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Closed Sets in Occurrence Nets with Conflicts

Luca Bernardinello*
DISCo, Università degli studi di Milano - Bicocca
viale Sarca 336 U14, Milano, Italia
bernardinello@disco.unimib.it

Carlo Ferigato
JRC, Joint Research Centre of the European Commission
via E. Fermi, 1 21027 Ispra, Italia

Stefan Haar
LSV, CNRS & ENS de Cachan
avenue du Président Wilson
61 94235 CACHAN Cedex, France

Lucia Pomello
DISCo, Università degli studi di Milano - Bicocca
viale Sarca 336 U14, Milano, Italia

Abstract. The semantics of concurrent processes can be defined in terms of partially ordered sets. Occurrence nets, which belong to the family of Petri nets, model concurrent processes as partially ordered sets of occurrences of local states and local events. On the basis of the associated concurrency relation, a closure operator can be defined, giving rise to a lattice of closed sets. Extending previous results along this line, the present paper studies occurrence nets with forward conflicts, modelling families of processes. It is shown that the lattice of closed sets is orthomodular, and the relations between closed sets and some particular substructures of an occurrence net are studied. In particular, the paper deals with runs, modelling concurrent histories, and trails, corresponding to possible histories of sequential components. A second closure operator is then defined by means of an iterative procedure. The corresponding closed sets, here called 'dynamically closed', are shown to form a complete lattice, which in general is not orthocomplemented. Finally, it is shown that, if an occurrence net satisfies a property called B-density, which essentially says that any antichain meets any trail, then the two notions of closed set coincide, and they form a complete, algebraic orthomodular lattice.

Keywords: Concurrency theory, partial order semantics, closure operators, orthomodular lattices

*Address for correspondence: DISCo, Università degli studi di Milano - Bicocca, viale Sarca 336 U14, Milano, Italia

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1. Introduction

Partially ordered sets (posets) are a traditional tool for modelling concurrent processes, in which the notions of *causal dependence* and *independence*, or *concurrency*, are clearly represented by the order relation and its complementary (non-order) relation.

We consider a special class of partially ordered sets: *occurrence nets*. In occurrence nets, the support set is split into two distinct sets: *conditions* and *events* representing, respectively, local states and (local) state changes.

The term "occurrence net" is sometimes applied to objects of two kinds: the first one represents a single history of a system (a *run*), while the second represents all of the system's possible histories of execution. In the first case, the occurrence net cannot have branches while, in the second case, forward-oriented branches are allowed.

This paper continues the work initiated in [3] on *runs*, possibly infinite in both directions, and extends it to a class of occurrence nets allowing for *forward branching* representing the *conflict* between alternative histories of a system. In these nets, seen as partially ordered sets, three relations between pairs of elements are possible: *concurrency, causal dependence* and *conflict*.

Under an assumption of *local finiteness*, a structure of *closed sets* obtained from the *concurrency* relation is defined and studied. Properties of this structure — an *orthocomplemented lattice* in which the partial order is the set inclusion and a special case of modularity holds — are subsequently related to a notion of density for discrete partially ordered sets. This notion was introduced first for non-branching occurrence nets by Petri, who called it *K-density* (see, for example, [4] and [12]). Here, we adapt it to branching occurrence nets, and rename it *B-density*.

The characterization of the algebraic structures associated to the closure operator and their relations to B-density are the main results presented. Besides this, *dynamically closed* sets are introduced. *Dynamically closed* sets are the operational counterpart of the closed sets mentioned above.

The conditions for the equivalence of closed and dynamically closed sets are presented and discussed.

This work has been influenced by Petri's ideas on a combinatorial representation of flows of information in non-sequential processes, constrained by physical laws, specifically by the theory of relativity (see [12]). A different source of inspiration came from the study of lattices of closed sets in Minkowski spacetime ([8, 7]).

The paper is structured as follows. Section 2 collects basic definitions. In Section 3, we turn to a class of occurrence nets, define closed sets, and analyze some of their properties, showing in particular that they form an orthomodular lattice. In a first subsection, B-density is characterized on the basis of a relation between trails and closed sets; in a second subsection the connections between the structure of closed sets in a single run and the structure of the closed sets in the whole occurrence net are analyzed. In Section 4 dynamically closed sets are introduced, giving also a constructive characterization. The relations between closed sets and dynamically closed sets are then studied in a subsection showing that, for B-dense occurrence nets, they coincide and form an algebraic orthomodular lattice. Finally, in Section 5, we briefly comment on the main results, and suggest further developments.

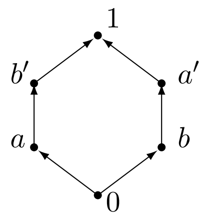


Figure 1. The ortholattice O_6

2. Preliminary Definitions

In this section we recall the definition of the main objects of interest, namely orthomodular posets [11], Petri nets [12, 4], occurrence nets [10], and closure operators [5].

