Forward Analysis of Updatable Timed Automata

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Abstract. Timed automata are a widely studied model. Its decidability has been proved using the so-called region automaton construction. This construction provides a correct abstraction for the behaviours of timed automata, but it suffers from a state explosion and is thus not used in practice. Instead, algorithms based on the notion of zones are implemented using adapted data structures like DBMs. When we focus on forward analysis algorithms, the exact computation of all the successors of the initial configurations does not always terminate. Thus, some abstractions are often used to ensure termination, among which, a widening operator on zones.

In this paper, we study in detail this widening operator and the corresponding forward analysis algorithm. This algorithm is most used and implemented in tools like Kronos and Uppaal. One of our main results is that it is hopeless to find a forward analysis algorithm for general timed automata, that uses such a widening operator, and which is correct. This goes really against what one could think. We then study in detail this algorithm in the more general framework of updatable timed automata, a model which has been introduced as a natural syntactic extension of classical timed automata. We describe subclasses of this model for which a correct widening operator can be found.

Keywords: (updatable) timed automata, forward analysis algorithm, widening operator, correctness, data structure

1 Introduction

Real-Time Systems. Since their introduction by Alur and Dill in [AD90,AD94], timed automata are one of the most studied models for real-time systems. Numerous works have been devoted to the "theoretical" comprehension of timed automata: determinization [AFH94], minimization [ACD+92], power of clocks [ACH94,HKWT95], power of ε-transitions [BDGP98], extensions of the model [DZ98,HRS98,CG00,BFH+01], logical characterizations [Wil94,HRS98],... have in particular been investigated. Practical aspects of the model have also been studied and several model-checkers are now available (HyTech[1] [HHWT97], Kronos[2] [DOTY96], Uppaal[3] [LPY97]). Timed automata afford to model many real-time systems and the existing model-checkers have been used to verify a lot of industrial case studies (see the web pages of the tools or, for example, [HSLL97,TY98]).

Implementation of Timed Automata. Decidability properties for the timed automata model have been proved by Alur and Dill in [AD90,AD94]. It is based on the construction of the so-called region automaton: it abstracts finitely and in a correct way the behaviours of timed automata. However, in practice, such a construction is not implemented, because it suffers from an enormous combinatorics explosion, but algorithms searching on-the-fly through the automaton are

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[1] http://www-cad.eecs.berkeley.edu:80/~tah/HyTech/

[2] http://www-verimag.imag.fr/TEMPORISE/kronos/

[3] http://www.uppaal.com/

preferred. These algorithms are based on the notion of zones and can be efficiently implemented using data structures like DBMs [Dil89] and CDDs [BLP+99]. Among these algorithms are the forward analysis algorithms that compute all the reachable configurations. However, the exact forward computation does not always terminate. Abstractions have been proposed to avoid this termination problem (see for example [DT98]) and they are implemented in tools like Kronos and Uppaal.

Our contributions. In this paper, we study in detail one of the abstractions proposed in [DT98]. It consists in a widening operator that is applied to zones to enforce the termination of the forward searching in the timed automaton. The most important result is that, against what one can think, there is no way to find a correct forward analysis algorithm for timed automata that uses this widening operator.
We then consider the more general framework of updatable timed automata, as defined in [BDFP00a]. This model is a syntactic extension of the classical timed automata coming from a model built for a communication protocol in [BF99] and its decidability has been precisely settled in [BDFP00a]. We then describe several subclasses of this model for which we can find a correct widening operator and thus a correct forward analysis algorithm. These subclasses contain in particular all the timed automata that do not use comparisons between clocks. From this study, it appears that problems in the correctness of the widening operator come only from the nature of the constraints which are used in the automata, but not from the syntactic macros that have been added in the updatable timed automata model.

Outline of the paper. The structure of the paper is the following: we first recall some background on the implementation of timed automata and we describe the forward analysis algorithm that uses the widening operator we will study (section 2) ; we then discuss the correctness of this algorithm and get that, even if surprising, the algorithm is not correct in general (section 3) ; we then present the updatable timed automata framework (section 4) and we study very precisely the correctness of the widening operator (section 5) ; we also propose an implementation of the algorithm for the subclasses of updatable timed automata for which a correct widening operator can be found (section 5) ; we discuss some other methods for all the automata for which we can not propose a correct widening operator ; we finally conclude with some remarks (section 6).

Some technical proofs of this paper are presented in Appendix (page 29).

2 Implementation of Timed Automata, State of the Art

In this section, we present the basic notions for timed automata [AD90,AD94], and we focus on implementation issues for this model. In particular, we present an algorithm which is implemented in some tools.

2.1 Preliminaries

If Z is any set, let Z^* be the set of *finite* sequences of elements in Z. We consider as time domain $\mathbb{T}$ the set $\mathbb{Q}^+$ of non-negative rationals or the set $\mathbb{R}^+$ of non-negative reals and Σ as a finite set of *actions*. A *time sequence* over $\mathbb{T}$ is a finite non decreasing sequence $\tau = (t_i)_{1\leq i\leq p} \in \mathbb{T}^*$. A *timed word* $\omega = (a_i, t_i)_{1\leq i\leq p}$ is an element of $(\Sigma \times \mathbb{T})^*$, also written as a pair $\omega = (\sigma, \tau)$, where $\sigma = (a_i)_{1\leq i\leq p}$ is a word in Σ^* and $\tau = (t_i)_{1\leq i\leq p}$ a time sequence in $\mathbb{T}^*$ of same length.

Clock Valuations. We consider a finite set X of variables, called *clocks*. A *clock valuation* over X is a mapping $v : X \to \mathbb{T}$ that assigns to each clock a time value. The set of all clock valuations over X is written $\mathbb{T}^X$. Let $t \in \mathbb{T}$, the valuation $v + t$ is defined by $(v + t)(x) = v(x) + t$, $\forall x \in X$. When $|X| = n$, we also use the notation $(\alpha_i)_{1\leq i\leq n}$ for the valuation v such that $v(x_i) = \alpha_i$. For a subset C of X, we denote by $[C \leftarrow 0]v$ the valuation such that for each $x \in C$, $([C \leftarrow 0]v)(x) = 0$ and for each $x \in X \setminus C$, $([C \leftarrow 0]v)(x) = v(x)$.

Clock Constraints. Given a set of clocks X, we introduce two sets of clock constraints over X. The most general one, denoted by $C(X)$, is defined by the following grammar:

$$\begin{aligned} \varphi \ ::= \ & x \sim c \mid x - y \sim c \mid \varphi \wedge \varphi \mid \mathit{true} \\ & \text{where } x,\ y \in X,\ c \in \mathbb{Z},\ \sim \in \{<, \leq, =, \geq, >\} \end{aligned}$$

We also use the proper subset of *diagonal-free* constraints where the comparison between two clocks is not allowed. This set is denoted by $C_{df}(X)$ and is defined by the grammar:

$$\begin{aligned} \varphi \ ::= \ & x \sim c \mid \varphi \wedge \varphi \mid \mathit{true}, \\ & \text{where } x \in X,\ c \in \mathbb{Z} \text{ and } \sim \in \{<, \leq, =, \geq, >\} \end{aligned}$$

We write $v \models \varphi$ when the clock valuation v satisfies the clock constraint φ.

A *k-bounded clock constraint* is a clock constraint that involves only constants between $-k$ and $+k$.

Timed Automata. A *timed automaton* over $\mathbb{T}$ is a tuple $\mathcal{A} = (\Sigma, Q, T, I, F, X)$, where Σ is a finite alphabet of actions, Q is a finite set of states, X is a finite set of clocks, $T \subseteq Q \times [C(X) \times \Sigma \times 2^X] \times Q$ is a finite set of transitions, $I \subseteq Q$ is the subset of initial states and $F \subseteq Q$ is the subset of final states.

A *path* in $\mathcal{A}$ is a finite sequence of consecutive transitions:

$$P = q_0 \xrightarrow{\varphi_1, a_1, C_1} q_1 \ \ldots \ q_{p-1} \xrightarrow{\varphi_p, a_p, C_p} q_p \text{ where } (q_{i-1}, \varphi_i, a_i, C_i, q_i) \in T, \text{ for each } 1 \leq i \leq p$$

The path is said to be *accepting* if it starts in an initial state ($q_0 \in I$) and ends in a final state ($q_p \in F$). A *run* of the automaton through the path P is a sequence of the form:

$$\langle q_0, v_0 \rangle \xrightarrow[t_1]{\varphi_1, a_1, C_1} \langle q_1, v_1 \rangle \ \ldots \ \xrightarrow[t_p]{\varphi_p, a_p, C_p} \langle q_p, v_p \rangle$$

where $\tau = (t_i)_{1 \leq i \leq p}$ is a time sequence and $(v_i)_{1 \leq i \leq p}$ are clock valuations such that:

$$\begin{cases} v_0(x) = 0, \ \forall x \in X \\ v_{i-1} + (t_i - t_{i-1}) \models \varphi_i \\ v_i = [C_i \leftarrow 0]\,(v_{i-1} + (t_i - t_{i-1})) \end{cases}$$

The label of the run is the timed word $w = (a_1, t_1) \ldots (a_p, t_p)$. If the path P is accepting then the timed word w is said to be accepted by the timed automaton. The set of all timed words accepted by $\mathcal{A}$ is denoted by $L(\mathcal{A})$.

2.2 Practical Issues

For verification purposes, a fundamental question about timed automata is to decide whether the accepted language is empty. This problem is called the *emptiness problem*. A class of timed automata is said *decidable* if the emptiness problem is decidable for this class. Note that this problem is equivalent to the *reachability problem* which tests whether a state can be reached in a timed automaton.

Alur and Dill proved in [AD90,AD94] that the emptiness problem is decidable for timed automata. The proof of this result is based on a "region automaton construction". In practice, this construction is not implemented because it suffers from an enormous combinatorics explosion. The idea of the region automaton is to build a finite simulation graph for the automaton based on an equivalence relation (of finite index) defined on the clock valuations. Some works have been done to reduce the size of this simulation graph by enlarging the equivalence relation, see for example [ACD+92,YL97,TY01]. However, in practice, such graphs are not constructed and on-the-fly zone algorithms searching symbolically through the graph are implemented.

One of the advantages of these algorithms is that they can easily be implemented using the *Difference Bounded Matrices* data structure (DBM for short), initially proposed by [Dil89]. For example, forward analysis algorithms (that is algorithms computing step-by-step the successors of the initial configurations) [Alu99] are implemented in tools like UPPAAL [BL96,LPY97] or KRONOS [BTY97,Daw97,Yov98]. The forward analysis algorithm computing the exact set of successors does not always terminate; in [MP99], subclasses of classical timed automata, for which termination is guaranteed, are proposed, but these classes are a bit restrictive. Different kinds of abstractions [DT98] are thus proposed to enforce the termination, among which a widening operator on zones.

The aim of this paper is to study in detail the widening operator proposed in [DT98] and implemented in tools like KRONOS and UPPAAL. We will thus now present this widening operator and describe the associated forward analysis algorithm. This algorithm is implemented in some tools for the verification of classical timed automata. We also explain how this algorithm is implemented using the DBM data structure. Note that this data structure, apart from being adapted to the implementation of algorithms for timed automata, is also very useful for proving properties of timed automata.

2.3 A Forward Analysis Algorithm for Timed Automata

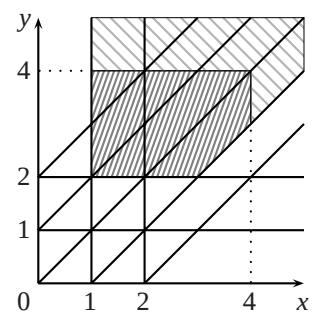
Zones. A *zone* is a subset of $\mathbb{T}^n$ defined by a general clock constraint. Let k be a constant. A *k-bounded zone* is a zone defined by a k-bounded clock constraint. Let Z be a zone. The set of k-bounded zones containing Z is finite and not empty ($\mathbb{T}^n$ is a k-bounded zone and contains Z), the intersection of these k-bounded zones is a k-bounded zone containing Z, and is thus the smallest one having this property. It is called the *k-approximation* of Z and is written $\mathsf{Approx}_k(Z)$.

Example 1. Consider the zone Z drawn with on the figure beside: Z is defined by the clock constraint

$$1 < x < 4 \wedge 2 < y < 4 \wedge x - y < 1.$$

Taking $k = 2$, the k-approximation of Z is obtained by adding the part ; it is defined by the clock constraint

$$1 < x \wedge 2 < y \wedge x - y < 1.$$



The Algorithm. Let $\mathcal{A}$ be a classical timed automaton. If $e = (q \xrightarrow{g,a,C:=0} q')$ is a transition of $\mathcal{A}$ and if Z is a zone, then $\mathsf{Post}(Z, e)$ denotes the set $[C \leftarrow 0](g \cap \overrightarrow{Z})$ where $\overrightarrow{Z}$ represents the *future* of Z, defined by

$$\overrightarrow{Z} = \{v + t \mid v \in Z \text{ and } t \geq 0\}$$

$\mathsf{Post}(Z, e)$ is the set of valuations which can be reached by waiting in the current state, q, and then taking the transition e. We associate with $\mathcal{A}$ the largest constant, k, appearing in $\mathcal{A}$ (*i.e.* the largest constant c such that there is a constraint $x \sim c$ for some clock x or $x - y \sim c$ for some clocks x and y). A maximal constant can be computed for each clock x (in a similar way), but for our purpose, it does not change anything, the presentation would just be a bit more complicated.

