the cumulative effect of transforming the tuple of counters, initially $\overline{0}$, to a tuple belonging to a prescribed semilinear set.

Definition 3 (Affine Parikh automaton, introduced in [5])**.** An *affine Parikh automaton (APA)* of dimension d is a triple (A, U, C) where A is an automaton with transition set δ, U is a morphism from δ^* to $\mathscr{F}_d$ and $C \subseteq \mathbb{N}^d$ is a semilinear set; recall that U need only be defined on δ. The language of the APA is $L(A, U, C) = \{\mu(\pi) \mid \pi \in \mathsf{Run}(A) \wedge (U(\pi))(\overline{0}) \in C\}$.

The APA is said to be *deterministic (DetAPA)* if A is. We write $\mathscr{L}_{\text{APA}}$ (resp. $\mathscr{L}_{\text{DetAPA}}$) for the class of languages recognized by APA (resp. DetAPA).

The *monoid of the APA*, written $\mathscr{M}(U)$, is the multiplicative monoid of matrices generated by $\{M \mid (\exists t)(\exists \overline{v})[U(t) = (M, \overline{v})]\}$.

Lemma 5. *An ε-CA (A, C) having the constraint-determinism property (see Theorem 4) can be simulated (meaning that the language of the ε-CA is preserved) by a DetAPA (A', U, E) whose monoid is finite and such that $L(A) = L(A')$.*

Proof. We outline the idea before giving the details. Let (A, C) be the ε-CA. We first apply the standard subset construction and obtain a deterministic FA $\underline{A}$ equivalent to A. Consider a state $\underline{q}$ of $\underline{A}$. Suppose that after reading some word w leading $\underline{A}$ into state $\underline{q}$ we had, for each $q \in \underline{q}$, the Parikh image $\overline{c_{w,q}}$ (counting transitions in A, i.e., recording the occurrences of each transition in A) of *some* initial w-labeled path leading A into state q. Suppose that $(\underline{q}, a, \underline{r})$ is a transition in $\underline{A}$. How can we compute, for each $q' \in \underline{r}$, the Parikh image $\overline{c_{wa,q'}}$ of *some* initial wa-labeled path leading A into q'? It suffices to pick any $q \in \underline{q}$ for which some a-labeled path leads A from q to q' (possibly using the ε-transitions in A) and to add to $\overline{c_{w,q}}$ the contribution of this a-labeled path. A DetAPA transition on a is well-suited to mimic this computation, since an affine transformation can first "flip" the current Parikh q-count tuple "over" to the Parikh q'-count tuple and then add to it the q-to-q' contribution. Hence a DetAPA $(\underline{A}, \cdot, \cdot)$ upon reading a word w leading to its state $\underline{q}$ is able to keep track, for each $q \in \underline{q}$, of the Parikh image of *some* initial w-labeled path leading A into q. We need constraint-determinism only to reach the final conclusion: if a word w leads $\underline{A}$ into a final state $\underline{q}$, then some $q \in \underline{q}$ is final in A, and because of constraint-determinism, imposing membership in C for the Parikh image of the particular initial w-labeled path leading A to q kept track of by the DetAPA is as good as imposing membership in C for the Parikh image of any other initial w-labeled path leading A to q.

We now give the details. Say $A = (Q, \Sigma, \delta, q_0, F)$, and identify Q with $\{1, \ldots, |Q|\}$.

For $p, q \in Q$, and $a \in \Sigma$, define $S(p, q, a)$ to be a shortest path (lexicographically smallest among shortest paths, for definiteness; this path can be longer than one because of ε-transitions) from p to q labeled by a, or $\perp$ if none exists. Let $\underline{A} = (2^Q, \Sigma, \underline{\delta}, \underline{q_0}, \underline{F})$ be the deterministic version of A defined by $\underline{q_0} = \{q_0\}$, $\underline{\delta}(\underline{p}, a) = \{q \in Q \mid (\exists p \in \underline{p})[S(p, q, a) \neq \perp]\}$ and $\underline{F} = \{\underline{q} \mid \underline{q} \cap F \neq \emptyset\}$; thus $L(\underline{A})$ is bounded. Note that, by construction, for any path π in A from q_0 to q, there exists a path $\underline{\pi}$ in $\underline{A}$ from $\underline{q_0}$ to a state $\underline{q}$ such that $q \in \underline{q}$ and $\mu_A(\pi) = \mu_{\underline{A}}(\underline{\pi})$.