2.1. Partially Ordered Sets, Orthocomplements and Orthomodular Lattices

A poset is a structure $\mathcal{P}$ composed of a set P and a partial order relation $\leq \subseteq P \times P$. From $\leq$ we derive — when they exist — the binary operators $\wedge$ and $\vee$: respectively, the greatest lower bound (*meet*) and the least upper bound (*join*). For $x, y \in P$, $x \leq y$, $[x, y]$ denotes the interval between x and y: $[x, y] = \{z \in P \mid x \leq z \leq y\}$. Given $A \subseteq P$, $\mathrm{future}(A)$ denotes the *future* of A: $\mathrm{future}(A) = \{x \in P \setminus A \mid a < x$ for some $a \in A\}$. The *past* of A, $\mathrm{past}(A)$, is defined similarly with $x < a$. A set $A \subseteq P$ is *convex* iff for all $x, y \in A$ such that $x \leq y$, $[x, y] \subseteq A$.

Definition 2.1. An *orthocomplemented poset* $\mathcal{P} = \langle P, \leq, 0, 1, (.)' \rangle$ is a poset $\langle P, \leq \rangle$, bounded by a minimum (0) and a maximum (1) element and with a map $(.)' : P \to P$, such that the following conditions are satisfied: $\forall x, y \in P$

i. $(x')' = x$

ii. $x \leq y \Rightarrow y' \leq x'$

iii. $x \wedge x' = 0$ *and* $x \vee x' = 1$

The map $(.)' : P \to P$ is called an *orthocomplementation* in P.

A lattice $\mathcal{L}$ is a poset in which for any pair of elements meet and join always exist. Furthermore, $\mathcal{L}$ is *complete* when the meet and the join of any subset of $\mathcal{L}$ exist. An orthocomplemented lattice is also called an *ortholattice*.

Definition 2.2. An *orthomodular lattice* is an ortholattice $\mathcal{L} = (L, 0, 1, \leq, \wedge, \vee, (\ )')$ in which, in addition to properties i, ii and iii in Definition 2.1 above, the following property, known as *orthomodular law*, holds:

$$x \leq y \Rightarrow y = x \vee (y \wedge x').$$

A *subalgebra* of an ortholattice $\mathcal{L}$ is a subset of $\mathcal{L}$, closed under the operations $(\ )'$, $\wedge$, $\vee$ and containing 0 and 1.

The following characterization of orthomodular lattices (see, for instance, [11], page 22) will be used in later proofs.

Theorem 2.3. [11] Let L be an ortholattice. Then L is orthomodular if, and only if, the lattice O_6, shown in Fig. 1, is not a subalgebra of L.

2.2. Nets

Definition 2.4. A net is a triple $N = (B, E, F)$ such that B and E are countable sets, $B \cap E = \emptyset$ and $F \subseteq (B \times E) \cup (E \times B)$. The preset and postset of $x \in B \cup E$, denoted by ${}^\bullet x$ and $x^\bullet$, respectively, are defined by ${}^\bullet x = \{y \in B \cup E \mid (y, x) \in F\}$, and $x^\bullet = \{y \in B \cup E \mid (x, y) \in F\}$; the neighbourhood of x, denoted by ${}^\bullet x^\bullet$, is given by ${}^\bullet x \cup x^\bullet$.

The elements of B are called *local states* or *conditions*, the elements of E *local changes of state* or *events*, and F is called *flow relation*. We will use the standard graphical notation for nets.

Definition 2.5. Let $N = (B, E, F)$ be a net, and $x, y \in B \cup E$. Then x and y are in *conflict*, denoted by $x \# y$, if there exist two distinct events $e_x, e_y \in E$ such that $e_x F^* x$, $e_y F^* y$, and ${}^\bullet e_x \cap {}^\bullet e_y \neq \emptyset$.

Let $x \# y$, and e_x, e_y as in the previous definition. Define $\mathrm{cfs}(x, y) := {}^\bullet e_x \cap {}^\bullet e_y$.

When $\leq := F^*$ is a partial order, we can define further interesting relations on the elements of N. Two elements x and y are *causally dependent*, denoted by x **li** y, if either xF^*y or yF^*x. An immediate consequence of definition 2.5 is *conflict heredity*:

$$\left.\begin{array}{ccc} x & \# & y \\ x & \leq & z \\ y & \leq & w \end{array}\right\} \Rightarrow \quad w \# z.$$

Nodes x and y are *concurrent*, written x **co** y, if neither $x \# y$ nor x **li** y hold. A clique D of the concurrency relation will be called *coset*. A coset D such that $D \subseteq B$ will be called B-coset.