One of the forward analysis algorithms for classical timed automata uses the k-approximation (*cf* section 2.3) as a widening operator. It is presented as Algorithm 1 (q_0 is the initial location of the automaton whereas Z_0 represents the initial zone, it usually contains the valuation where all the clocks are set to zero).
We first point out that this algorithm terminates because there are finitely many k-bounded zones and thus finitely many k-approximations of zones that can be computed for each control state of the automaton. This algorithm computes step-by-step an overapproximation of the set of

Algorithm 1 Algorithm using k-Approximations for Classical Timed Automata

```
# TA-Algorithm (𝒜:TA) {
#     Define k as the maximal constant appearing in 𝒜;
#     Visited := ∅;                                  (* Visited stores the visited states *)
#     Waiting := {(q0,Approx_k(Z0))};
#     Repeat
#         Get and Remove (q,Z) from Waiting;
#         If q is final
#             then {Return "Yes";}
#             else {If there is no (q,Z') ∈ Visited such that Z ⊆ Z'
#                     then {Visited := Visited ∪ {(q,Z)};
#                           Successor := {(q',Approx_k(Post(Z,e))) | e transition from q to q'};
#                           Waiting := Waiting ∪ Successor;}}
#     Until (Waiting = ∅);
#     Return "No"; }
```

reachable states and tests whether this approximation intersects the set of final states or not. Thus, if the answer of the algorithm is "No", then no final state can be reached. However, if the answer is "Yes", it can *a priori* be the case that the overapproximation intersects the set of final states whereas the exact set of reachable states does not intersect this set. We say that Algorithm 1 is *correct with repect to reachability* if the set of (discrete) states computed by Algorithm 1 is precisely the set of reachable (discrete) states. We will discuss in detail the correctness of Algorithm 1 in Section 3.

2.4 The Implementation: the DBM Data Structure

In order to implement Algorithm 1, an adapted data structure is needed to represent zones and this data structure must be convenient for testing inclusion of zones and computing easily the different operations used in the algorithm, that is the intersection of two zones, the future of a zone, the image of a zone by a reset and the k-approximation of a zone. Tools like Uppaal or Kronos use the data structure proposed by Dill in [Dil89], the DBM data structure. A detailed presentation of this data structure can be found in [CGP99] and in [Ben02]. However, we will recall some basic properties for it because it will be the core tool in the rest of the paper.

A *difference bounded matrice* (say *DBM* for short) for a set $X = \{x_1, \ldots, x_n\}$ of n clocks is an $(n+1)$-square matrice of pairs

$$(m; \prec) \in \mathbb{V} = (\mathbb{Z} \times \{<, \leq\}) \cup \{(\infty; <)\}.$$

A DBM $M = (m_{i,j}, \prec_{i,j})_{i,j=1\ldots n}$ defines the following subset of $\mathbb{T}^n$ (the clock x_0 is supposed to be always equal to zero, *i.e.* for each valuation v, $v(x_0) = 0$):

$$\left\{ v : X \longrightarrow \mathbb{T} \mid \forall\, 0 \leq i, j \leq n,\ v(x_i) - v(x_j) \prec_{i,j} m_{i,j} \right\}$$

where $\gamma < \infty$ simply means that γ is some real without bound.
This subset of $\mathbb{T}^n$ is a zone and will be denoted, in what follows, by $[\![M]\!]$. Each DBM on n clocks represents a zone of $\mathbb{T}^n$. Note that several DBMs can define the same zone.

Example 2. The zone defined by the equations $x_1 > 3 \wedge x_2 \leq 5 \wedge x_1 - x_2 < 4$ can be represented by the two DBMs

$$\begin{pmatrix} (0;\leq) & (-3;<) & (\infty;<) \\ (\infty;<) & (0;\leq) & (4;<) \\ (5;\leq) & (\infty;<) & (0;\leq) \end{pmatrix} \text{ and } \begin{pmatrix} (\infty;<) & (-3;<) & (\infty;<) \\ (\infty;\leq) & (\infty;<) & (4;<) \\ (5;\leq) & (\infty;<) & (0;\leq) \end{pmatrix}$$

Thus the DBMs are not a canonical representation of zones. Moreover, it isn't possible to test syntactically whether $[\![M_1]\!] = [\![M_2]\!]$. A normal form has thus been defined for the representation

of zones. Its computation uses Floyd algorithm and some syntactic rewritings (see [Dil89,CGP99] for a description of this procedure). In what follows, we denote by $\phi(M)$ the *normal form* of M. Before stating some very important properties of the normal form, we define a total order on $\mathbb{V}$ in the following way: if $(m;\prec),(m';\prec') \in \mathbb{V}$, then

$$(m;\prec) \leq (m';\prec') \iff \begin{cases} m < m' \\ \text{or} \\ m = m' \text{ and either } \prec = \prec' \text{ or } \prec' = \leq . \end{cases}$$

Of course, for each $m \in \mathbb{Z}$, we have $m < \infty$. We define $>$, $\geq$ and $<$ in a natural way. These orders are extended to DBMs $M = (m_{i,j};\prec_{i,j})_{i,j=0\ldots n}$ and $M' = (m'_{i,j};\prec'_{i,j})_{i,j=0\ldots n}$ by

$$M \leq M' \iff \text{for every } i,j = 0\ldots n,\ (m_{i,j};\prec_{i,j}) \leq (m'_{i,j};\prec'_{i,j}).$$

We can now state some (very useful) properties of normal forms. If M and M' are DBMs, then:

(i) $[\![M]\!] = [\![\phi(M)]\!]$ and $\phi(M) \leq M$,
(ii) $[\![M]\!] \subseteq [\![M']\!] \iff \phi(M) \leq M' \iff \phi(M) \leq \phi(M')$.

The last point expresses the fact that the test for inclusion of zones can be checked syntactically on the normal forms of the DBMs (representing the zones).
Normal forms of DBMs can be characterized in a natural way. If $M = (m_{i,j};\prec_{i,j})_{i,j=0\ldots n}$ is a DBM such that $[\![M]\!] \neq \emptyset$, then the two following properties are equivalent:

(i) M is in normal form,
(ii) for every $i,j = 0\ldots n$, for every real $-m_{j,i} \prec_{j,i} r \prec_{i,j} m_{i,j}$, there exists a valuation $v \in [\![M]\!]$ such that $v(x_j) - v(x_i) = r$ (still assuming that $v(x_0) = 0$).

This property expresses the fact that if a DBM is in normal form, then no constraint of this DBM can be tightened using Floyd algorithm.

The emptiness can be checked on the DBMs, even if they are not in normal form. For this, we define an addition on the set $\mathbb{V}$:

$$(m;\prec) + (m';\prec') = (m'';\prec'')$$

where $m'' = m + m'$ and $\prec''$ is $\leq$ if both $\prec$ and $\prec'$ are $\leq$ and $\prec''$ is $<$ otherwise. Now, if $M = ((m_{i,j};\prec_{i,j})_{i,j})$ is a DBM, $[\![M]\!] = \emptyset$ if and only if there exists a negative cycle in M, which means that there exists a sequence of distinct indices $(i_1, i_2, \ldots, i_{l-1}, i_l = i_1)$ such that

$$(m_{i_1,i_2};\prec_{i_1,i_2}) + (m_{i_2,i_3};\prec_{i_2,i_3}) + \cdots + (m_{i_{l-1},i_l};\prec_{i_{l-1},i_l}) < (0;\leq)$$

This condition will often be used in the remainder of the paper.

Computation of Some Operations on DBMs. As we argued at the beginning of the section, the data structure used to represent zones must also be appropriate to compute the four operations used by Algorithm 1, namely future, intersection, image by resets and k-approximation. These operations on DBMs are described in [CGP99] and [Ben02], we only recall them quickly.

Intersection. Let $M = (m_{i,j};\prec_{i,j})_{i,j=1\ldots n}$ and $M' = (m'_{i,j};\prec'_{i,j})_{i,j=1\ldots n}$ be two DBMs and define $M'' = (m''_{i,j};\prec''_{i,j})_{i,j=1\ldots n}$ by

$$(m''_{i,j};\prec''_{i,j}) = \min((m_{i,j};\prec_{i,j}),(m'_{i,j};\prec'_{i,j})) \text{ for all indices } i,j = 1\ldots n.$$

Then $[\![M'']\!] = [\![M]\!] \cap [\![M']\!]$. Note that it can be the case that M'' is not in normal form, even if M and M' are in normal form.

Future. Assume that $M = (m_{i,j}; \prec_{i,j})_{i,j=1\ldots n}$ is a DBM in normal form. Define the DBM $\overrightarrow{M} = (m'_{i,j}; \prec'_{i,j})_{i,j=1\ldots n}$ by:

$$\begin{cases} (m'_{i,j}; \prec'_{i,j}) = (m_{i,j}; \prec_{i,j}) & \text{if } j \neq 0 \\ (m'_{i,0}; \prec'_{i,0}) = (\infty; <) \end{cases}.$$

Then $[\![\overrightarrow{M}]\!] = \overrightarrow{[\![M]\!]}$ and the DBM $\overrightarrow{M}$ is in normal form.

Image by resets. Assume that $M = (m_{i,j}; \prec_{i,j})_{i,j=1\ldots n}$ is a DBM in normal form and define the DBM $M_{x_k:=0} = (m'_{i,j}; \prec'_{i,j})_{i,j=1\ldots n}$ by:

$$\begin{cases} (m'_{i,j}; \prec'_{i,j}) = (m_{i,j}; \prec_{i,j}) & \text{if } i, j \neq k \\ (m'_{k,k}; \prec'_{k,k}) = (m'_{k,0}; \prec'_{k,0}) = (m'_{0,k}; \prec'_{0,k}) = (0; \leq) \\ (m'_{i,k}; \prec'_{i,k}) = (m_{i,0}; \prec_{i,0}) & \text{if } i \neq k \\ (m'_{k,i}; \prec'_{k,i}) = (m_{0,i}; \prec_{0,i}) & \text{if } i \neq k \end{cases}$$

Then $[\![M_{x_k:=0}]\!] = [x_k \leftarrow 0][\![M]\!]$ and that the DBM $M_{x_k:=0}$ is in normal form.

k-approximation. Assume that $M = (m_{i,j}; \prec_{i,j})_{i,j=1\ldots n}$ is a DBM in normal form. We define the DBM $M_k = (m'_{i,j}; \prec'_{i,j})_{i,j=1\ldots n}$ by:

$$\begin{cases} (m'_{i,j}; \prec'_{i,j}) = (m_{i,j}; \prec_{i,j}) & \text{if } -k \leq m_{i,j} \leq k \\ (m'_{i,j}; \prec'_{i,j}) = (\infty; <) & \text{if } m_{i,j} > k \\ (m'_{i,j}; \prec'_{i,j}) = (-k; <) & \text{if } m_{i,j} < -k \end{cases}$$

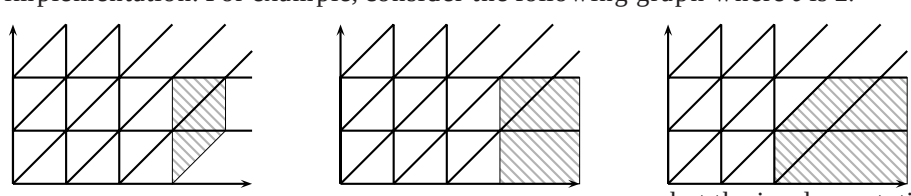
We get that $[\![M_k]\!] = \mathsf{Approx}_k([\![M]\!])$, but M_k is not in normal form.

Combining all these constructions, if e is a transition of the form $q \xrightarrow{g,a,C:=0} q'$ and if M is a DBM representing a zone Z, using the properties above, it is easy to compute a DBM which represents the zone $\mathsf{Approx}_k([C \leftarrow 0](g \cap \overrightarrow{Z})) = \mathsf{Approx}_k(\mathsf{Post}(Z, e))$. This DBM may not be in normal form. The DBM data structure can thus be used to compute all the operations needed by Algorithm 1.

3 Correctness Problems

As argued in Section 2.3, Algorithm 1 raises a correctness problem. Indeed, it computes an overapproximation of the set of reachable states which can *a priori* be strictly larger than the set of reachable states. This algorithm is largely implemented and used, nevertheless, we did not find any complete proof of its correctness in the literature. There exist of course several papers claiming that they present such proofs but these proofs are incomplete and even incorrect:

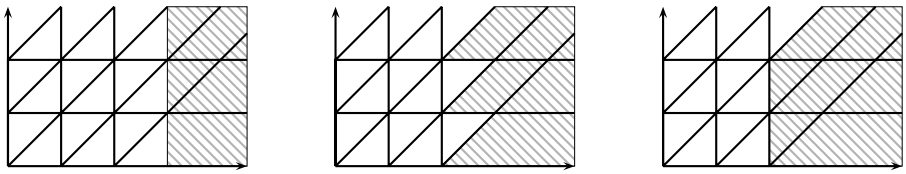
– In the PhD. Thesis [Tri98], the implementation of the *c-closure* presented page 127 (of [Tri98]) corresponds to our Approx_c approximation. Thus, their Yes/No Algorithm should correspond to Algorithm 1. However, the definition of the *c*-closure given page 21 does not correspond to the implementation. For example, consider the following graph where c is 2:



zone Z — c-closure of Z — what the implementation really computes

The proof of correctness of the Yes/No Reachability Algorithm (Lemma 5.9 page 58) is done using the *c*-closure which is defined and not the one which is implemented. Thus, the correctness of the implemented algorithm is not complete in this work.

– In the PhD. Thesis [WT94], if Z is a zone then $round(Z)$ corresponds to our Approx_k overapproximation. The theorem which aims at proving that Algorithm 1 is correct is Theorem 4.8 on page 90. The proof is done in the following way: considering a zone Z, it is decomposed into finitely many subzones that are assumed to be included in regions, then the computation of the *round* operator is done first for the subzones (this step is easy); $round(Z)$ is obtained by computing the union of the *round* of these subzones. But *round* is not compatible with union, for example have a look at the following zone:



zone Z | *round* applied to the subzones of Z | $round(Z)$

Besides, this proof only aimed at proving the correctness of Algorithm 1 when we restrict to diagonal-free timed automata.

A surprising observation...

In trying to write a complete proof for the correctness of Algorithm 1, we had some troubles, and investigating precisely what were the problems we were confronted to, we have been forced to face the fact that **Algorithm 1 can not be correct!** Consider for example the automaton Cex depicted on figure 1.