We now attach a function to each transition of $\underline{A}$, where the functions are of dimension $(|Q|.|\delta| + 1)$. We first define $V \colon \underline{\delta} \to \mathscr{F}_{|Q|.|\delta|}$, and will later add the extra component. The intuition is as follows. Consider a path $\underline{\pi}$ on $\underline{A}$ from the initial state to a state $\underline{q}$ — the empty path is considered to be from $\underline{q_0}$

to $\underline{q_0}$. We view $(V(\underline{\pi}))(\overline{0})$ as a list of counters $(\overline{c_1}, \ldots, \overline{c_{|Q|}})$ where $\overline{c_q} \in \mathbb{N}^{|\delta|}$. We will ensure that for any $q \in \underline{q}$, $\overline{c_q}$ is the Parikh image of a path π in A from q_0 to q such that $\mu_A(\pi) = \mu_{\underline{A}}(\underline{\pi})$. If two such paths π_1 and π_2 exist, we may choose one arbitrarily, as they are equivalent in the following sense: if ρ is such that $\pi_1\rho \in \mathsf{Run}(A)$ and $\Phi(\pi_1\rho) \in C$, then the same holds for π_2.

For $\underline{p} \subseteq Q$, $q \in Q$, and $a \in \Sigma$, let $P(\underline{p}, q, a)$ be the smallest $p \in \underline{p}$ such that $S(p,q,a) \neq \perp$ and $\perp$ if none exists. Let $\underline{t} = (\underline{p}, a, \underline{q})$ be a transition of $\underline{A}$. Define:

$$V(\underline{t}) = \left(\sum_{q \in \underline{q}} M(P(\underline{p},q,a),q), \qquad \sum_{q \in \underline{q}} N(q, \Phi(S(P(\underline{p},q,a),q,a))) \right)$$

where $M(p,q)$ is the matrix which transfers the p-th counter to the q-th, and zeroes the others, and $N(q,\overline{d})$ is the shift of $\overline{d} \in \mathbb{N}^{|\delta|}$ to the q-th counter. More precisely, $M(p,q)_{i,j} = 1$ iff there exists $1 \leq e \leq |\delta|$ such that $i = (q-1).|\delta| + e$ and $j = (p-1).|\delta| + e$; likewise, $N(q,\overline{d}) = (0^{(q-1).|\delta|}) \cdot \overline{d} \cdot (0^{(|Q|-q).|\delta|})$. The matrices appearing in V are 0-1 matrices with at most one nonzero entry per row; composing such matrices preserves this property, thus $\mathscr{M}(V)$ is finite.

Claim. *Let $\underline{\pi}$ be a path on $\underline{A}$ from $\underline{q_0}$ to some state $\underline{q}$. Let $(\overline{c_1}, \ldots, \overline{c_{|Q|}}) = (V(\underline{\pi}))(\overline{0})$, where $\overline{c_q} \in \mathbb{N}^{|\delta|}$. Then for all $q \in \underline{q}$, $\overline{c_q}$ is the Parikh image of a path in A from q_0 to q labeled by $\mu(\underline{\pi})$.*

We show this claim by induction. If $|\underline{\pi}| = 0$, then the final state is $\{q_0\}$ and $\overline{c_{q_0}}$ is by definition all-zero. This describes the empty path in A, from q_0 to q_0. Let $\underline{\pi}$ be such that $|\underline{\pi}| > 0$, and consider a state q in the final state of $\underline{\pi}$. Write $\underline{\pi} = \underline{\rho} \cdot \underline{t}$, with $\underline{t} \in \underline{\delta}$, and let $p = P(\mathsf{To}(\underline{\rho}), q, \mu(t))$ and $\zeta = S(p,q,\mu(t))$. The induction hypothesis asserts that the p-th counter of $(V(\underline{\rho}))(\overline{0})$ is the Parikh image of a path ρ on A from q_0 to p labeled by $\mu(\underline{\rho})$. Thus, the q-th counter of $(V(\underline{\pi}))(\overline{0})$ is $\Phi(\rho) + \Phi(\zeta)$, which is the Parikh image of $\rho\zeta$, a path from q_0 to q labeled by $\mu(\underline{\pi})$. This concludes the proof of this claim.