Definition 2.6. An occurrence net is a net $N = (B, E, F)$ such that for all $b \in B$, $|{}^\bullet b| \leq 1$, F^* is a partial order, the conflict relation is irreflexive, the minimal elements with respect to F^* belong to B, and for all $x \in B \cup E$, $|\{y \in B \cup E : yF^+x\}| < \infty$.

In an occurrence net, the minimal nodes with respect to F^* form a B-coset; equivalently, there is no event $e \in E$ such that ${}^\bullet e = \emptyset$. We obtain a poset $(X, \leq)$ by defining $X = (B \cup E)$ and $\leq = F^*$. For any subset A of elements of an occurrence net $N = (B, E, F)$, define $\min(A) = \{x \in A \mid {}^\bullet x \cap A = \emptyset\}$.

The definition of occurrence net implies that $\forall x \in B \cup E : |{}^\bullet x| < \infty$ and $\forall x, y \in B \cup E : |[x, y]| < \infty$. In the following we assume also that all occurrence nets considered are *condition bordered*: every event $e \in E$ in an occurrence net has at least one output condition, i.e. $|e^\bullet| \geq 1$. Moreover, we assume that $\forall e \in E \; |e^\bullet| < \infty$.

The following lemma will be used in several proofs.

Lemma 2.7. Let $e \in E$, $z \in B \cup E$. Then

1. z **co** $e \Leftrightarrow \forall b \in {}^\bullet e : b$ **co** z

2. z **co** $e \Leftrightarrow \forall b \in e^\bullet : b$ **co** z

Proof:
Suppose first that e **co** z. Let $b \in {}^\bullet e$. If zF^*b, then zF^*e; if bF^*z, then either eF^*z or $e \# z$; in all such cases, we have a contradiction with the hypothesis that e **co** z.

Let now $b \in e^\bullet$. If bF^*z, then eF^*z; since (e, b) is the only arc entering b, if zF^*b then zF^*e, and if $b \# z$ then $e \# z$, again contradicting e **co** z.

Suppose b **co** z for all $b \in {}^\bullet e$. Then z cannot follow e, nor can it precede it, because any path from z to e should pass through some $b \in {}^\bullet e$. Also $z \# e$ must be excluded, because in such case, either there is a $b \in {}^\bullet e$ which is also in ${}^\bullet z$, or $z \# b$ for some $b \in {}^\bullet e$.

Finally, suppose b **co** z for all $b \in e^\bullet$. Then z cannot follow e, nor can precede it, like in the previous case. Also $z \# e$ must be excluded, because conflict is inherited, so $z \# b$ for all $b \in e^\bullet$. □

Within an occurrence net, we have the following objects of interest (see [4, 10]):

Definition 2.8. Line, cut, trail, run.

i. A maximal clique $\lambda \subseteq B \cup E$ of **li** is a *line*.

ii. A maximal clique $\gamma \subseteq B \cup E$ of **co** $\cup$ **id** is a *cut*.

iii. A maximal clique $\tau \subseteq B \cup E$ of $(\# \cup$ **li**$)$ is a *trail*.

iv. A maximal clique $\rho \subseteq B \cup E$ of $($**li** $\cup$ **co**$)$ is a *run*.

As a consequence of the definition of occurrence nets and of the definitions above, the assumption of local finiteness does not hinder from the creation of forward infinite lines and infinite cuts. The interpretation for a line is as a, possibly infinite, history of a sequential process while cuts can be interpreted as system snapshots. Cuts composed exclusively by B elements are a special case; these cuts can be interpreted as maximal sets of system properties valid in a mutually independent way.

A trail, in which only the relations $\#$ and **li** occur, can be interpreted as the complete history of a sequential process, including all the possible alternatives consequent to different choices operated in the represented system. On the contrary, a run can be interpreted as an execution of the system since all of the choices are effectively solved. Runs induce conflict-free nets. A run ρ is called *K-dense* iff every line λ of ρ intersects each of ρ's cuts. An occurrence net N is called *R-dense* iff each of its runs is K-dense.

Definition 2.9. An occurrence net $N = (B, E, F)$ is *B-dense* iff, for every trail τ and for every cut γ, $\tau \cap \gamma \neq \emptyset$.

By results in [10], we have:

the intersection of a trail with a run is a line, and

N is R-dense iff it is B-dense.

For posets derived from conflict-free occurrence nets, as it is the case for runs, K-density can be characterized by the absence of the substructures shown in Fig. 2 [4]. In order to formalize this fact we need to say that a poset $(A', \leq')$ is embeddable into $(A, \leq)$ iff there exists an injection $\pi : A' \to A$ such that $\forall x, y \in A' : x \leq' y \Leftrightarrow \pi(x) \leq \pi(y)$.