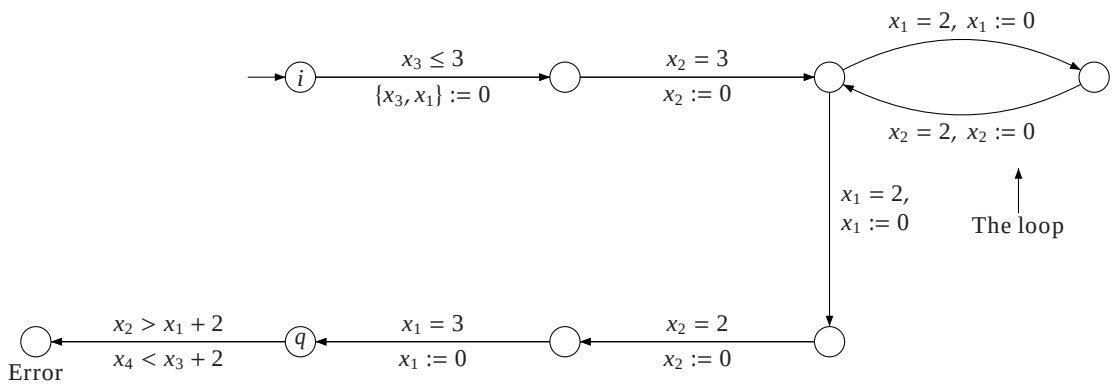


Fig. 1. A surprising counter-example, Cex

It is just a matter of computing successors of zones to obtain the set of zones reached in state q, starting in state i with the valuation where all clocks are set to zero:

$$Z_\alpha = \begin{cases} x_3 \geq 2\alpha + 5 \\ x_2 \geq 1 \\ x_4 \geq 2\alpha + 6 \\ 1 \leq x_2 - x_1 \leq 3 \\ x_3 - x_1 = 2\alpha + 5 \\ 2\alpha + 6 \leq x_4 - x_1 \leq 2\alpha + 8 \\ 2\alpha + 2 \leq x_3 - x_2 \leq 2\alpha + 4 \\ x_4 - x_2 = 2\alpha + 5 \\ 1 \leq x_4 - x_3 \leq 3 \end{cases}$$

when the loop is taken α times. A zone Z_α is depicted on figure 3.

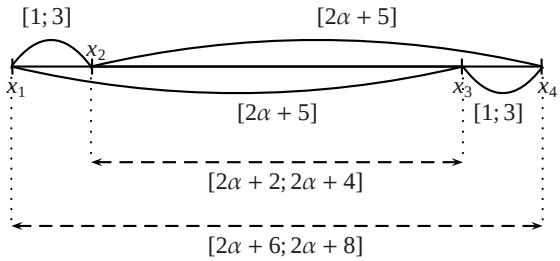


Fig. 2. The zone Z_α

We can notice that, for every α, if we require the condition $x_4 - x_3 < 2$ in Z_α, then $x_2 - x_1 < 2$ because

$$\begin{aligned} x_2 - x_1 &= (x_2 - x_4) + (x_4 - x_3) + (x_3 - x_1) \\ &= (-2\alpha - 5) + (x_4 - x_3) + (2\alpha + 5) \\ &= x_4 - x_3 \end{aligned}$$

Thus, the state "Error" of Cex is not reachable. However, if we fix a maximal constant k and if we use the widening operator with this constant k (that is if we use the operator Approx_k), then for a number of loops α large enough (more precisely for an α such that $2\alpha + 2 > k$), the constraint $(x_3 - x_1 \leq 2\alpha + 5)$ will be removed when the widening operator is applied and thus Algorithm 1 will answer that the state "Error" is reachable. Thus, we conclude that for the automaton Cex, there exists no k such that an algorithm involving Approx_k is correct with respect to reachability. In particular, Algorithm 1 is not correct with respect to reachability in general.

From this counter-example, we find that the widening operator is not correct with respect to reachability and this was a big surprise! We thus argue for the following thesis:

Thesis. *There is no way to have a correct forward analysis algorithm that uses a widening operator* Approx_k *on zones for general timed automata.*

Notice that this can be extended to more general widening operators. Assume Abs is an abstraction operator such that for every zone Z, $Abs(Z)$ is finite and $\{Abs(Z) \mid Z \text{ zone}\}$. Replacing the widening operator Approx_k by Abs in Algorithm 1, we get that this algorithm is not correct with respect to reachability. Indeed, if we take k as the maximal absolute value of a coefficient of DBMs in the set $\{M \text{ DBM in normal form} \mid [\![M]\!] = Abs(Z) \text{ for some zone } Z\}$, it is easy to prove that $\mathsf{Approx}_k(Z) \subseteq Abs(Z)$ for every zone Z. As Approx_k leads to an incorrect Algorithm 1, we get what we claimed.

The aim of the rest of this paper is to study more precisely where the problems appear and to bring out subclasses for which a correct widening operator can be found. We decided to do this study in the more general framework of updatable timed automata, as defined in [BDFP00a].

4 The Updatable Timed Automata Framework

Updatable timed automata are a syntactic extension of timed automata, introduced in [BDFP00a]. We present the model as well as some basic results and we propose a first algorithm which is correct for all the updatable timed automata that are decidable.

4.1 Definitions

Updates. An *update* is a function from $\mathbb{T}^X$ to $\mathcal{P}(\mathbb{T}^X)$ which assigns to each valuation a set of valuations. In this work, updates are restricted according to the following definitions.

A *simple update* over a clock z has one of the two following forms:

$$up \ ::= \ z :\sim c \ \mid \ z :\sim y + c$$
$$\text{where } c \in \mathbb{Z},\ y \in X \text{ and } \sim \in \{<, \leq, =, \geq, >\}$$

Let v be a valuation and up be a simple update over z. A valuation v' is in $up(v)$ if $v'(y) = v(y)$ for any clock $y \neq z$ and if

$$\begin{cases} v'(z) \sim c \text{ and } v'(z) \geq 0 & \text{if } up = z :\sim c \\ v'(z) \sim v(y) + d \text{ and } v'(z) \geq 0 & \text{if } up = z :\sim y + d \end{cases}$$

where $\sim \in \{<, \leq, =, \neq, \geq, >\}$:
In what follows, an *update* over a set of clocks X is a collection $up = (up_i)_{1 \leq i \leq k}$, where each up_i is a simple update over some clock $x_i \in X$ (note that it could happen that $x_i = x_j$ for some $i \neq j$). Let $v,\ v' \in \mathbb{T}^n$ be two clock valuations. Then we say $v' \in up(v)$ if for all i, the clock valuation v_i defined by

$$\begin{cases} v_i(x_i) = v'(x_i) \\ v_i(y) \ = v(y) & \text{for any } y \neq x_i \end{cases}$$

is in $up_i(v)$. The set of updates over the set of clocks X is denoted by $\mathcal{U}(X)$.

Example 3. For the update $up = (x :> y, x :< 7)$, a valuation v' is in $up(v)$ if the value $v'(x)$ satisfies $v'(x) > v(y) \wedge v'(x) < 7$. Note that $up(v)$ may be empty. For instance, the update $(x :< 1, x :> 1)$ leads to the empty set.

An update is a reset whenever it is a collection of simple updates of the form $x := 0$. The set of resets is denoted by $\mathcal{U}_0(X)$. For a subset C of X, if up is the reset $\bigwedge_{x \in C} x := 0$ (we write $C := 0$), then for each valuation v, the valuation $up(v)$ is also written $[C \leftarrow 0]v$.

Updatable Timed Automata. An *updatable timed automaton* over $\mathbb{T}$ is a 6-tuple $\mathcal{A} = (\Sigma, Q, T, I, F, X)$, where Σ is a finite alphabet of actions, Q is a finite set of states, X is a finite set of clocks, $T \subseteq Q \times [C(X) \times \Sigma \times \mathcal{U}(X)] \times Q$ is a finite set of transitions, $I \subseteq Q$ is the subset of initial states and $F \subseteq Q$ is the subset of final states.
A *path* in $\mathcal{A}$ is a finite sequence of consecutive transitions:

$$P = q_0 \xrightarrow{\varphi_1, a_1, up_1} q_1 \ \ldots \ q_{p-1} \xrightarrow{\varphi_p, a_p, up_p} q_p \text{ where } (q_{i-1}, \varphi_i, a_i, up_i, q_i) \in T, \text{ for each } 1 \leq i \leq p$$

The path is said to be *accepting* if it starts in an initial state ($q_0 \in I$) and ends in a final state ($q_p \in F$). A *run* of the automaton through the path P is a sequence of the form:

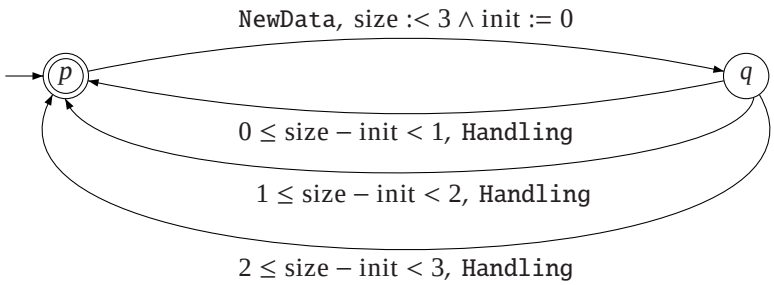
$$\langle q_0, v_0 \rangle \xrightarrow[t_1]{\varphi_1, a_1, up_1} \langle q_1, v_1 \rangle \ \ldots \ \xrightarrow[t_p]{\varphi_p, a_p, up_p} \langle q_p, v_p \rangle$$

where $\tau = (t_i)_{1 \leq i \leq p}$ is a time sequence and $(v_i)_{1 \leq i \leq p}$ are clock valuations defined by:

$$\begin{cases} v_0(x) = 0,\ \forall x \in X \\ v_{i-1} + (t_i - t_{i-1}) \models \varphi_i \\ v_i \in up_i\,(v_{i-1} + (t_i - t_{i-1})) \end{cases}$$

The label of the run is the timed word $w = (a_1, t_1) \ldots (a_p, t_p)$. If the path P is accepting then the timed word w is said to be accepted by the timed automaton. The set of all timed words accepted by $\mathcal{A}$ is denoted by $L(\mathcal{A})$.

Example 4. Consider a system that handles data which come from an external environment. The time needed to handle the data depends on its (bounded) size. Updatable timed automata provide an elegant and concise way to model such a system: assuming that the maximum size for the data is 3, it suffices to store the size in a variable taking non-deterministically a value less than 3.



Let $C \subseteq \mathcal{C}(X)$ be a set of clock constraints and $\mathcal{U} \subseteq \mathcal{U}(X)$ be a set of updates, the class $Aut(C, \mathcal{U})$ is the set of all updatable timed automata whose transitions only use clock constraints in C and updates in $\mathcal{U}$. The usual class of timed automata, defined in [AD90,AD94], is the family $Aut(\mathcal{C}_{df}(X), \mathcal{U}_0(X))$. However, what we now usually call the family of *classical timed automata*, as presented in section 2.1, is $Aut(\mathcal{C}(X), \mathcal{U}_0(X))$.

Known Results about the Emptiness Problem. In [BDFP00a], we characterized the decidable subclasses of updatable timed automata. The following tabular roughly summarizes the results:

$\mathcal{U}_0(X) \cup \{\dots\}$	**Diagonal-free clock constraints**	**General clock constraints**
$x := c$, $x := y$	PSPACE-complete	PSPACE-complete
$x := x + 1$		*Undecidable*
$x := y + c$		
$x := x - 1$	*Undecidable*	
$x := y - c$		
$x :< c$	PSPACE-complete	PSPACE-complete
$x :> c$		*Undecidable*
$x :\sim y + c$		
$y + c <: x :< y + d$		
$y + c <: x :< z + d$	*Undecidable*	

with $\sim \in \{\leq, <, >, \geq\}$ and $c,\ d \in \mathbb{N}$.

The table reads as follows: the class of updatable timed automata that use only diagonal-free clock constraints but also decrementation updates (*i.e.* updates of the form $x := x - 1$) is undecidable whereas the class of updatable timed automata with general clock constraints and updates of the form $x :< c$ is decidable using a PSPACE algorithm.

The proof of these decidability results is based on the generalization of the region automaton construction for classical timed automata (as presented in [AD90,AD94]). For the rest of the paper, we need more precise decidability results, we thus present them now.

4.2 Regions and Precise Results about Decidability [BDFP00a]

Regions. The definition of regions given in this paper is the one presented in [BDFP00a]. There is a slight difference from the original definition of Alur and Dill [AD90,AD94], which will be seen in Example 5. Let X be a set of clocks, $n = |X|$ and

$$\alpha = ((\max_x)_{x \in X}; (\max_{x,y})_{x,y \in X}) \in \mathbb{N}^n \times (\mathbb{Z} \times \mathbb{Z})^{n^2}$$

be a tuple such that for every $x, y \in X$, $\begin{cases} -\max_{y,x} \leq \max_{x,y} \\ \max_{x,y} \leq \max_x \end{cases}$.

The tuple α defines a set of regions $\mathcal{R}_\alpha$: a *region* R in $\mathcal{R}_\alpha$ is defined by a tuple

$$((I_x)_{x \in X}; (J_{x,y})_{x \neq y \in X}; \prec)$$

where:

– I_x is one of the following intervals:

$$]-\infty;-\max_x[\quad ; \quad]\max_x;+\infty[\quad ; \quad]c;c+1[\quad \text{where} \quad -\max_x \le c < \max_x$$
$$\text{or} \quad \{c\} \quad \text{where} \quad -\max_x \le c \le \max_x$$

– $J_{x,y}$ is one of the following intervals:

$$]-\infty;-\max_{y,x}[\quad ; \quad]\max_{x,y};+\infty[\quad ; \quad]c;c+1[\quad \text{where} \quad -\max_{y,x} \le c < \max_{x,y}$$
$$\text{or} \quad \{c\} \quad \text{where} \quad -\max_{y,x} \le c \le \max_{x,y}$$

– $\prec$ is a total preorder defined on the set $\{x \mid \exists\, c \in [-\max_x; \max_x[\text{ such that } I_x =]c;c+1[\}$.