We now define $U \colon \underline{\delta} \to \mathscr{F}_{|Q|.|\delta|+1}$. We add a component to the functions of V, such that for $\underline{\pi} \in \mathsf{Run}(\underline{A})$, the last component of $(U(\underline{\pi}))(\overline{0})$ is 0 if $\mathsf{To}(\underline{\pi}) \cap F = \emptyset$ and $\min(\mathsf{To}(\underline{\pi}) \cap F)$ otherwise. For $\underline{t} = (\underline{p}, a, \underline{q}) \in \underline{\delta}$, let:

$$U(\underline{t}) : (\overline{x}, s) \mapsto \left((V(\underline{t}))(\overline{x}), \qquad \begin{cases} q & \text{if } q \text{ is the smallest s.t. } q \in \underline{q} \cap F, \\ 0 & \text{if no such } q \text{ exists.} \end{cases} \right)$$

Now define $E \subseteq \mathbb{N}^{|Q|\cdot|\delta|+1}$ to be such that $(\overline{v_1}, \ldots, \overline{v_{|Q|}}, q) \in E$ iff $\overline{v_q} \in C$. We adjoin $\overline{0}$ to E iff $\overline{0} \in C$, in order to deal with the empty word. A word w is accepted by the DetAPA $(\underline{A}, U, E)$ iff there exists a path in A from q_0 to $q \in F$, labeled by w, and whose Parikh image belongs in C, i.e., $w \in L(A, C)$.

Finally, note that $L(\underline{A}) = L(A)$ and that $\mathscr{M}(U)$ is finite: the extra component of U only adds a column and a row of 0's to the matrices. □

Before concluding with Lemma 9, let us first recall the following results and definitions:

Lemma 6 ([12, Lemmata 5.5.1 and 5.5.4])**.** *Let $u, v \in \Sigma^*$. Then $(u+v)^*$ is bounded iff there exists $z \in \Sigma^*$ such that $u, v \in z^*$.*

Definition 4 ([10])**.** Let Γ be an alphabet. A *semilinear regular expression (SLRE)* is a finite set of *branches*, defined as expressions of the form $y_0 x_1^* y_1 x_2^* y_2 \cdots x_n^* y_n$, where $x_i \in \Gamma^+$ and $y_i \in \Gamma^*$. The language of an SLRE is the union of the languages of each of its branches.

Theorem 7 ([10])**.** *A regular language is bounded iff it is expressible as a SLRE.*

We will need the following technical lemma. Bounded languages being closed under morphisms, for all automata A if $\mathsf{Run}(A)$ is bounded then so is $L(A)$. The converse is true when A is deterministic (and false otherwise):

Lemma 8. *Let A be a deterministic automaton for a bounded language, then* $\mathsf{Run}(A)$ *is bounded. Moreover,* $\mathsf{Run}(A)$ *is expressible as a SLRE whose branches are of the form $\rho_0\pi_1^*\rho_1\cdots\pi_n^*\rho_n$ where $\rho_i \neq \varepsilon$ for all $1 \leq i < n$ and the first transition of π_i differs from that of ρ_i for every i (including $i = n$ if $\rho_i \neq \varepsilon$).*

Proof. Recall that bounded languages are closed under *deterministic* rational transduction (see, e.g., [12, Lemma 5.5.3]). Now consider A' the automaton defined as a copy of A except that its transitions are $(q,(a,t),q')$ for $t = (q,a,q')$ a transition of A. Then $\tau_{A'}$, the deterministic rational transduction defined by A', is such that $\mathsf{Run}(A) = \tau_{A'}(L(A))$, and thus, $\mathsf{Run}(A)$ is bounded.