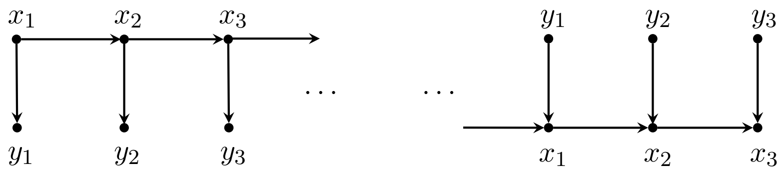


Figure 2. Posets which are not K-dense.

Proposition 2.10. ([4], prop. 2.3.9) Let $(X, \leq)$ be the poset associated to a conflict-free occurrence net $N = (B, E, F)$, $X = (B \cup E)$. If none of the posets shown in *Fig. 2* is embeddable into $(X, \leq)$, then $(X, \leq)$ is K-dense.

Actually, in our case, the substructure shown in the right side of Fig. 2 will never be embeddable into the posets considered here since they have no backwards infinite chains.

2.3. Closure Operators

References for this section are [5] and [9].

Definition 2.11. Let X be a set and $\mathbb{P}(X)$ the powerset of X. A map $\mathcal{C} : \mathbb{P}(X) \to \mathbb{P}(X)$ is a *closure operator* on X if, for all $A, B \subseteq X$,

i. $A \subseteq \mathcal{C}(A)$,

ii. $A \subseteq B \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B)$,

iii. $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$.

Note that, with this definition, $\mathcal{C}$ is not a topological closure operator, since, in general, the union of two closed sets is not a closed set. A subset A of X is called *closed* with respect to $\mathcal{C}$ if $\mathcal{C}(A) = A$. If $\mathcal{C}$ is a closure operator on a set X, the family $L_{\mathcal{C}} = \{A \subseteq X \mid \mathcal{C}(A) = A\}$ of closed subsets of X forms a complete lattice, when ordered by inclusion, in which

$$\bigwedge\{A_i : i \in I\} = \bigcap_{i\in I} A_i, \quad \bigvee\{A_i : i \in I\} = \mathcal{C}(\bigcup_{i\in I} A_i).$$

The proof of this statement can be found in [5].

We now recall a construction from binary relations to closure operators. Let X be a set of elements, and $\alpha \subseteq X \times X$ be a symmetric relation. Given $A \subseteq X$ we can define an operator $(.)^{\perp}$ on the powerset of X

$$A^{\perp} = \{x \in X \mid \forall y \in A : (x, y) \in \alpha\}.$$

By applying twice the operator $(.)^{\perp}$, we get a map on the powerset of X, which is a closure operator on X. A subset A of X is called *closed* if $A = (A^{\perp})^{\perp}$. The family $L(X)$ of all closed sets of X, ordered by set inclusion, is then a complete lattice.

When α is also irreflexive, the operator $(.)^{\perp}$, applied to elements of $L(X)$, is an orthocomplementation; the structure $\mathcal{L}(X) = \langle L(X), \subseteq, \emptyset, X, (.)^{\perp}\rangle$ then forms an orthocomplemented complete lattice [5].

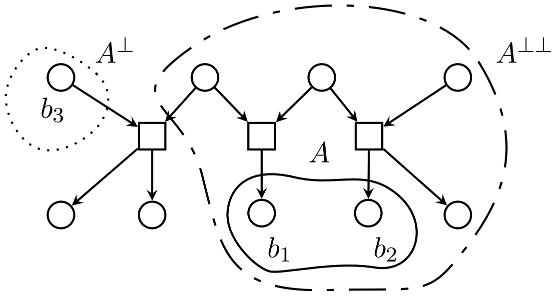


Figure 3. Application of the closure operator.

3. Closure Operators on Occurrence Nets

In this section we define closed sets on occurrence nets on the basis of the concurrency relation and we study some of their properties, showing in particular that they form an orthomodular lattice. In Subsection 3.1 we study the relations between closed sets and trails, and we give a characterization of B-density in terms of closed sets; in Subsection 3.2 we show the connections between the structure of closed sets in a single run and the structure of the closed sets in the whole occurrence net.

Throughout this section, let $N = (B, E, F)$ be an occurrence net. For any $A \subseteq B \cup E$, define $A^\perp = \{x \in B \cup E \mid \forall y \in A : x \textbf{ co } y\}$. Since the concurrency relation is symmetric and irreflexive, the operator $(.)^{\perp\perp}$ defined by $A^{\perp\perp} = (A^\perp)^\perp$ is a closure operator on $B \cup E$, and $(.)^\perp$ is an orthocomplement ([5]).

Example 3.1. In Figure 3 an application of the closure operator on an occurrence net is represented. Let $A = \{b_1, b_2\}$ then $A^\perp = \{b_3\}$ and $A^{\perp\perp}$ is the set enclosed in the dotted line.