A valuation v is in the region R if:[4]

$$\begin{cases} v(x) \in I_x & \text{for each } x \in X \\ I_x =]\max_x;+\infty[\text{ or } I_y =]\max_y;+\infty[\implies v(x)-v(y) \in J_{x,y} & \text{for every } x \neq y \in X \\ x \prec y \implies \mathtt{frac}(v(x)) \le \mathtt{frac}(v(y)) & \text{for every } x, y \in X. \end{cases}$$

Note that for a valuation v, there is a unique region in $\mathcal{R}_\alpha$ containing v; it is denoted by $[v]_\alpha$.

Example 5. We define $\alpha = ((\max_x, \max_y), (\max_{x,y}, \max_{y,x}))$ as

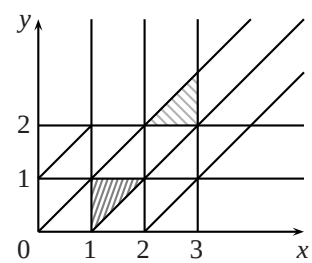
$$\max_x = 3,\ \max_y = 2,\ \max_{x,y} = 2,\ \max_{y,x} = 0$$

The set of regions $\mathcal{R}_\alpha$ is represented on the figure beside. The region drawn with ▧ is represented by the intervals

$$\begin{cases} I_x =]2,3[\\ I_y =]2,+\infty[\end{cases} \quad \text{and} \quad \begin{cases} I_{x,y} =]0,1[\\ I_{y,x} =]-1,0[\end{cases}$$

The region drawn with ▨ is represented by the following intervals and preorder:

$$\begin{cases} I_x =]1,2[\\ I_y =]1,0[\end{cases} \quad \text{and } \prec \text{ is defined by } y \prec x$$



The difference with the traditional set of regions from [AD90,AD94] is that some diagonal constraints may appear in the non-bounded regions.

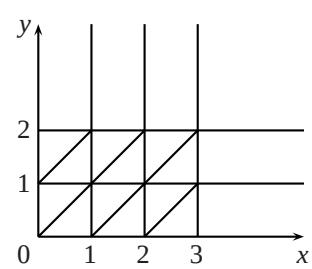
We can note that every convex union of regions of $\mathcal{R}_\alpha$ is a zone. The set of such convex unions of regions of $\mathcal{R}_\alpha$ is denoted by $\mathcal{Z}_\alpha$. If Z is a zone, we define $\mathsf{Approx}_\alpha(Z)$ as the smallest zone of $\mathcal{Z}_\alpha$ that contains Z (this zone is well-defined because there are finitely many zones in $\mathcal{Z}_\alpha$, thus it can be argued as in section 2.3 for the k-approximation that this smallest zone exists). We say that $\mathsf{Approx}_\alpha(Z)$ is the *α-overapproximation* of Z. We can note that the k-overapproximation of a zone defined in Section 2.3 corresponds to its α-overapproximation with $\alpha = ((k,\dots,k),(k,\dots,k))$.

A set of regions $\alpha = ((\max_x))_{x\in X}; (\max_{x,y})_{x,y\in X})$ is called *diagonal-free* if for all clocks $x,\ y \in X$, $\max_{x,y} = -\infty$. In this case we simply write $\alpha = (\max_x)_{x\in X}$.

Example 6. We define α as the (diagonal-free) tuple $(\max_x, \max_y)$ defined by:

$$\max_x = 3,\ \max_y = 2$$

The set of regions $\mathcal{R}_\alpha$ is represented on the figure beside.



[4] $\mathtt{frac}(\gamma)$ denotes the fractional part of γ

Precise Decidability Results. In Section 4.1, we presented roughly the decidability results obtained in [BDFP00a]. We will now make these results more precise. We distinguish two different cases, the updatable timed automata with diagonal-free clock constraints and the updatable timed automata with general clock constraints.

Diagonal-free updatable timed automata. Let $C \subseteq C_{df}(X)$ be a set of diagonal-free clock constraints and let $\mathcal{U} \subseteq \mathcal{U}(X)$ a set of updates such that

$$up = \bigwedge_{x \in X} up_x \in \mathcal{U} \iff \text{for each } x,\ up_x \in \{\det_x, \inf_x, \sup_x, \mathrm{int}_x\} \text{ where:}$$

$$\begin{cases} \det_x ::= x := c \mid x := z + d \text{ with } c \in \mathbb{N},\ d \in \mathbb{Z} \text{ and } z \in X \\ \inf_x ::= x :\lhd c \mid x :< z + d \mid \inf_x \wedge \inf_x \text{ with } \lhd \in \{<, \leq\},\ c \in \mathbb{N},\ d \in \mathbb{Z} \text{ and } z \in X \\ \sup_x ::= x :\rhd c \mid x :> z + d \mid \sup_x \wedge \sup_x \text{ with } \rhd \in \{>, \geq\},\ c \in \mathbb{N},\ d \in \mathbb{Z} \text{ and } z \in X \\ \mathrm{int}_x ::= x :\in (c; d) \mid x :\in (c; z + d) \mid x :\in (z + c; d) \mid x :\in (z + c; z + d) \\ \qquad\qquad \text{where (and) are either [or],\ } z \text{ is a clock and } c,\ d \text{ are in } \mathbb{Z}. \end{cases}$$

If the system of linear Diophantine inequations over the variables $(\max_x)_{x \in X}$

$$\{c \leq \max_x \mid x \sim c \in C \text{ or } x :\sim c \in \mathcal{U}\} \cup \{\max_z \leq \max_y + c \mid z :\sim y + c \in \mathcal{U}\} \tag{1}$$

has a solution, then the class $Aut(C, \mathcal{U})$ is decidable. Moreover, if $\alpha = (\max_x)_{x \in X}$ is a solution of this system, then we can test for emptiness of any updatable timed automaton from $Aut(C, \mathcal{U})$ using the region automaton based on the diagonal-free set of regions $\mathcal{R}_\alpha$. Of course, this condition is not *necessary*, it can be the case that this system has no solution, but its emptiness can nevertheless be decided, for example when clocks are bounded, but the updates involved lead to systems which do not have any solution.

Remark 1. The system of equations (1) has been set in order for a solution α to define a set of regions $\mathcal{R}_\alpha$ such that the region automaton using $\mathcal{R}_\alpha$ as set of regions is bisimilar to the original updatable timed automaton. All the conditions described by (1) aim at ensuring this property. See [BDFP00a] for detailed explanations on how these conditions ensure the bisimulation property.

General updatable timed automata. Let $C \subseteq C(X)$ be a finite set of clock constraints and $\mathcal{U} \subseteq \mathcal{U}(X)$ be a finite set of updates such that

$$up = \bigwedge_{x \in X} up_x \in \mathcal{U} \iff \forall x \in X,\ up_x \in \{x := c, x :< c, x :\leq c \mid c \in \mathbb{N}\} \cup \{x := y \mid y \in X\}.$$

Then the class $Aut(C, \mathcal{U})$ is decidable. Moreover, if $\alpha = ((\max_x)_{x \in X}; (\max_{x,y})_{x,y \in X})$ is a solution of the following linear Diophantine inequation system:

$$\begin{array}{ll} & \{c \leq \max_x \mid x \sim c \in C\} \\ \cup & \{c \leq \max_{x,y} \mid x - y \sim c \in C\} \\ \cup & \{c \leq \max_x,\ \max_z \geq c + \max_{z,x} \mid x :< c \text{ or } x :\leq c \text{ or } x := c\ \in \mathcal{U}, \text{ and } z \in X\} \\ \cup & \{\max_x \leq \max_y,\ \max_{z,y} \geq \max_{z,x},\ \max_{x,z} \leq \max_{x,y} \mid x := y \in \mathcal{U} \text{ and } z \in X\} \end{array} \tag{2}$$

then we can test for emptiness of every updatable timed automaton from $Aut(C, \mathcal{U})$ using the region automaton based on the set of regions $\mathcal{R}_\alpha$.

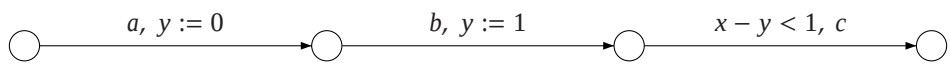
Remark 2. Consider a classical timed automaton $\mathcal{A}$ and define the constant k as the largest constant appearing in $\mathcal{A}$. Then the set of regions $\alpha = ((k, \ldots, k), (k, \ldots, k))$ is a solution for the system associated with $\mathcal{A}$ as above.

Remark 3. In both cases (diagonal-free or not), the set of solutions of the linear system corresponding to an updatable timed automaton could have several solutions. However, if there is a solution, then there is a unique minimal solution (for the natural order on the tuples): indeed, all the inequations are of one of the forms

$$X \geq c,\ X - Y \geq c$$

where X and Y are variables and c is a constant. Thus, if we denote by $(X_i)_{i=1\ldots n}$ all the variables involved in the system, it can be represented by a DBM M and the unique minimal solution is the *offset*-point of the zone represented by M, that is the point with minimal coordinates belonging to the zone represented by M.

Example 7. Consider the following updatable timed automaton:



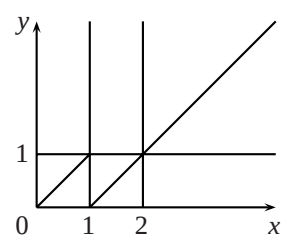
The system of inequations associated with this automaton is the following:

$$\begin{cases} \max_y \geq 0 \\ \max_x \geq 0 + \max_{x,y} \\ \max_y \geq 1 \\ \max_x \geq 1 + \max_{x,y} \\ \max_{x,y} \geq 1 \end{cases}$$

The minimal solution α of this system is:

$$\begin{cases} \max_x = 2 \\ \max_y = 1 \\ \max_{x,y} = 1 \end{cases}$$

The set of regions defined by these constants is drawn on the figure beside.



In what follows we will only consider decidable updatable timed automata, that is updatable timed automata that belong to some decidable class $Aut(\mathcal{C}, \mathcal{U})$. We will associate with each decidable updatable timed automaton a set of regions, $\mathcal{R}_\alpha$ as described above, where α is the minimal solution of the previous system (the smaller the solution is, the smaller is the number of regions of $\mathcal{R}_\alpha$ and the more abstract is the region automaton).

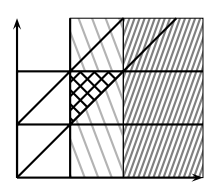
4.3 A First Correct Algorithm for Decidable Updatable Timed Automata

For each zone Z, for each tuple α, we define the closure by regions from $\mathcal{R}_\alpha$ of Z by

$$\mathsf{Closure}_\alpha(Z) = \bigcup \{R \in \mathcal{R}_\alpha \mid R \cap Z \neq \emptyset\}.$$

We call $\mathsf{Closure}_\alpha(Z)$ the *α-closure* of Z. First note that $\mathsf{Closure}_\alpha(Z)$ can be different from $\mathsf{Approx}_\alpha(Z)$: $\mathsf{Approx}_\alpha(Z)$ is convex whereas $\mathsf{Closure}_\alpha(Z)$ can be non-convex.

Example 8. Let us consider the set of regions depicted as on the picture beside. If we consider the zone Z drawn using ▨ then the α-closure $\mathsf{Closure}_\alpha(Z)$ is drawn adding the part ▧: it is thus non convex. The α-approximation $\mathsf{Approx}_\alpha(Z)$ is obtained by adding the part ⊠.



The first algorithm is presented as Algorithm 2.

Algorithm 2 A First Algorithm for Decidable Updatable Timed Automata

```
# First UTA-Algorithm (𝒜:UTA) {
#     Compute α using 𝒰 and C;
#     Visited := ∅;                                  (* Visited stores the visited states *)
#     Waiting := {(q0,Closure_α(Z0))};
#     Repeat
#         Get and Remove (q,Z) from Waiting;
#         If q is final
#             then {Return "Yes";}
#             else {If there is no (q,Z') ∈ Visited such that Z ⊆ Z'
#                     then {Visited := Visited ∪ {(q,Z)};
#                           Successor := {(q',Closure_α(Post(Z,e))) | e transition from q to q'};
#                           Waiting := Waiting ∪ Successor;}}
#     Until (Waiting = ∅);
#     Return "No"; }
```

This algorithm of course terminates (if α is fixed, when Z varies, there are finitely many different $\mathsf{Closure}_\alpha(Z)$). We will prove its correctness.

Let $\mathcal{U}$ be a set of updates and C a set of clock constraints. Let α be the tuple defining the set of regions $\mathcal{R}_\alpha$ as in Section 4.2. It implies in particular (see [BDFP00a]) that for each clock constraint $g \in C$, for each update $up \in \mathcal{U}$, for each region $R \in \mathcal{R}_\alpha$, for each valuation $v \in R$:

$$\begin{cases} R \subseteq g \quad \text{or} \quad R \subseteq \neg g \\ up(v) \cap R \neq \emptyset \implies \forall v' \in [v]_\alpha,\ up(v') \cap R \neq \emptyset \end{cases} \tag{3}$$

This ensures that the emptiness problem is decidable for the class $Aut(C, \mathcal{U})$ (see [BDFP00a]).

Proposition 1. *The "First UTA-Algorithm" tests for emptiness of every updatable timed automaton belonging to a decidable class as described before.*

The proof of this proposition relies on several intermediate results. We first claim that:

Lemma 1. *The two following conditions hold:*

$$\begin{cases} \mathsf{Closure}_\alpha(g) = g & \textit{for each clock constraint } g \textit{ in } C \\ up(\mathsf{Closure}_\alpha(Z)) \subseteq \mathsf{Closure}_\alpha(up(Z)) & \textit{for each zone } Z \textit{ and each update } up \textit{ in } \mathcal{U} \end{cases} \tag{4}$$

➜ *See the proof in Appendix.*

We also claim that:

Lemma 2. *For each zone Z, $\overrightarrow{\mathsf{Closure}_\alpha(Z)} \subseteq \mathsf{Closure}_\alpha(\overrightarrow{Z})$ and for each clock constraint g such that $\mathsf{Closure}_\alpha(g) = g$, $g \cap \mathsf{Closure}_\alpha(Z) \subseteq \mathsf{Closure}_\alpha(g \cap Z)$.*

➜ *See the proof in Appendix.*

With these two lemmas, we are now ready to prove the proposition. Using the condition (4), if $e = (q, g, a, up, q')$ is a transition, we compute:

$$\begin{aligned} \mathsf{Post}(\mathsf{Closure}_\alpha(Z), e) &= up(g \cap \overrightarrow{\mathsf{Closure}_\alpha(Z)}) \\ &\subseteq up(g \cap \mathsf{Closure}_\alpha(\overrightarrow{Z})) \text{ from Lemma 2} \\ &\subseteq up(\mathsf{Closure}_\alpha(g \cap \overrightarrow{Z})) \text{ from Lemma 2} \\ &\subseteq \mathsf{Closure}_\alpha(up(g \cap \overrightarrow{Z})) \text{ from condition (4)} \\ &\subseteq \mathsf{Closure}_\alpha(\mathsf{Post}(Z, e)) \end{aligned}$$

In particular, whenever $\mathsf{Post}(Z, e)$ is empty, $\mathsf{Post}(\mathsf{Closure}_\alpha(Z), e)$ is empty (because $\mathsf{Closure}_\alpha(Z)$ is empty if and only if Z is empty).