It will be useful to note the claim that if $X\pi_1^*\pi_2^*Y \subseteq \mathsf{Run}(A)$ for some nontrivial paths π_1,π_2 and some bounded languages X and Y, then for some path π, $X\pi_1^*\pi_2^*Y \subseteq X\pi^*Y \subseteq \mathsf{Run}(A)$. To see this, note that if $X\pi_1^*\pi_2^*Y \subseteq \mathsf{Run}(A)$ then $X(\pi_1+\pi_2)^*Y \subseteq \mathsf{Run}(A)$ because π_1 and π_2 are loops on a same state. Now $X(\pi_1+\pi_2)^*Y$ is bounded because $\mathsf{Run}(A)$ is bounded, hence $(\pi_1+\pi_2)^*$ is bounded. So pick π such that $\pi_1,\pi_2 \in \pi^*$ (by Lemma 6). Then $X(\pi_1+\pi_2)^*Y \subseteq X\pi^*Y$. But π is a loop in A because $\pi_1 = \pi^j$ for some $j > 0$ is a loop so that $\mathsf{From}(\pi) = \mathsf{To}(\pi)$ in A. Hence $X\pi^*Y \subseteq \mathsf{Run}(A)$. Thus $X\pi_1^*\pi_2^*Y \subseteq X(\pi_1+\pi_2)^*Y \subseteq X\pi^*Y \subseteq \mathsf{Run}(A)$.

Let E be a SLRE for $\mathsf{Run}(A)$, and consider one of its branches $P = \rho_0\pi_1^*\rho_1\cdots\pi_n^*\rho_n$. We assume n to be minimal among the set of all n' such that $\rho_0\pi_1^*\rho_1\cdots\pi_n^*\rho_n \subseteq \rho_0'\pi_1'^*\rho_1'\cdots\pi_{n'}'^*\rho_{n'}' \subseteq \mathsf{Run}(A)$ for some $\rho_0',\pi_1',\ldots,\pi_{n'}',\rho_{n'}'$.

First we do the following for $i = n, n-1, \ldots, 1$ in that order. If $\pi_i = \zeta\pi$ and $\rho_i = \zeta\rho$ for some maximal nontrivial path ζ and for some paths π and ρ, we rewrite $\rho_{i-1}\pi_i^*\rho_i$ as $\rho_{i-1}'\pi_i'^*\rho_i'$ by letting $\rho_{i-1}' = (\rho_{i-1}\zeta)$, $\pi_i' = (\pi\zeta)$ and $\rho_i' = \rho$. This leaves the language of P unchanged and ensures at the ith stage that the first transition of π_j' (if any) differs from that of ρ_j' (if any) for $i \leq j \leq n$. Note that n has not changed.

Let $\rho_0'\pi_1'^*\rho_1'\cdots\pi_n'^*\rho_n'$ be the expression for P resulting from the above process. By the minimality of n, $\pi_i' \neq \varepsilon$ for $1 \leq i \leq n$. And for the same reason, $\rho_i' \neq \varepsilon$ for $1 \leq i < n$, since $\rho_i' = \varepsilon$ implies $X\pi_i'^*\pi_{i+1}'^*Y \subseteq Xz^*Y \subseteq \mathsf{Run}(A)$ for some z by the claim above, where $X = \rho_0'\cdots\pi_{i-1}'^*\rho_{i-1}'$ and $Y = \rho_{i+1}'\pi_{i+2}'^*\cdots\rho_n'$ are bounded languages. □

We are now ready to prove the following, thus concluding the proof of Theorem 4:

Lemma 9. *Let (A,U,C) be a DetAPA such that $L(A)$ is bounded and $\mathscr{M}(U)$ is finite. Then there exist a finite number of flat DetCA the union of the languages of which is $L(A,U,C)$.*

Proof. Let $A = (Q_A,\Sigma,\delta,q_0,F_A)$ be a deterministic automaton whose language is bounded, let $U\colon \delta \to \mathscr{F}_d$ for some $d > 0$ such that $\mathscr{M}(U)$ is finite, and let $C \subseteq \mathbb{N}^d$ be a semilinear set.