Define
$$L(N) = \{A \subseteq B \cup E \mid A = A^{\perp\perp}\}$$
as the set of closed sets of N. By the results recalled in the previous section,
$$\mathcal{L}(N) = \langle L(N), \subseteq, \emptyset, B \cup E, (.)^\perp \rangle$$
is a complete orthocomplemented lattice, where, in particular, $\emptyset^\perp = B \cup E$. Note that, for any $A \subseteq B \cup E$, $A^\perp \in L(N)$.

The following proposition collects some elementary properties of closed sets.

Proposition 3.2. Let $A \in L(N)$, and $x, y \in A$.

i. If $x \in E$, then ${}^\bullet x^\bullet \subseteq A$
ii. If $z \in E$ and ${}^\bullet z \subseteq A$ then $z \in A$
iii. If $z \in E$ and $z^\bullet \subseteq A$ then $z \in A$
iv. If xF^*y, then $z \in A$ for all $z \in [x, y]$
v. If $x \# y$, then $z \in A$ for all $z \in \text{cfs}(x, y)$

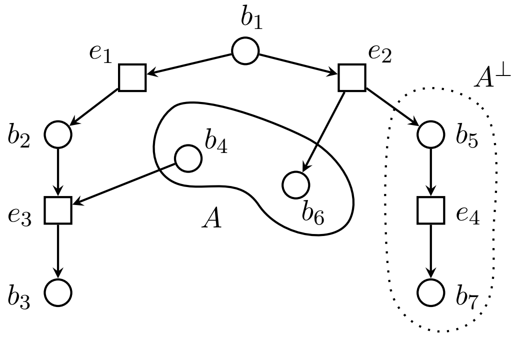


Figure 4. Future is not shared.

Proof:
The first three items follow immediately from Lemma 2.7. In the case of item (iii) note that $z^\bullet$ is never empty because we assume the occurrence net to be condition-bordered.

To prove statement (iv), let xF^*y. Then, for all v such that v **co** x and v **co** y, it holds also v **co** z for all z such that xF^*zF^*y, namely for all $z \in [x, y]$. Hence z is concurrent with all elements in $A^\perp$, for each A containing x and y.

To prove (v), let $x, y \in B \cup E$, and $x \# y$. Take $b \in \text{cfs}(x, y)$, and $v \in B \cup E$ such that v **co** x and v **co** y. We show that v **co** b. In fact, if $b \# v$, then $v \# x$, contradicting the assumption v **co** x; if bF^*v, then either $x \# v$ or $y \# v$, again a contradiction with the hypothesis. If vF^*b, then vF^*x, again contradicting the hypothesis. Hence b is concurrent with all elements in $A^\perp$, for each A containing x and y. □

Define the *border* of $A \subseteq B \cup E$ as $\mu(A) = \{x \in A \mid \exists y \in (B \cup E) \setminus A : (x, y) \in F \cup F^{-1}\}$. Prop. 3.2(i) implies that the border of a closed set is made of B-elements; moreover, Prop. 3.2(iv) implies that closed sets are convex subsets.

The *past* of a subset A of a poset has been defined in Section 2.1 as the set of elements which do not belong to A and precede at least one element in A (the *future* of A has been defined similarly). A closed set A and its orthocomplement $A^\perp$ share the past, as shown below.

Lemma 3.3. Let $A \in L(N)$. Then $\text{past}(A) = \text{past}(A^\perp)$.

Proof:
It suffices to show that $\text{past}(A) \subseteq \text{past}(A^\perp)$, since the converse follows by the fact that $A^\perp \in L(N)$ and $(A^\perp)^\perp = A$. Let first $x \in \text{past}(A)$ such that xF^+y for some $y \in A$. If we assume that $x \in \text{future}(A^\perp)$, there is an F-chain from $A^\perp$ to A, contradicting the definition of $A^\perp$. Hence assume there is no $z \in A^\perp$ such that xF^+z. Then, if for all $z \in A^\perp$ we have x **co** z, and this contradicts $x \notin (A^\perp)^\perp = A$; therefore, there must be $z \in A^\perp$ such that $x \# z$. This implies by conflict heredity that $y \# z$, contradicting the definition of $A^\perp$; hence $x \in \text{past}(A^\perp)$. □

A closed set and its orthocomplement do not share their future, in general. Consider for example the occurrence net N given in Fig. 4. Let $A = \{b_4, b_6\}$. Then $A \in L(N)$, and $A^\perp = \{b_5, e_4, b_7\}$. The future of A is given by $\mathrm{future}(A) = \{e_3, b_3\}$, whereas $\mathrm{future}(A^{\perp}) = \emptyset$. Notice that, if $\# = \emptyset$, then a closed set and its orthocomplement do share their future ([3]).