In fact, the "First UTA-Algorithm" computes for some consecutive transitions $(e_i)_{i=1\ldots p}$, the zone

$$\mathsf{Closure}_\alpha(\mathsf{Post}(\mathsf{Closure}_\alpha(\mathsf{Post}(\mathsf{Closure}_\alpha(\mathsf{Post}(\ldots, e_{p-2})), e_{p-1})), e_p)).$$

Using iteratively the above relation and the fact that $\mathsf{Closure}_\alpha$ is involutive, we get that:

$$\mathsf{Closure}_\alpha(\mathsf{Post}(\mathsf{Closure}_\alpha(\mathsf{Post}(\ldots, e_{p-1})), e_p)) \subseteq \mathsf{Closure}_\alpha(\mathsf{Post}(\mathsf{Post}(\ldots, e_{p-1}), e_p))$$

Thus, if $\Re(\langle q_0, Z_0\rangle)$ and $\mathfrak{F}(\langle q_0, Z_0\rangle)$ denote respectively the exact set of reachable states and the set of states computed by the "First UTA-Algorithm", we have:

$$\Re(\langle q_0, Z_0\rangle) \subseteq \mathfrak{F}(\langle q_0, Z_0\rangle) \subseteq \mathsf{Closure}_\alpha(\Re(\langle q_0, Z_0\rangle))$$

Hence, $\mathfrak{F}(\langle q_0, Z_0\rangle) = \emptyset \iff \Re(\langle q_0, Z_0\rangle) = \emptyset$, which concludes the proof of the theorem. □

Finally, for each updatable timed automaton $\mathcal{A}$ belonging to some decidable class we described before, we can test for emptiness of $L(\mathcal{A})$ using Algorithm 2. Although this algorithm can not be implemented efficiently (the α-closure of a zone is not a zone and its computation requires to check all regions intersecting the zone, which might need a time exponential in the number of clocks), its correctness is fundamental for deriving the correctness of other algorithms.

5 Focusing on Correctness

In this section, we extend Algorithm 1 to (decidable) updatable timed automata and study precisely its correctness. We know that a correct widening operator can not be found for every automaton, but we want to point out what are exactly the problems and provide subclasses for which such an algorithm is correct and can thus be used safely.

5.1 The Natural Extension of Algorithm 1 for Decidable Updatable Timed Automata

In this section, we present a forward analysis algorithm for updatable timed automata. This algorithm is parameterized by a function f, which associates with each automaton a set of regions. The widening operator used in the algorithm will be this set of regions. Of course, starting from the class of decidable updatable timed automata, we want to bring out subclasses for which we can find a function f yielding a correct algorithm. The algorithm is described by Algorithm 3.

Algorithm 3 Algorithm for Updatable Timed Automata

```
# UTA-Algorithm [f] (A:UTA) {
#     Define β as f(A);
#     Visited := ∅;                          (* Visited stores the visited states *)
#     Waiting := {(q0,Approx_β(Z0))};
#     Repeat
#         Get and Remove (q,Z) from Waiting;
#         If q is final
#             then {Return "Yes";}
#             else {If there is no (q,Z') ∈ Visited such that Z ⊆ Z'
#                     then {Visited := Visited ∪ {(q,Z)};
#                           Successor := {(q',Approx_β(Post(Z,e))) | e transition from q to q'};
#                           Waiting := Waiting ∪ Successor;}}
#     Until (Waiting = ∅);
#     Return "No"; }
```

If we come back to classical timed automata and define the function f as the one which associates with every classical timed automaton $\mathcal{A}$ the tuple $((k,\dots,k),(k,\dots,k))$ where k is the maximum constant appearing in $\mathcal{A}$, Algorithm 3 parametrized by f is exactly Algorithm 1. Like Algorithm 1, Algorithm 3 terminates because there are finitely many overapproximations of zones that can be associated with each control state. From the counter-example of section 3, we know that it is hopeless to find a parameter function f such that for every decidable updatable timed automaton, Algorithm 3 is correct, but our aim is to distinguish classes together with appropriate parameters such that it is the case (in which case we will say that Algorithm 3 is *correct with respect to reachability* for this class of automata using the given parameter).

We first study in detail the (partial) correctness of Algorithm 3, and we then present how the DBM data structure can be used in order to implement Algorithm 3.

5.2 Partial Correctness

As we have seen in section 3, Algorithm 3 is not correct in general. However, we will propose some subclasses for which it is correct. We will write the proofs very carefully and with details because, as it will appear, some properties of zones and DBMs are not intuitive at all, and, in the past, mistakes have been done (see section 3), maybe because of this bad intuition we have about zones. Our aim is thus to explain where the problems exactly come from.

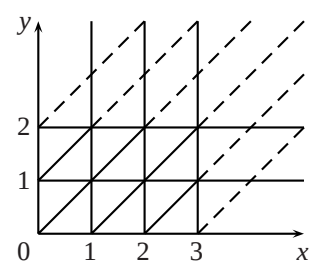
As we already proved that Algorithm 2 is correct with respect to reachability, in order to prove that Algorithm 3 (parameterized by a function f) is correct for a subclass of timed automata, it is sufficient to prove the following: given a timed automaton $\mathcal{A}$ of this class, α associated to $\mathcal{A}$ as in section 4.2 and $\beta = f(\mathcal{A})$, then for every "computed zone"[5],

$$\begin{array}{ccccc} Z & \subseteq & \mathsf{Approx}_\beta(Z) & \subseteq & \mathsf{Closure}_\alpha(Z) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Exact set} & & \text{Algorithm 3} & & \text{Algorithm 2} \end{array} \tag{5}$$

Indeed, it would mean that the widening operator Approx_β used in Algorithm 3 is coarser than the α-closure used in Algorithm 2. Thus, since Algorithm 2 is correct with respect to reachability, Algorithm 3 will also be correct for the corresponding class of automata.

Diagonal-Free Updatable Timed Automata. Let $\mathcal{A}$ be a diagonal-free decidable updatable timed automaton. As described in Section 4.2, we associate with $\mathcal{A}$ a diagonal-free set of regions $\mathcal{R}_\alpha$ where $\alpha = (\max_x)_{x\in X}$. We then define a second set of regions, $\mathcal{R}_\beta$ where β is the tuple $((\mathrm{Max}_x)_{x\in X};(\mathrm{Max}_{x,y})_{x,y\in X})$ with $\mathrm{Max}_{x,y} = \mathrm{Max}_x = \max_x$ for all clocks $x, y \in X$.

Example 9. The sets of regions $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ can be both represented (in a two-dimensional time-space) as on the figure beside, where α is drawn with solid lines whereas the refinement β also uses dashed lines ($\mathcal{R}_\beta$ refines $\mathcal{R}_\alpha$).



If α and β are defined as above, we have the following property:

Proposition 2. *For each zone Z, the following holds:*

$$Z \subseteq \mathsf{Approx}_\beta(Z) \subseteq \mathsf{Closure}_\alpha(Z).$$

[5] We will make the notion of "computed zone" more precise later.

Proof. It is equivalent to prove that for every zone Z, for every region R of $\mathcal{R}_\alpha$,

$$R \cap Z = \emptyset \implies R \cap \mathsf{Approx}_\beta(Z) = \emptyset$$

Let $R = ((I_{x_i})_{x_i \in X}; \prec)$ be a region of $\mathcal{R}_\alpha$ and let $M_R = ((r_{i,j}; \prec_{i,j})_{i,j=0\ldots n})$ be the DBM representing R such that:

$$\begin{cases} (r_{i,0}; \prec_{i,0}) = \begin{cases} (c+1; <) & \text{if } I_{x_i} =]c; c+1[\\ (c; \leq) & \text{if } I_{x_i} = \{c\} \\ (+\infty; <) & \text{if } I_{x_i} =]\max_{x_i}; +\infty[\end{cases} \\ (r_{0,i}; \prec_{0,i}) = \begin{cases} (-c; <) & \text{if } I_{x_i} =]c; c+1[\\ (-c; \leq) & \text{if } I_{x_i} = \{c\} \\ (-\max_{x_i}; <) & \text{if } I_{x_i} =]\max_{x_i}; +\infty[\end{cases} \\ (r_{i,j}; \prec_{i,j}) = \begin{cases} (c-d+1; <) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} =]d; d+1[\text{ and } x_j \prec x_i,\ x_i \not\prec x_j \\ (c-d; <) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} =]d; d+1[\text{ and } x_i \prec x_j,\ x_j \not\prec x_i \\ (c-d; \leq) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} =]d; d+1[\text{ and } x_i \prec x_j,\ x_j \prec x_i \\ (c-d; \leq) & \text{if } I_{x_i} = \{c\} \text{ and } I_{x_j} = \{d\} \\ (c+1-d; <) & \text{if } I_{x_i} =]c; c+1[,\ I_{x_j} = \{d\} \\ (c-d-1; <) & \text{if } I_{x_i} = \{c\} \text{ and } I_{x_j} =]d; d+1[\\ (+\infty; <) & \text{in all other cases} \end{cases} \end{cases}$$

Note that the diagonal-free hypothesis on $\mathcal{R}_\alpha$ has the important consequence that if $(r_{i,0}; \prec_{i,0}) = (+\infty; <)$, then for every $j \neq 0$, $(r_{i,j}; \prec_{i,j}) = (r_{j,i}; \prec_{j,i}) = (+\infty; <)$. This property will be the core of the proof.

Let Z be a zone and $M = ((m_{i,j}; \prec_{i,j})_{i,j})$ a DBM in normal form representing Z. Assume that $Z \cap R = \emptyset$. It means that there exists a sequence of distinct indices $(i_1, i_2, \ldots, i_l = i_1)$ such that

$$\alpha_{i_1,i_2} + \alpha_{i_2,i_3} + \cdots + \alpha_{i_{l-1},i_l} < (0; \leq) \tag{6}$$

where $\alpha_{i_j,i_k} = \min((m_{i_j,i_k}; \prec_{i_j,i_k}); (r_{i_j,i_k}; \prec_{i_j,i_k}))$. Since M is in normal form, we can assume that two successive $\alpha_{h,\ell}$ do not come from M, otherwise we could simplify the sum (6). Note that if $(m_{i_j,i_k}; \prec_{i_j,i_k}) < (r_{i_j,i_k}; \prec_{i_j,i_k})$ and if I_{x_i} and I_{x_j} are both bounded, then we can distinguish two cases:

- <u>First case:</u> $(r_{i_j,i_k}; \prec_{i_j,i_k}) = (r_{i_j,i_k}; \leq) = (-r_{i_k,i_j}; \leq) = (-r_{i_k,i_j}; \prec_{i_k,i_j})$.
  In this case, $(m_{i_j,i_k}; \prec_{i_j,i_k}) < (-r_{i_k,i_j}; \prec_{i_k,i_j})$, hence

  $$(m_{i_j,i_k}; \prec_{i_j,i_k}) + (r_{i_k,i_j}; \prec_{i_k,i_j}) < (0; \leq)$$

  Thus, $Z \subseteq (x_{i_j} - x_{i_k} < -r_{i_k,i_j})$ and $R \cap (x_{i_j} - x_{i_k} < -r_{i_k,i_j}) \neq \emptyset$. We obtain $\mathsf{Approx}_\beta(Z) \subseteq (x_{i_j} - x_{i_k} < -r_{i_k,i_j})$ and then $\mathsf{Approx}_\beta(Z) \cap R = \emptyset$. Note that, obviously, the zone defined by $(x_{i_j} - x_{i_k} < -r_{i_k,i_j})$ is β-bounded because $\mathcal{R}_\beta$ refines $\mathcal{R}_\alpha$.
- <u>Second case:</u> $(r_{i_j,i_k}; \prec_{i_j,i_k}) = (c; <)$ and $(r_{i_k,i_j}; \prec_{i_k,i_j}) = (-c+1; <)$.
  In this case, $(m_{i_j,i_k}; \prec_{i_j,i_k}) \leq (c-1; <)$. Thus, like above, we obtain $\mathsf{Approx}_\beta(Z) \cap R = \emptyset$.