Consider the set $\mathsf{Run}(A)$ of accepting paths in A; it is, by Lemma 8, a bounded language. Let P be the language defined as a branch $\rho_0\pi_1^*\rho_1\cdots\pi_n^*\rho_n$ of the SLRE for $\mathsf{Run}(A)$ given by Lemma 8. We will first construct a finite number of flat DetPA for this branch, such that their union has language $\{\mu(\pi) \mid \pi \in P \wedge (U(\pi))(\overline{0}) \in C\}$. For simplicity, we will assume that $\rho_0 = \varepsilon$ and $\rho_n \neq \varepsilon$. We will later show how to lift this restriction.

For $t \in \delta$, we write $U(t) = (M_t,\overline{v_t})$, and more generally, for $\pi = t_1\cdots t_k \in \delta^*$, we write $U(\pi) = (M_\pi,\overline{v_\pi})$; in particular, $M_\pi = M_{t_k}\ldots M_{t_1}$.

Let $p_i,r_i \in \mathbb{N}^+$, for $1 \leq i \leq n$, be integers such that $M_{\pi_i}^{p_i} = M_{\pi_i}^{p_i+r_i}$; such integers are guaranteed to exist as $\mathscr{M}(U)$ is finite. Now let a_i be in $\{0,\ldots,p_i+r_i-1\}$, for $1 \leq i \leq n$, and note, as usual, $\overline{a} = (a_1,\ldots,a_n)$. For succinctness, we give constructions where the labels of the transitions can be nonempty *words*. This is to be understood as a string of transitions, with fresh states in between, with the following specificity: for $w = \ell_1 \cdots \ell_k$, an extended-transition $(q, (w, \overline{d}), q')$ in a PA is a string of transitions the first of which is labeled $(\ell_1, \overline{d})$ and the other ones $(\ell_i, \overline{0})$, $i \geq 2$.

Let $B_{\overline{a}}$ be the ordinary flat automaton with state set $Q = \{q_i, q'_i \mid 1 \leq i \leq n\} \cup \{q_{n+1}\}$, and with transition set δ_B defined by:

$$\delta_B = \{(q_i, \pi_i^{a_i}, q'_i), (q'_i, \rho_i, q_{i+1}) \mid 1 \leq i \leq n\} \cup \{(q'_i, \pi_i^{r_i}, q'_i) \mid 1 \leq i \leq n \wedge a_i \geq p_i\}.$$

We let the initial state be q_1 and the final state be q_{n+1}; note that $B_{\overline{a}}$ is deterministic, thanks to the form of P given by Lemma 8.

So $L(B_{\overline{a}}) = \{\pi_1^{a_1+m_1.r_1}\rho_1 \cdots \pi_n^{a_n+m_n.r_n}\rho_n | (\forall i)[m_i \in \mathbb{N} \wedge (a_i < p_i \rightarrow m_i = 0)]\}$, and the union of the languages of the $B_{\overline{a}}$'s for all valid values of $\overline{a}$ is P. We now make a simple but essential observation on $(U(\pi))(\overline{0})$. For paths π and ρ, we write $V(\pi, \rho)$ for $M_\rho.\overline{v_\pi}$. Then let $\pi \in L(B_{\overline{a}})$, and let m_i, for $1 \leq i \leq n$, be such that $\pi = \pi_1^{a_1+m_1.r_1}\rho_1 \cdots \pi_n^{a_n+m_n.r_n}\rho_n$. First note that $V(\cdot, \cdots \pi_i^{a_i+m_i.r_i} \cdots) = V(\cdot, \cdots \pi_i^{a_i} \cdots)$, then:

$$\begin{aligned}
(U(\pi))(\overline{0}) &= U(\rho_n) \circ U(\pi_n^{a_n+m_n.r_n}) \circ \cdots \circ U(\pi_1^{a_1+m_1.r_1})(\overline{0}) \\
&= M_{\rho_n}(M_{\pi_n}(\ldots(\underbrace{M_{\pi_1}(\ldots(M_{\pi_1}}_{a_1+m_1.r_1 \text{ times}}.\overline{0} + \overline{v_{\pi_1}})\ldots) + \overline{v_{\pi_1}})\ldots) + \overline{v_{\pi_n}}) + \overline{v_{\rho_n}} \\
&= \textstyle\sum_{i=1}^{n} \left((\sum_{l=0}^{a_i+m_i.r_i-1} V(\pi_i, \pi_i^l \rho_i \pi_{i+1}^{a_{i+1}} \cdots \rho_n)) + V(\rho_i, \pi_{i+1}^{a_{i+1}} \cdots \rho_n) \right).
\end{aligned}$$

Moreover, letting $\overline{v_i}(l) = V(\pi_i, \pi_i^l \rho_i \pi_{i+1}^{a_{i+1}} \cdots \rho_n)$, we have for all $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\textstyle\sum_{l=0}^{a_i+m_i.r_i-1} \overline{v_i}(l) &= \textstyle\sum_{l=0}^{a_i-1} \overline{v_i}(l) + \sum_{l=a_i}^{a_i+m_i.r_i-1} \overline{v_i}(l) \\
&= \textstyle\sum_{l=0}^{a_i-1} \overline{v_i}(l) + \sum_{m=0}^{m_i-1} \sum_{l=a_i+m.r_i}^{a_i+(m+1).r_i-1} \overline{v_i}(l) \\
&= \textstyle\sum_{l=0}^{a_i-1} \overline{v_i}(l) + m_i \times \sum_{l=a_i}^{a_i+r_i-1} \overline{v_i}(l).
\end{aligned}$$

We now define a PA $D_{\overline{a}}$ which is a copy of $B_{\overline{a}}$ except for the labels of its transition set δ_D. Each transition of $D_{\overline{a}}$ incorporates the relevant value of $V(\cdot, \cdot)$ so that the sum in the last equation above will be easily computable. For each transition $(q, \pi, q'_i) \in \delta_B$, there is a transition $(q, (\pi, \overline{d}), q'_i) \in \delta_D$ such that if $q = q_i$ then $\overline{d} = \sum_{l=0}^{a_i-1} \overline{v_i}(l)$, and if $q = q'_i$ then $\overline{d} = \sum_{l=a_i}^{a_i+r_i-1} \overline{v_i}(l)$. Finally, for each transition $(q'_i, \rho_i, q_{i+1}) \in \delta_B$, there is a transition $(q'_i, (\rho_i, \overline{d}), q_{i+1}) \in \delta_D$ with $\overline{d} = V(\rho_i, \pi_{i+1}^{a_{i+1}} \cdots \rho_n)$.

Let $\omega \in L(D_{\overline{a}})$, there exists $\pi \in L(B_{\overline{a}})$ such that $\Psi(\omega) = \pi$. We show by induction on n, the number of π_i's, that $\tilde{\Phi}(\omega) = (U(\pi))(\overline{0})$. If $n = 0$, then it is trivially true. Suppose $n > 0$, and write $\pi = \pi_1^{a_1+m.r_1}\rho_1\pi'$ for some m and π'. Write $\omega = \omega_1\omega_2\omega_3\omega'$ with $\Psi(\omega_1) = \pi_1^{a_1}$, $\Psi(\omega_2) = \pi_1^{m_1.r_1}$, $\Psi(\omega_3) = \rho_1$. Then:

$$\begin{aligned}
(U(\pi))(\overline{0}) &= \underbrace{\textstyle\sum_{l=0}^{a_i-1} \overline{v_i}(l)}_{\tilde{\Phi}(\omega_1)} + \underbrace{m_i \times \textstyle\sum_{l=a_i}^{a_i+r_i-1} \overline{v_i}(l)}_{\tilde{\Phi}(\omega_2)} + \underbrace{V(\rho_1, \pi')}_{\tilde{\Phi}(\omega_3)} + \underbrace{(U(\pi'))(\overline{0})}_{\tilde{\Phi}(\omega') \text{ by ind. hyp.}} \\
&= \tilde{\Phi}(\omega).
\end{aligned}$$

Thus, for a path π, $\pi \in L(B_{\overline{a}})$ iff there exists $\omega \in L(D_{\overline{a}})$ with $\Psi(\omega) = \pi$, and in this case, $\tilde{\Phi}(\omega) = (U(\pi))(\overline{0})$.