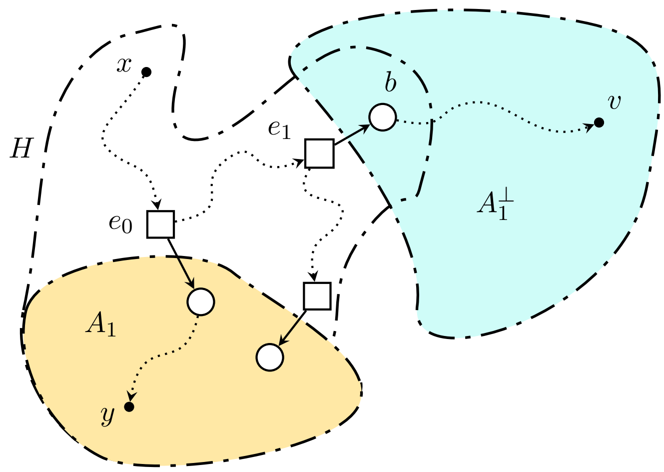
We can now state the main theorem of this section.

Theorem 3.4. $L(N)$ is orthomodular.

Thorem 3.4 is actually an immediate consequence of Theorem 2.3 and of the following Lemma 3.5, which highlights the orthomodular law, as pointed after the proof.

Lemma 3.5. Let $A_1 \in L(N)$ and $x \notin A_1$. Set $H = (A_1 \cup \{x\})^{\perp\perp}$. Then $H \cap A_1^{\perp} \neq \emptyset$.

Proof:
If x **co** A_1, then we are done. Otherwise, we consider three cases.



Case 1 (see picture above): assume there exists $y \in A_1$ such that xF^+y. Since H is closed, $[x, y] \subseteq H$. From Lemma 3.3, it follows that there exists $v \in A_1^{\perp}$ with xF^+v. Let e_1 be the event along a path from x to v such that $e_1 \notin A_1^{\perp}$, and a post-condition of e_1, let us call it b, is in $A_1^{\perp}$. Again by Lemma 3.3, e_1 is in the past of A_1, so, by convexity of closed sets, $e_1 \in H$; hence, by Prop. 3.2, $b \in H$, and b **co** A_1.

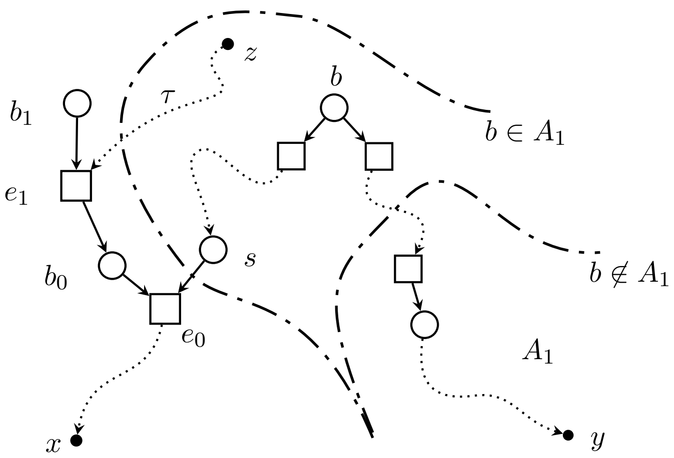
Case 2: assume there exists $y \in A_1$ such that $y \# x$. Then there is $b \in \mathrm{cfs}(x, y)$. By Prop. 3.2, $b \in H$. If $b \notin A_1$, then we fall in Case 1 above with $x = b$, since there is a path from b to y. If instead $b \in A_1$, then a path from b to x must leave A_1 at $(s, e_0) \in F$ for some $s \in B$, $e_0 \in E$. Since $b, x \in H$, we have $e_0 \in H$. From $e_0 \notin A_1$ it follows that there is a precondition b_0 of e_0 such that $b_0 \notin A_1$. If b_0 **co** A_1, then we are done, since, by Prop. 3.2, $b_0 \in H$; otherwise there are three cases:

2.a) b_0F^+z for some $z \in A_1$;

2.b) zF^+b_0 for some $z \in A_1$;

2.c) $b_0 \# z$ for some $z \in A_1$.

In Case 2.a, since $b_0 \in H$, apply Case 1 with $x = b_0$. For Case 2.b and Case 2.c, we show that either we find an element in $H \cap A_1^\perp$ or we find $b_1 \in H$ such that $b_1 F^+ b_0 F^+ x$. Since the set of predecessors of x in N is finite by definition of occurrence net, we eventually fall into the first case, or we find a condition which is concurrent with A_1. In Case 2.b (see picture below), there is a path τ from z to b_0. Let e_1 be the first point not in A_1 along this path; e_1 must be an event, and it must have a precondition, b_1, which does not belong to A_1.