Let us assume now that we are not in one of the two previous cases.
Thus, if $(m_{i_j,i_{j+1}}; \prec_{i_j,i_{j+1}}) < (r_{i_j,i_{j+1}}; \prec_{i_j,i_{j+1}})$, it means that either $I_{x_{i_j}} =]\max_{x_{i_j}}; +\infty[$ or $I_{x_{i_{j+1}}} =]\max_{x_{i_{j+1}}}; +\infty[$. If $I_{x_{i_j}} =]\max_{x_{i_j}}; +\infty[$, from the construction of M_R, we get $i_{j-1} = 0$, because if $i_{j-1} \neq 0$, then $(r_{i_{j-1},i_j}; \prec_{i_{j-1},i_j}) = (+\infty; <)$. If $I_{x_{i_{j+1}}} =]\max_{x_{i_{j+1}}}; +\infty[$, we obtain a contradiction, because every coefficient $r_{i_{j+1},*}$ is infinite. After this analysis, we know that in the sum (6), we cannot have two coefficients coming from M. Thus, since R is not empty, we have exactly one coefficient coming from M in the sum (6). Collecting all these informations, we obtain a sequence of indices $(k_1, \ldots, k_p)$ such that

$$(r_{0,k_1}; \prec_{0,k_1}) + (m_{k_1,k_2}; \prec_{k_1,k_2}) + (r_{k_2,k_3}; \prec_{k_2,k_3}) + \cdots + (r_{k_p,0}; \prec_{k_p,0}) < (0; \leq)$$

Now, by computing the normal form of M_R, the only coefficients of M_R that can be tightened are the $(r_{x,y}; \prec_{x,y})$ where $r_{x,0}$ is bounded. Thus, simplifying the previous sum yields

$$(r_{0,k_1}; \prec_{0,k_1}) + (m_{k_1,k_2}; \prec_{k_1,k_2}) + (r_{k_2,0}; \prec_{k_2,0}) < (0; \leq)$$

Setting $(c; \prec) = (r_{k_2,0}; \prec_{k_2,0}) + (r_{0,k_1}; \prec_{0,k_1})$, we get

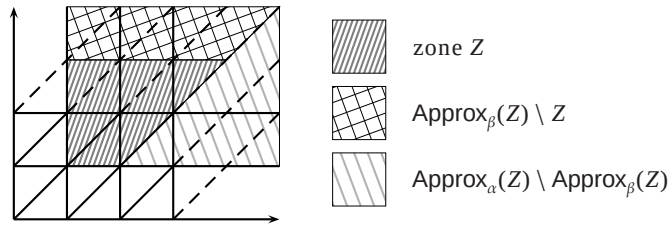
$$\begin{cases} Z \subseteq (x_{k_1} - x_{k_2} \prec -c) \\ R \cap (x_{k_1} - x_{k_2} \prec -c) = \emptyset \end{cases}$$

Then

$$-\mathrm{Max}_{x_{k_1},x_{k_2}} = -\max_{x_{k_1}} \leq c \leq \max_{x_{k_2}} = \mathrm{Max}_{x_{k_2},x_{k_1}}$$

implies that the zone $(x_{k_1} - x_{k_2} \prec -c)$ is β-bounded. As previously, we conclude that $\mathsf{Approx}_\beta(Z) \subseteq (x_{k_1} - x_{k_2} \prec -c)$ and thus $R \cap \mathsf{Approx}_\beta(Z) = \emptyset$. □

Remark 4. Note that if we consider Approx_α instead of Approx_β, the previous property does not hold anymore. See for example the figure beside.



Using what was said before, we get the following theorem:

Theorem 1. *Algorithm 3 is correct (with respect to reachability) for decidable diagonal-free updatable timed automata (taking as parameter the function defined just before the previous proof).*

This theorem, even if it is not as general as we could expect, is already interesting because in many practical cases, the automata built for real systems are diagonal-free (see as examples [HSLL97] or [BBP02]). As a direct consequence of Theorem 1, we get the following correctness result for classical timed automata, as defined in section 2.

Corollary 1. *Algorithm 1 is correct (with respect to reachability) for diagonal-free classical timed automata. The algorithm implemented in* Uppaal *and* Kronos *is correct for diagonal-free classical timed automata.*

General Updatable Timed Automata. From the counter-example of section 3, we know that it is hopeless to prove the correctness of Algorithm 3 in the general case. However, we will bring out a subclass of the general updatable timed automata for which we can find a parameter function f such that Algorithm 3 is correct. Consider a decidable updatable timed automaton $\mathcal{A}$ and the set of regions $\mathcal{R}_\alpha$ defined (as in section 4.2) by $\alpha = ((\max_x)_{x\in X}; (\max_{x,y})_{x,y\in X})$. The parameter function f associates with $\mathcal{A}$ the tuple $\beta = ((\mathrm{Max}_x)_{x\in X}; (\mathrm{Max}_{x,y})_{x,y\in X})$ where $\mathrm{Max}_x = \mathrm{Max}_{x,y} = \sum_{z\in X} \max_z$ for all clocks $x, y \in X$. Roughly speaking, the constant $\sum_{z\in X} \max_z$ is the maximum constant that can appear in a DBM in normal form that represents a region of $\mathcal{R}_\alpha$.

Theorem 2. *Algorithm 3 is correct (with respect to reachability) for decidable general updatable timed automata with less than three clocks (the parameter is the one defined just above).*

This theorem is not in contradiction with what we announced in section 3. Indeed, the counter-example of figure 1 uses four clocks. The limit between correctness and non-correctness locates thus precisely between three and four clocks.

Proof. Proving property (5) is equivalent to prove that for every region R of $\mathcal{R}_\alpha$, for every "computed" zone Z, if $R \cap Z = \emptyset$ then $R \cap \mathsf{Approx}_\beta(Z) = \emptyset$. Let $\mathcal{A}$ be a decidable general updatable timed automaton with less than three clocks, as defined in section 4.2. We will prove that, for each zone

Z which is computed while Algorithm 3 runs on $\mathcal{A}$, the following holds: for each region R of $\mathcal{R}_\alpha$, whenever

$$R \cap Z = \emptyset$$

then there exists a β-bounded zone V such that

$$\begin{cases} Z \subseteq V \\ R \cap V = \emptyset \end{cases} \tag{7}$$

This property will imply what we want because $\mathsf{Approx}_\beta(Z)$ is the smallest β-bounded zone that contains the zone Z.
Assume that a zone Z satisfies condition (7) and suppose we are given a clock constraint g and an update up of the automaton $\mathcal{A}$. We will prove that $up(g \cap \overrightarrow{Z})$ also satisfies condition (7). Having proved that, we will deduce that if Z is one of the zones computed by the algorithm and if R is any region of $\mathcal{R}_\alpha$, if $Z \cap R = \emptyset$, then $\mathsf{Approx}_\beta(Z) \cap R = \emptyset$ because $\mathsf{Approx}_\beta(Z)$ is the smallest β-bounded zone containing Z.

Computing the future. Assume that $R \cap \overrightarrow{Z} = \emptyset$. It means that $\overleftarrow{R} \cap Z = \emptyset$ where $\overleftarrow{R}$ is the past of R, that is $\overleftarrow{R} = \{v \mid \exists t.\ v + t \in R\}$. Moreover, using the bisimulation property of the set of regions $\mathcal{R}_\alpha$ (see remark 1, page 13), $\overleftarrow{R}$ is a union of regions $\bigcup_i R_i$. For each i, we can find a β-bounded zone V_i such that

$$\begin{cases} Z \subseteq V_i \\ R_i \cap V_i = \emptyset \end{cases}$$

We then define $V = \bigcap_i V_i$. It satisfies:

$$\begin{cases} Z \subseteq V \\ \overleftarrow{R} \cap V = \emptyset \end{cases}$$

Let us prove that we can eliminate all the constraints of V which are of the form "x bounded by c". Let $M_R = ((r_{i,j}); \prec_{i,j})_{i,j})$ be a DBM for $\overleftarrow{R}$ such that for each i, $(m_{0,i}; \prec_{0,i})$ is $(0; \leq)$ (see [Ben02], part A). Moreover, assume that V can be rewritten as a DBM $((v_{i,j}; \prec_{i,j})_{i,j})$. Since $\overleftarrow{R} \cap V = \emptyset$, if we assume that a constraint "x bounded by c" is needed in V, it means that there exists $(i_1, i_2, \ldots, i_p = 0, i_{p+1}, \ldots i_l = i_1)$ such that

$$\alpha_{i_1,i_2} + \cdots + \alpha_{i_{p-2},i_{p-1}} + (v_{i_{p-1},0}; \prec_{i_{p-1},0}) + \alpha_{0,i_{p+1}} + \cdots + \alpha_{i_{l-1},i_l} < (0; \leq) \tag{8}$$

where $\alpha_{k,h} = \min((r_{k,h}; \prec_{k,h}); (v_{k,h}; \prec_{k,h}))$. In particular,

$$\alpha_{0,i_{p+1}} = \min((r_{0,i_{p+1}}; \prec_{0,i_{p+1}}); (v_{0,i_{p+1}}; \prec_{0,i_{p+1}})) = (v_{0,i_{p+1}}; \prec_{0,i_{p+1}})$$

because $(r_{0,i_{p+1}}; \prec_{0,i_{p+1}}) = (0; \leq)$. Thus, the sum $(v_{i_{p-1},0}; \prec_{i_{p-1},0}) + (v_{0,i_{p+1}}; \prec_{0,i_{p+1}})$ in (8) can be safely replaced by a constraint $(v'_{i_{p-1},i_{p+1}}; \prec'_{i_{p-1},i_{p+1}}) = (v_{i_{p-1},0}; \prec_{i_{p-1},0}) + (v_{0,i_{p+1}}; \prec_{0,i_{p+1}})$. We then define the zone

$$V' = \bigwedge \{x_i - x_j \prec_{i,j} v_{i,j} \in V \mid j \neq 0\} \wedge (x_{i_{p-1}} - x_{i_{p+1}} \prec'_{i_{p-1},i_{p+1}} v'_{i_{p-1},i_{p+1}})$$

which is a β-bounded zone. We have

$$\begin{cases} \overrightarrow{Z} \subseteq V' \\ R \cap V' = \emptyset \end{cases}$$

Intersection with a clock constraint. Assume that g is a clock constraint and that $R \cap (g \cap Z) = \emptyset$. We distinguish two cases:
<u>First case:</u> $R \cap g = \emptyset$, from the definition of regions, there exists a constraint $x - y \prec c$ (constraint "defining α") such that $g \subseteq (x - y \prec c)$ and $R \cap (x - y \prec c) = \emptyset$. Thus, we get that $g \cap Z \subseteq (x - y \prec c)$ and $R \cap (x - y \prec c) = \emptyset$. Note that since $x - y = c$ is a constraint defining the set of region $\mathcal{R}_\alpha$, $(x - y \prec c)$ is a β-bounded zone.
<u>Second case:</u> $R \subseteq g$. It means that $R \cap Z = \emptyset$, and we can use the induction hypothesis to prove that there exists a β-bounded zone V such that $Z \subseteq V$ and $R \cap V = \emptyset$. In particular, $Z \cap g \subseteq V$ and $R \cap V = \emptyset$.

Computing the image by an update. Let up be an update such as the ones of Section 4.2 (general case). It can be written as

$$up = \bigwedge_{x \in Y} (x := y_x) \wedge \bigwedge_{x \in X \setminus Y} (x :\sim c_x).$$

Then the image of a zone by up can be computed by first applying the update $\bigwedge_{x \in Y}(x := y_x)$ (which is equivalent to the update $\bigwedge_{x \in Y}(x := y_x) \wedge \bigwedge_{x \in X \setminus Y}(x := x)$) and then applying successively each update $x :\sim c_x$ (for $x \in X \setminus Y$).

First assume that up is $\bigwedge_{x \in X}(x := y_x)$ and $R \cap up(Z) = \emptyset$. First of all, if $y_{x_1} = y_{x_2}$, then $up(Z)$ is included in the zone $(x_1 = x_2)$. If for one of such pairs, $R \cap (x_1 = x_2) = \emptyset$, then $up(Z) \subseteq (x_1 = x_2)$ and $(x_1 = x_2)$ is β-bounded. We thus assume that it is not the case. The set $E = \{R_i \in \mathcal{R}_\alpha \mid up(R_i) \subseteq R\}$ is not empty and we have moreover that $\{v \mid up(v) \in R\} = \bigcup_{R_i \in E} R_i$ (because of the bisimulation property of the set of regions $\mathcal{R}_\alpha$, see remark 1 page 13). For each i, $R_i \cap Z = \emptyset$, so that there exists a β-bounded zone V_i with $Z \subseteq V_i$ and $V_i \cap R_i = \emptyset$. We distinguish two cases (**we have only three clocks**, say $\{x_1, x_2, x_3\}$):

- <u>First case:</u> $X = \{y_{x_1}, y_{x_2}, y_{x_3}\}$. We define

  $$V = \bigwedge_i \{x_i - x_j \prec m \mid y_{x_i} - y_{x_j} \prec m \in V_i\}$$

  We have that $up(Z) \subseteq V$ and that $R \cap V = \emptyset$, because if $v \in R \cap V$, then we can define safely v' such that $v'(y_{x_i}) = v(x_i)$ for each $i = 1, 2, 3$ and v' will be in $\bigcap_i V_i$, which is not possible because $v = up(v')$ and thus v' would be in one of the R_i's.
- <u>Second case:</u> $Y = \{y_{x_1}, y_{x_2}, y_{x_3}\} \neq X$. We define $R' = R \cap \bigcap_{y_{x_i} = y_{x_j}} (x_i = x_j)$, hence $Z \cap R' = \emptyset$. The zones Z and R' can now be considered as one or two-dimensional zones (depending on the cardinality of Y). To conclude, we apply the following (obvious) claim to the zones Z and R':

  Claim. Let Z_1 and Z_2 be one or two-dimensional zones such that $Z_1 \cap Z_2 = \emptyset$. Assume that $M_1 = ((m_{i,j}; \prec_{i,j}))$ is a DBM in normal form defining Z_1, then there exists i, j such that $Z_2 \subseteq \neg(x_i - x_j \prec_{i,j} m_{i,j})$. Note that one of the indices i or j may be 0.

  There exists thus a constraint $x_i - x_j \prec c$ that is a subconstraint defining the normal form of R' and such that $Z \subseteq \neg(x_i - x_j \prec c)$. Of course, from the definition of β, we get that the zone $(x_i - x_j \prec c)$ is β-bounded, and defining

  $$V = \neg(x_i - x_j \prec c) \wedge \bigwedge_{y_{x_k} = y_{x_l}} (x_k = x_l)$$

  we have $up(Z) \subseteq V$ and $R \cap V = \emptyset$.