We now lift the restrictions we set at the beginning of the proof. First, if $\rho_0 \neq \varepsilon$, we add a fresh state q'_0 to $B_{\overline{a}}$ together with a transition (q'_0, ρ_0, q_1). In the construction of $D_{\overline{a}}$, the label ρ_0 is changed to $(\rho_0, V(\rho_0, \pi_1^{a_1}\rho_1 \cdots \pi_n^{a_n}\rho_n))$. Now, for $\omega \in L(D_{\overline{a}})$, let $\pi = \Psi(\omega)$ and write $\pi = \rho_0\pi'$ for some π'; likewise, write $\omega = \omega_1\omega'$ where $\Psi(\omega_1) = \rho_0$. Then:

$$(U(\pi))(\overline{0}) = V(\rho_0, \pi_1^{a_1}\rho_1 \cdots \pi_n^{a_n}\rho_n) + (U(\pi'))(\overline{0}).$$

The previous results show that $(U(\pi'))(\overline{0}) = \tilde{\Phi}(\omega')$, and thus $(U(\pi))(\overline{0}) = \tilde{\Phi}(\omega)$.

Next, if $\rho_n = \varepsilon$, then the transition (q'_n, ρ_n, q_{n+1}) is removed from the construction of $B_{\overline{a}}$, and the final state changed to q_n. The absence of this transition does not influence the computations in $D_{\overline{a}}$, thus we still have that for any $\omega \in L(D_{\overline{a}})$, $\tilde{\Phi}(\omega) = (U(\Psi(\omega)))(\overline{0})$.

Now define $D'_{\overline{a}}$ to be a copy of $D_{\overline{a}}$ which differs only in the transition labels: a transition labeled $(t, \overline{d})$ in $D_{\overline{a}}$ becomes a transition labeled $(\mu_A(t), \overline{d})$ in $D'_{\overline{a}}$. The properties of the SLRE for $\mathsf{Run}(A)$ extracted from Lemma 8 imply that $(D'_{\overline{a}}, C)$ is a flat DetPA. Let $L_P \in \mathscr{L}_{\text{DetPA}}$ be the union of the languages of all flat DetPA $(D'_{\overline{a}}, C)$, for any $\overline{a}$ with $a_i \in \{0, \dots, p_i + r_i - 1\}$. Then $L_P = \{\mu(\pi) \mid \pi \in P \wedge (U(\pi))(\overline{0}) \in C\}$. Thus the union of all L_P for all branches P is $L(A, U, C)$. The flat DetPA involved can be made into flat DetCA because the equivalence in Theorem 1 carries over the case of flat automata. □

We note that we can show that there exist nonsemilinear bounded languages in $\mathscr{L}_{\text{DetAPA}}$, thus there exist bounded languages in $\mathscr{L}_{\text{DetAPA}} \setminus \mathscr{L}_{\text{PA}}$.

6 Discussion

We showed that PA and DetPA recognize the same class of bounded languages, namely BSL. Moreover, we note that the union of flat DetCA is a concept that has already been defined in [4] as *1-constrained-queue-content-decision-diagram (1-CQDD)*, and we may thus conclude, thanks to the specific form of the DetCA obtained in Theorem 4, that 1-CQDD capture exactly BSL.

A related model, *reversal-bounded multicounter machines (RBCM)* [14], has been shown to have the same expressive power as PA [18]. We can show that *one-way* deterministic RBCM are strictly more powerful than DetPA, thus our result carries over to the determinization of bounded RBCM languages. This generalizes [15, Theorem 3.5] to bounded languages where the words of the socle are of length other than one. The fact that *two-way* deterministic RBCM are equivalent to *two-way* RBCM over bounded languages can be found in [7, Theorem 3] without proof and in [20] with an extremely terse proof.

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