In Case 2.c, we have a configuration analogous to the initial configuration of Case 2, with y replaced by z and x replaced by b_0. By iterating the argument in Case 2, we either can apply Case 1 or find b_1 such that $b_1 F^+ b_0$.

Case 3: assume there exists $y \in A_1$ such that $y F^+ x$. Let π be a path from y to x, and $(b, e_0) \in F$ the arc where π leaves A_1. Then, by convexity of closed sets, $e_0 \in H$. Since $e_0 \notin A_1$, there exists a precondition b_0 of e_0 which does not belong to A_1; note that $b_0 \in H$. If b_0 **co** A_1, then we are done; otherwise there are three cases:

3.a) $b_0 \# z$ for some $z \in A_1$;

3.b) $b_0 F^+ z$ for some $z \in A_1$;

3.c) $z F^+ b_0$ for some $z \in A_1$.

In Case 3.a, apply Case 2 with $x = b_0$. In Case 3.b, apply Case 1, with $x = b_0$; In Case 3.c, let π_1 be a path from z to b_0; by repeating the previous construction, we find an event e_1 which is in H, and has at least one precondition b_1 which is not in A_1. Apply to b_1 the argument used for b_0. Either b_1 **co** A_1, or we fall into Case 1 or Case 2, or we find e_2, and b_2 with the same properties. This last case cannot occur indefinitely, since any element of N has a finite past. □

Lemma 3.5 implies that any closed set can be made bigger only by adding at least an element concurrent to it, in other words: given $A_1, A_2 \in L(N)$, if $A_1 \subset A_2$, then $A_2 \cap A_1^\perp \neq \emptyset$. Because of Theorem 2.3, this corresponds to the *orthomodularity* of the ortholattice $L(N)$.

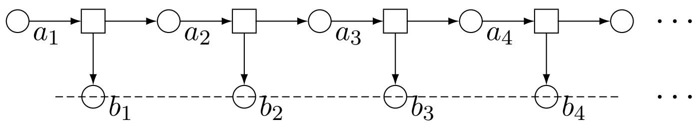


Figure 5. Non B-dense net.

3.1. Closed Sets and Trails

In this section we look at the relations between closed sets and trails. Trails represent branching histories of sequential components of a system. The results in this section generalize similar results holding for lines in causal nets (see [3]).

Let us start from a simple fact. Let A be a closed set, and τ a trail. Since a trail cannot contain a pair of concurrent elements, if $A \cap \tau \neq \emptyset$, then $A^{\perp} \cap \tau = \emptyset$.

In general, it is possible that a trail crosses neither a closed set, nor its orthocomplement, as shown in the following example. Define $A = \{b_i \mid i \text{ is even}\}$ in the net shown in Figure 5. Then A is closed, and $A^{\perp} = \{b_i \mid i \text{ is odd}\}$. The elements in the upper line form a trail, which crosses neither A nor $A^{\perp}$.

However, if a net is B-dense, the following theorem holds, giving a characterization of B-density in terms of closed sets: an occurrence net is B-dense if, and only if, for any closed set, any trail crosses either the given set or its orthocomplement.

Theorem 3.6. N is B-dense if, and only if, for all $A \in L(N)$, for any trail τ of N, $\tau \cap A \neq \emptyset$ or $\tau \cap A^{\perp} \neq \emptyset$.

Proof:
$\Rightarrow$) Let N B-dense and $A \in L(N)$; then $\min(A)$ is a B-coset of N, which can be extended to a B-cut γ of N. Let $Y = \gamma \setminus \min(A)$. We prove that $Y \subseteq A^{\perp}$ and then, since N is B-dense, we get the thesis.

Let $y \in Y$; then it cannot be the case that $y \in \mathrm{past}(A)$, otherwise there would be a path from y to an element in $\min(A)$, in contradiction with y **co** $\min(A)$. If $y \in \mathrm{future}(A)$, then there is a path leading from a place b in the border of A to y. Also in this case we get a contradiction since there would be a path from an element in $\min(A)$ to y through b. If y is in conflict with some element of A then the conflict can be originated either in the past of A or inside A. In the first case, y must be in conflict with at least one element in $\min(A)$, while in the second case there would be a path from some element in $\min(A)$ to y. In both cases we get a contradiction.

$\Leftarrow$) We show that, if N is not B-dense, then there exists a trail τ and a closed set S such that $\tau \cap (S \cup S^{\perp}) = \emptyset$. If N is not B-dense, then, by Proposition 2.10, the poset on the left side of Figure 2 embeds into N, or, more precisely, embeds into a run ρ of N. Let λ be a line of N, and then also of ρ, extending the set formed by all the elements denoted x_i in Figure 2; such a line exists, because those elements form a clique of li. Let $Y_P = \{y_i | i \text{ is even}\}$ and $Y_D = \{y_i | i \text{ is odd}\}$. Clearly, $Y_D \subseteq Y_P^{\perp}$, where $Y_P^{\perp}$ is closed. Since each element of λ is in relation li with at least one element of Y_P, we also have $\lambda \cap Y_P^{\perp} = \emptyset$ and $\lambda \cap Y_P^{\perp\perp} = \emptyset$.