Assume that up is $x_k :\prec c$ and

$$R \cap up(Z) = \emptyset$$

Since $up(Z)$ is included in $x_k \prec c$, if R is not included in $x_k \prec c$, we have finished because $(x_k \prec c)$ is a β-bounded zone. Assume now that $R \subseteq (x_k \prec c)$. We define $\mathrm{Free}_{x_k}(R)$ as the set $\{v \mid \exists v' \in R \text{ s.t. } \forall x \in X \setminus \{x_k\},\ v(x) = v'(x)\}$. In the same way, we define $\mathrm{Free}_{x_k}(Z)$. By construction of the set of regions $\mathcal{R}_\alpha$ (because of the bisimulation property), $\mathrm{Free}_{x_k}(R)$ is a union of regions and for each v in $\mathrm{Free}_{x_k}(R)$, there exists $v' \in up(v) \cap R$. Thus, it is easy to see that $\mathrm{Free}_{x_k}(R) \cap \mathrm{Free}_{x_k}(Z) = \emptyset$. **Since we only have 3 clocks**, $\mathrm{Free}_{x_k}(Z)$ and $\mathrm{Free}_{x_k}(R)$ can be considered as two-dimensional zones. Applying the previous claim, we find a constraint $x_i - x_j \prec c$ that does not involve the clock x_k, which is a constraint of the normal form of $\mathrm{Free}_{x_k}(R)$ and such that $\mathrm{Free}_{x_k}(Z) \subseteq \neg(x_i - x_j \prec c)$. The constraint $\neg(x_i - x_j \prec c)$ is of course a β-bounded zone (because of the definition of β), $up(Z) \subseteq \neg(x_i - x_j \prec c)$ and $R \cap \neg(x_i - x_j \prec c) = \emptyset$.

We can now conclude: we have proved that if Z satisfies condition (7), then $up(\overrightarrow{Z} \cap g)$ also satisfies condition (7). Applying the widening operator Approx_β at each step of the computation is thus correct: if R is a region and Z a computed zone, $R \cap Z = \emptyset$ implies that $R \cap \mathsf{Approx}_\beta(Z) = \emptyset$. □

Remark 5. In the previous proof, the hypothesis on the small number of clocks is used. Indeed, updating a clock is "equivalent" to projecting on a smaller dimension, and when we only have three clocks, the projection is one or two-dimensional and has thus nice properties. An error which maybe explains the numerous mistakes done in the literature is that zones in big dimensions are not similar to what we can "see" in two dimensions, thus it is not correct to project and to visualize what happens in two dimensions. In the previous proof, for the reset, if we want to be general and to reason with n clocks, we get the following: applying the induction hypothesis to the projection of Z and any region "belonging" to $\mathrm{Free}_{x_k}(R)$ will yield a β-bounded zone V which "separates" them (in the sense that $Z \subseteq V$ and $\mathrm{Free}_{x_k}(R) \cap V = \emptyset$). However, we have no argument to say that this β-bounded zone does not use the clock x_k. Thus, from V, it is not possible to compute a β-bounded zone that "separates" $up(Z)$ and R. This is the reason why Theorem 2 is not true in general.

5.3 Implementation of the Algorithm

From what precedes, we can see that, even if it is not correct in general, Algorithm 3 can however be used safely for some subclasses of systems. We will thus show that the DBM data structure is appropriate to implement Algorithm 3: we will show how to compute the new operations on zones which have appeared in Algorithm 3 using the DBMs. These new operations are of two types: image by updates and widening operator Approx_β.

Image by updates. Let up be an update and M be a DBM (in normal form). The DBM M represents a clock constraint φ over the set of clocks X. From up, we construct a clock constraint φ_{up} over the set of clocks $X \cup X'$ (where X' is a disjoint copy of X) by induction on up in the following manner:

$$\begin{cases} \varphi_{x:\sim c} = x' \sim c \\ \varphi_{x:\sim y+c} = x' \sim y + c \\ \varphi_{up_1 \wedge up_2} = \varphi_{up_1} \wedge \varphi_{up_2}. \end{cases}$$

We now define the formula $\varphi'_{up} = \varphi \wedge \varphi_{up}$. This formula represents the update up in the sense that if V is a valuation for the clocks $X \cup X'$, then

$$V \models \varphi'_{up} \iff V_{|X} \models \varphi \text{ and } V_{|X'} \in up(V_{|X}).$$

We assume that $up = \bigwedge_{i=1}^{n} up_i$ where up_i updates the clock x_i. We define $N = (\eta_{i,j})_{i,j=0\ldots 2n}$ as the DBM of size $2n+1$ (over the set of clocks $X \cup X'$) as follows:

- for $i, j = 0 \ldots n$, $\eta_{i,j} = (m_{i,j}; \prec_{i,j})$,
- for each update up_i ($1 \leq i \leq n$), we distinguish the different possible values of up_i:

if up_i is:	then we set:
$x_i := c$	$\eta_{i+n,0} = (c, \leq)$ and $\eta_{0,i+n} = (-c; \leq)$
$x_i := x_j + c$	$\eta_{i+n,j} = (c; \leq)$ and $\eta_{j,i+n} = (-c; \leq)$
$x_i :\prec c$ with $\prec \in \{<; \leq\}$	$\eta_{i+n,0} = (c; \prec)$ and $\eta_{0,i+n} = (0; \leq)$
$x_i :\succ c$ with $\succ \in \{>; \geq\}$	$\eta_{0,i+n} = (-c; \prec)$
	(where $\prec$ is $<$ if $\succ$ is $>$ and $\prec$ is $\leq$ if $\succ$ is $\geq$)
$x_i :\prec x_j + c$ with $\prec \in \{<; \leq\}$	$\eta_{i+n,j} = (c; \prec)$ and $\eta_{j,i+n} = (\infty; \prec)$
$x_i :\succ x_j + c$ with $\succ \in \{>; \geq\}$	$\eta_{j,i+n} = (-c; \prec)$
	(where $\prec$ is $<$ if $\succ$ is $>$ and $\prec$ is $\leq$ if $\succ$ is $\geq$)

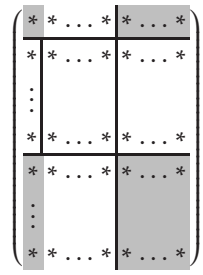
  If up_i is a conjunction of simple updates, we take the smallest constraints,
- for every $i, j = 0 \ldots 2n+1$, if $\eta_{i,j}$ has not been initialized yet , we set $\eta_{i,j} = (+\infty; <)$.

By construction, the DBM N represents the clock constraint φ'_{up} (indices $1 \leq i \leq n$ correspond to the set of clocks X whereas indices $n+1 \leq i \leq 2n+1$ correspond to the set of clocks X'):

$$[\![N]\!] = \{V \text{ valuation over } X \cup X' \mid V \models \varphi'_{up}\}.$$

We define $(\phi(N))_{|X'}$ as the square submatrice of size $n+1$ of N where we erase everything about the set of clocks X. It can be represented by the picture beside (the white part correspond to the clocks from X). Then we get that:

$$[\![(\phi(N))_{|X'}]\!] = up([\![M]\!]).$$



If we want to compute the image by the update up of a zone Z represented by a DBM M (in normal form), we first compute the DBM N, then its normal form and finally we erase some coefficients and the result of this algorithm gives a DBM (in normal form) which represents $up(Z)$.

β-overapproximation. Let $\beta = ((\max_x)_{x \in X}; (\max_{x,y})_{x,y \in X})$ define a set of regions. All the β which are used in one of the correct algorithms described before verifies in addition that for every $x, y \in X$, $\max_{x,y} = \max_x$. We write $X = \{x_1, \ldots, x_n\}$. A DBM $M = (m_{i,j}; \prec_{i,j})_{i,j}$ is said to be *β-bounded* if:

$$\begin{cases} 0 \leq m_{i,0} \leq \max_{x_i} \text{ or } m_{i,0} = \infty & \text{for every } 1 \leq i \leq n \\ -\max_{x_i} \leq m_{0,i} \leq 0 & \text{for every } 1 \leq i \leq n \\ -\max_{x_j,x_i} \leq m_{i,j} \leq \max_{x_i,x_j} \text{ or } m_{i,j} = \infty & \text{for every } 1 \leq i, j \leq n \end{cases}$$

Lemma 3. *Let Z be a zone. Then*

$$Z \in \mathcal{Z}_\beta \iff \textit{there exists a } \beta\textit{-bounded DBM } M_\beta \textit{ such that } Z = [\![M_\beta]\!].$$

➜ *See the proof in Appendix.*

We consider the following algorithm, with $M = (m_{i,j}; \prec_{i,j})$ and $M' = (m'_{i,j}; \prec'_{i,j})$:

```
# β-Overapproximation Algorithm (M: n-dimensional DBM)
#     M' = φ(M);                          (* Normal Form of M *)
#     for i = 1 to n do
#         for j = 1 to n do
#             if (m'_{i,j};≺_{i,j})>(max_{x_i,x_j};≤) then (m'_{i,j};≺_{i,j}) becomes (+∞;<);
#             if (m'_{i,j};≺_{i,j})<(− max_{x_j,x_i};<) then (m'_{i,j};≺_{i,j}) becomes (− max_{x_j,x_i};<);
#         endfor;
#         if (m'_{i,0};≺_{i,0})>(max_{x_i};≤) then (m'_{i,0};≺_{i,0}) becomes (+∞;<);
#         if (m'_{0,i};≺_{0,i})<(− max_{x_i};<) then (m'_{0,i};≺_{0,i}) becomes (− max_{x_i},<);
#     endfor;
#     Return M';
```

If M is a DBM, applying this algorithm to M, we obtain a DBM that we denote by $\phi_\beta(M)$. We can note that $\phi_\beta(M)$ is a β-bounded DBM and thus that $[\![\phi_\beta(M)]\!]$ is in $\mathcal{Z}_\beta$.

Property 1. If M is a DBM, then$[\![\phi_\beta(M)]\!] = \mathsf{Approx}_\beta([\![M]\!])$.

➜ *See the proof in Appendix.*

The "β-Overapproximation Algorithm" applied to M computes a DBM which represents the β-overapproximation of $[\![M]\!]$. The complexity of the previous computation is in $O(n^2)$. However, the resukting DBM might not be in normal form which increases the complexity to $O(n^3)$.

Combining the constructions of section 2.4 with what precedes, we get that if $e = q \xrightarrow{g,a,up} q'$ is a transition and M a DBM representing the zone Z, we can compute a DBM M' which represents the zone $\mathsf{Approx}_\beta(up(g \cap \overrightarrow{Z})) = \mathsf{Approx}_\beta(\mathsf{Post}(Z, e))$ but which might not be in normal form. Normal forms are computed before performing all the operations on DBMs, apart from the intersection for which it is not required, and apart from when we just performed either an image by update or a future operation (after such operations, DBMs are automatically in normal form). Thus, each operation in Algorithm 3 can be computed using the DBM data structure, which is thus appropriate to the implementation. Moreover, the complexity of each step of the algorithm is, as for classical timed automata, polynomial in the number of clocks.

5.4 Discussion

The automata for which we have troubles with the widening operator have diagonal clock constraints (and more than four clocks). We will propose some methods for alleviating this problem. Of course, these solutions will *a priori* not be as efficient as Algorithm 3.

First Proposal with a Preprocessing Step. This method concerns the decidable updatable timed automata with general clock constraints and deterministic updates: we allow updates of the form $x := y$ and $x := c$ where x and y are clocks and c is a constant. We know that every (classical) timed automaton with general clock constraints can be transformed into a (classical) diagonal-free timed automaton [BDGP98]. In the same way, every updatable timed automaton which uses only updates of the form $x := y$ and $x := c$ can be transformed into a diagonal-free updatable timed automaton that uses also updates of the form $x := y$ and $x := c$. This transformation can not be done if we use non-deterministic updates. As we know, this transformation suffers from a combinatorics explosion. However, if the number of diagonal constraints used in the automaton is small, the size of the automaton obtained after this preprocessing step will not be too big. Thus, Algorithm 3 can then be applied to the resulting diagonal-free automaton.

Second Proposal. The second method can be used for every decidable updatable timed automaton. In practice, it may be expensive, in memory as well as in time.

Let $\mathcal{A}$ be an updatable timed automaton. We associate with $\mathcal{A}$ the set of regions $\mathcal{R}_\alpha$ as in section 4.2. We assume that $\alpha = ((\max_x)_{x\in X}; (\max_{x,y})_{x,y\in X})$ and we define the tuple $\beta = ((\mathrm{Max}_x)_{x\in X}; (\mathrm{Max}_{x,y})_{x,y\in X})$ such that $\mathrm{Max}_x = \mathrm{Max}_{x,y} = \max_x$ for every $x, y \in X$.

Let R be a region of $\mathcal{R}_\alpha$ and Z a zone. We assume that $M = ((m_{i,j}; \prec_{i,j})_{i,j})$ is a DBM in normal form for Z and $M_R = ((r_{i,j}; \prec_{i,j})_{i,j})$ is a DBM that represents R (computed directly from its definition using intervals, like in section 4.2, without tightening any constraint). Assume now that $Z \cap R = \emptyset$. It means that there exists a sequence of distincts indices $(i_1, \ldots, i_l = i_1)$ such that

$$\alpha_{i_1,i_2} + \cdots + \alpha_{i_{l-1},i_l} < (0; \leq) \tag{9}$$

where $\alpha_{h,\ell} = \min((r_{h,\ell}; \prec_{h,\ell}); (m_{h,\ell}; \prec_{h,\ell}))$ and two consecutives $\alpha_{h,\ell}$ do not belong to M (this is posssible because M is in normal form).

The idea is then to ensure that there is just one of the coefficient $\alpha_{i_j,i_{j+1}}$ which comes from the DBM M_R. It will be the case whenever for every pair (i, j) where $i.j \neq 0$, there exists an interval $I_{i,j} \in \mathcal{I}_{i,j}$ defined by

$$\begin{aligned}\mathcal{I}_{i,j} = \quad & \{]-\infty; -\max_{x_j}[,]\max_{x_i}; +\infty[\} \cup \{[c] \mid -\max_{x_j} \leq c \leq \max_{x_i}\} \\ & \cup \{]c; c+1[\mid -\max_{x_j} \leq c < \max_{x_i}\}\end{aligned}$$

such that $Z \subseteq I_{i,j}$. In that case, for every pair (i, j) such that $i.j \neq 0$, we have $(m_{i,j}; \prec_{i,j}) \leq (r_{i,j}; <_{i,j})$. Thus inequality (9) can be transformed into:

$$\alpha_{i,0} + \alpha_{0,j} + (m_{j,i}; \prec_{j,i}) < (0; \leq)$$

We distinguish three cases:

1. $\alpha_{i,0} = (m_{i,0}; \prec_{i,0})$ and $\alpha_{0,j} = (r_{0,j}; <_{0,j})$.
   In this case, we obtain $(r_{0,j}; <_{0,j}) + (m_{j,0}; \prec_{j,0}) < (0; \leq)$. Thus, $Z \subseteq \neg(-x_j <_{0,j} r_{0,j})$ and $R \subseteq (-x_j <_{0,j} r_{0,j})$
2. $\alpha_{i,0} = (r_{i,0}; <_{i,0})$ and $\alpha_{0,j} = (m_{0,j}; \prec_{0,j})$.
   In this case, we have $(r_{i,0}; <_{i,0}) + (m_{0,i}; \prec_{0,i}) < (0; \leq)$. Thus, $Z \subseteq \neg(x_i <_{i,0} r_{i,0})$ and $R \subseteq (x_i <_{i,0} r_{i,0})$.
3. $\alpha_{i,0} = (r_{i,0}; <_{i,0})$ and $\alpha_{0,j} = (r_{0,j}; <_{0,j})$.
   In this case, we get that $(r_{i,0}; <_{i,0}) + (r_{0,j}; <_{0,j}) + (m_{j,i}; \prec_{j,i}) < (0; \leq)$. Thus, $Z \subseteq \neg(x_i - x_j <' c)$ and $R \subseteq (x_i - x_j <' c)$ where $(c; <') = (r_{i,0}; <_{i,0}) + (r_{0,j}; <_{0,j})$

In all three cases, there exists a β-bounded zone V such that

$$\begin{cases} Z \subseteq V \\ R \cap V = \emptyset \end{cases}$$

The solution we propose starts from a decomposition a "computed zone" in the following way:

$$Z = \bigsqcup \left\{ Z \cap \bigcap_{i,j \neq 0} I_{i,j} \mid I_{i,j} \in \mathcal{I}_{i,j} \right\}$$

and compute the image by Approx_β of each such sub-zone of Z. Note that a similar idea of splitting zones was already presented in [Bou00,Ben02], but it was slightly different, and, here, it appears like some kind of "necessary" condition for having a correct algorithm.

A direction for future work is to look for more efficient algorithms that test emptiness of timed automata with diagonal clock constraints. Probably backward analysis algorithms would be more appropriate to this model, because there is no need of abstraction for assuring termination. However backward algorithms are not appropriate when we want to analyze models which use both clocks and discrete variables as Uppaal does. For example computing backwards the predecessor of an assignment $i := j.k + \ell.m$ where i, j, k, ℓ and m are integer variables can not be done symbolically, there is no simpler way than enumerating all the possible tuples of values.

A last proposal. The last proposal mixes usual forward techniques with backward counter-example checking. The idea is to use Algorithm 1 (with the usual widening operator). If the target is not reached, then it is not reachable, otherwise, we generate a counter-example from the target states which are reached. If we find a counter-example, then some target state is really reachable, otherwise, no target state is reachable. In order that generating a counter-example is not equivalent to a whole backward analysis, the forward analysis must remember the paths leading to the target states. This technique is close to techniques based on counter-example guided refinements [CGJ+00] but I believe that there is no need for further refinement (after the first counter-example checking). Of course, this needs to be proved because it might be the case that the algorithm with the widening operator reachs a target state in k steps, whereas this state is really reachable in more than k steps. Up to now, this is probably one of the most useful methods (together with the first method that has a preprocessing step) as in most cases, the widening operator will not introduce mistakes in the computation and its implementation only requires to add a counter-example generation module which performs a backward computation along finite given paths.

6 Conclusion

In this paper, we studied the widening operator proposed in [DT98] to overcome the non-termination problem of the forward analysis algorithm for timed automata. This operator is very nice in that it can be very easily computed using the standard DBM data structure. However, we have proved in this paper that it is hopeless to look for a correct forward analysis algorithm that uses such an operator for the class of general timed automata: there exist timed automata for which no widening operator is correct with respect to reachability. This result is very surprising because several (attempts of) correctness proofs can be found in the literature and the algorithm is implemented and used in model-checking tools like Kronos and Uppaal.

Having stated this result, we investigated the more general setting of updatable timed automata and brought out subclasses of this model for which we can propose a correct widening operator. Among these subclasses, we find all the decidable *diagonal-free* updatable timed automata, and in particular all the classical timed automata which do not use diagonal clock constraints. This class of timed automata is sufficiently large, and this is maybe the reason why the bug has not been detected before. For the timed automata for which no correct widening operator can be used, other methods are proposed.

This work is also the link that was missing between the theoretical work we did on this updatable timed automata model in [BDFP00a,BDFP00b] (namely decidability and expressiveness) and a tool that would deal with updatable timed automata. If we restrict to diagonal-free systems, updates can be safely added without more difficulty than implementing the image by updates of zones. Considering general updatable timed automata, some problems appear when we want to implement them. However, the problems come from the use of diagonal clock constraints and not from the updates which are used.

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Appendix: Technical Proofs of this Paper

4.3 A First Correct Algorithm for Decidable Updatable Timed Automata

Lemma 1. *The two following conditions hold:*

$$\begin{cases} \mathsf{Closure}_\alpha(g) = g & \textit{for each clock constraint } g \textit{ in } \mathcal{C} \\ up(\mathsf{Closure}_\alpha(Z)) \subseteq \mathsf{Closure}_\alpha(up(Z)) & \textit{for each zone } Z \textit{ and for each update } up \textit{ in } \mathcal{U} \end{cases} \tag{4}$$

Proof. Let g be in $\mathcal{C}$. Then $\mathsf{Closure}_\alpha(g) = \bigcup\{R \in \mathcal{R}_\alpha \mid R \cap g \neq \emptyset\} = g$ because for each region $R \in \mathcal{R}_\alpha$, if $R \cap g \neq \emptyset$ then $R \subseteq g$.
We will now prove the second condition, namely that for each zone Z, for each update $up \in \mathcal{U}$,

$$up(\mathsf{Closure}_\alpha(Z)) \subseteq \mathsf{Closure}_\alpha(up(Z)) \tag{5}$$

Consider a zone Z. We can write Z as $\bigcup_f Z_i$ where there exist regions R_i such that $Z_i = Z \cap R_i$ and Z_i is not empty. If we prove condition (5) for each zone Z_i, then condition (5) will also hold for Z:

$$\begin{aligned} up(\mathsf{Closure}_\alpha(Z)) &= up(\mathsf{Closure}_\alpha(\cup_f Z_i)) \\ &= up(\cup_f \mathsf{Closure}_\alpha(Z_i)) \\ &= \cup_f up(\mathsf{Closure}_\alpha(Z_i)) \\ &\subseteq \cup_f \mathsf{Closure}_\alpha(up(Z_i)) \quad \text{if condition (5) holds for each } Z_i \\ &\subseteq \mathsf{Closure}_\alpha(\cup_f up(Z_i)) \\ &\subseteq \mathsf{Closure}_\alpha(up(\cup_f Z_i)) \\ &\subseteq \mathsf{Closure}_\alpha(up(Z)) \end{aligned}$$

Thus, we just have to prove condition (5) for zones contained in a region (defined by α).
Let Z be such a zone. To prove (5), assume that there exists a valuation $v' \in up(\mathsf{Closure}_\alpha(Z)) \setminus \mathsf{Closure}_\alpha(up(Z))$. There exists $v \in \mathsf{Closure}_\alpha(Z)$ such that $v' \in up(v)$. Thus, we obtain that $up(v) \cap [v']_\alpha \neq \emptyset$. Let v_0 be in Z, then $[v]_\alpha = [v_0]_\alpha$ hence, by hypothesis, $up(v_0) \cap [v']_\alpha \neq \emptyset$ so that $[v']_\alpha \subseteq \mathsf{Closure}_\alpha(up(Z))$. This is a contradiction to the fact that $v' \notin \mathsf{Closure}_\alpha(up(Z))$. □

Lemma 2. *For each zone Z, $\overrightarrow{\mathsf{Closure}_\alpha(Z)} \subseteq \mathsf{Closure}_\alpha(\overrightarrow{Z})$ and for each clock constraint g such that $\mathsf{Closure}_\alpha(g) = g$, $g \cap \mathsf{Closure}_\alpha(Z) \subseteq \mathsf{Closure}_\alpha(g \cap Z)$.*

Proof. Let us take $v_0 \in \overrightarrow{\mathsf{Closure}_\alpha(Z)}$. There exists some $v_1 \in \mathsf{Closure}_\alpha(Z)$ and some $t \geq 0$ such that $v_0 = v_1 + t$. From the definition of $\mathsf{Closure}_\alpha$, there exists some $v_2 \in Z$ such that $[v_1]_\alpha = [v_2]_\alpha$. Thus, $[v_0]_\alpha$ is a time successor of $[v_2]_\alpha$: there exists $v_3 \in [v_0]_\alpha$ and $t' \geq 0$ such that $v_3 = v_2 + t'$. Thus, $v_3 \in \overrightarrow{Z}$ and it follows that $v_0 \in \mathsf{Closure}_\alpha(\overrightarrow{Z})$ because $[v_3]_\alpha = [v_0]_\alpha$, which gives the property.

The second property holds because of the relation $\mathsf{Closure}_\alpha(g) = g$: let us take some $v_0 \in g \cap \mathsf{Closure}_\alpha(Z)$. We have $[v_0]_\alpha \subseteq g$ and there exists $v_1 \in [v_0]_\alpha \cap Z$. We obtain $v_1 \in g \cap Z$ and thus $v_0 \in \mathsf{Closure}_\alpha(g \cap Z)$. □

5.3 Implementation of the Algorithm

Lemma 3. *Let Z be a zone. Then*

$$Z \in \mathcal{Z}_\beta \iff \textit{there exists a } \beta\textit{-bounded DBM } M_\beta \textit{ such that } Z = [\![M_\beta]\!].$$

Proof. If M_β is a β-bounded DBM, then $[\![M_\beta]\!]$ is in $\mathcal{Z}_\beta$.

Conversely, we first note that a region $R \in \mathcal{R}_\beta$ is β-bounded. Assume Z is the convex union of two zones Z_1 and Z_2 which belong to $\mathcal{Z}_\beta$ and such that there exist two β-bounded DBMs M_{Z_1} and M_{Z_2} such that $Z_1 = [\![M_{Z_1}]\!]$ and $Z_2 = [\![M_{Z_2}]\!]$. We assume that for $l = 1, 2$, $M_{Z_l} = (m^{(l)}_{i,j}; \prec^{(l)}_{i,j})_{i,j=1\ldots n}$ and we define $M = (m_{i,j}; \prec_{i,j})_{i,j=1\ldots n}$ by:

$$(m_{i,j}; \prec_{i,j}) = \max((m^{(1)}_{i,j}; \prec^{(1)}_{i,j}), (m^{(2)}_{i,j}; \prec^{(2)}_{i,j})).$$

First note that M is a β-bounded DBM. We want to prove that $[\![M]\!] = Z$. For $l = 1, 2$, $M_{Z_l} \leq M$, thus $Z_l \subseteq [\![M]\!]$ which implies $Z \subseteq [\![M]\!]$.

Let r be a real number such that $-m_{j,i} \prec_{j,i} r \prec_{i,j} m_{i,j}$. There exists k such that $(m_{i,j}; \prec_{i,j}) = (m^{(k)}_{i,j}; \prec^{(k)}_{i,j})$. We have $(m^{(k)}_{j,i}; \prec^{(k)}_{j,i}) \leq (m_{j,i}; \prec_{j,i})$, thus $-m^{(k)}_{j,i} \prec^{(k)}_{j,i} r \prec^{(k)}_{i,j} m^{(k)}_{i,j}$. This implies that there exists a valuation $v \in Z_k$ such that $v(x_i) - v(x_j) = r$. As a consequence, we cannot tighten any constant in M. Thus, $Z = [\![M]\!]$. This concludes the proof. □

Property 1. If M is a DBM, then that $[\![\phi_\beta(M)]\!] = \mathsf{Approx}_\beta([\![M]\!])$.

Proof. Let M be a DBM in normal form. We get immediately that $\mathsf{Approx}_\beta([\![M]\!]) \subseteq [\![\phi_\beta(M)]\!]$ because $\mathsf{Approx}_\beta([\![M]\!])$ is the smallest zone from $\mathcal{Z}_\beta$ that contains $[\![M]\!]$ and $[\![\phi_\beta(M)]\!]$ is such a zone.

There exists a β-bounded DBM M_β such that $\mathsf{Approx}_\beta([\![M]\!]) = [\![M_\beta]\!]$. We obtain $M \leq M_\beta$. Assume $M = (m_{i,j}; \prec_{i,j})_{i,j=1\ldots n}$ and $M_\beta = (m^{(\beta)}_{i,j}; \prec^{(\beta)}_{i,j})_{i,j=1\ldots n}$. We will prove, coefficient by coefficient, that $\phi_\beta(M) \leq M_\beta$.

As a first example, assume that $(m_{i,0}; \prec_{i,0}) > (\max_i; \leq)$ and that $(m_{i,j}; \prec_{i,j}) > (\max_{i,j}; \leq)$, then $M \leq M_\beta$ implies that $(m^{(\beta)}_{i,j}; \prec^{(\beta)}_{i,j}) = (\infty; <)$.

As an other example, assume that $(m_{i,0}; \prec_{i,0}) > (\max_i; \leq)$ and that $(m_{i,j}; \prec_{i,j}) < (-\max_{j,i}; <)$, then $M \leq M_\beta$ implies that $(m^{(\beta)}_{i,j}; \prec^{(\beta)}_{i,j}) \geq (-\max_{j,i}; <)$.

We do not write the many other cases, but, in this way, we can prove that $\phi_\beta(M) \leq M_\beta$, which implies that $[\![\phi_\beta(M)]\!] \subseteq \mathsf{Approx}_\beta([\![M]\!])$. This concludes the proof. □