We now show that, for each trail τ extending the line λ, we also get $\tau \cap Y_P^{\perp} = \emptyset$ and $\tau \cap Y_P^{\perp\perp} = \emptyset$. Assume, by contradiction, that there exists an element $z \in \tau \setminus \lambda$ such that: either $z \in Y_P^{\perp}$ or $z \in Y_P^{\perp\perp}$. Let $z \in Y_P^{\perp}$, since $z \in \tau \setminus \lambda$ and λ is downwards infinite, z should be in conflict with at least an element belonging to λ, denoted by x_j. Consequently, z should also be in conflict with the elements which belong to Y_P and are in the future of x_j, which is in contradiction with z **co** Y_P. The proof in the case $z \in Y_P^{\perp\perp}$ is analogous. $\square$

The next proposition gives a characterization of the closure of cosets, related to the set of trails of the net. Let $N = (B, E, F)$ be a B-dense occurrence net, and T the set of trails of N. For each $x \in B \cup E$, let T_x be the set of all trails passing through x: $T_x = \{\tau \in T \mid x \in \tau\}$; for $H \subseteq B \cup E$, define $T_H = \bigcup_{x \in H} T_x$.

Let A be a coset of N, namely a set of pairwise concurrent elements. The next proposition shows that a point x belongs to the closure of A if, and only if, every trail which passes through x passes also through at least one point in A.

Proposition 3.7. Let A be a coset of a B-dense occurrence net N. Then

$$\forall x \in B \cup E : x \in A^{\perp\perp} \Leftrightarrow T_x \subseteq T_A$$

Proof:
$\Leftarrow$) Let $T_x \subseteq T_A$ for $x \in B \cup E$. Let $y \in A^{\perp}$; then x **co** y, because, if x **li** y or $x \# y$, then there would exist a trail through both x and y, but this is impossible, because that trail would intersect both $A^{\perp}$ and $A^{\perp\perp}$. Since this holds for any $y \in A^{\perp}$, it follows that $x \in A^{\perp\perp}$.

$\Rightarrow$) Let $x \in A^{\perp\perp}$. Let $\tau \in T_x$, and γ a cut of N, such that $A \cap \gamma \neq \emptyset$. Then $\gamma \setminus A \subseteq A^{\perp}$, and $\tau \cap \gamma \neq \emptyset$, since N is B-dense.

Suppose $\tau \cap A = \emptyset$. Then, there is $y \in \gamma \setminus A$, with $y \in \tau$. This implies $(x, y) \notin$ **co** and $y \in A^{\perp}$, contradicting the hypothesis that $x \in A^{\perp\perp}$. Hence τ must pass through an element of A, so $T_x \subseteq T_A$. $\square$

Building on this proposition, we can describe a more concrete interpretation of the closure operator, when applied to a B-coset. Suppose that an occurrence net N models a system of interacting sequential components. Then the net can be decomposed into a set of trails, each one modelling the possible alternative histories of one sequential component, where the conditions represent local states, and the events can represent either autonomous changes of state or interactions among more components. This decomposition of the net into trails is made of a subset of all the trails of N which is a *minimal covering* of N.

Definition 3.8. A *minimal covering by trails* of an occurrence net $N = (B, E, F)$ is a family of trails of N, $(\tau_i)_{i \in I}$, which covers N, i.e.: such that: $\forall x \in B \cup E$, $\exists\, \tau_i = (B_i, E_i, F_i) : x \in B_i \cup E_i$, and such that it is minimal, i.e.: $\forall k \in I$, $(\tau_i)_{i \in I \setminus \{k\}}$ does not cover N.

Let $\Pi = (\tau_i)_{i \in I}$ be a minimal covering by trails of an occurrence net N, this induces an interpretation of N as the set of potential histories of a set of interacting sequential components.

On the other hand, given an occurrence net N, there may be different coverings/interpretations.

Suppose that we "observe" a B-coset of N, say β, i.e. a set of independent properties representing some local states.

Then, as an immediate consequence of Proposition 3.7, the closure of β, denoted $\beta^{\perp\perp}$, can be characterized as follows: an element $x \in B \cup E$ belongs to $\beta^{\perp\perp}$ if, and only if, for every minimal covering Π all the trails passing through x are also passing through one element of β. In terms of interacting sequential components, an element $x \in B \cup E$ belongs to $\beta^{\perp\perp}$ if, and only if, for every interpretation of N in terms of interacting sequential components, x certainly belongs only to some sequential components observed in β, i.e.: x certainly represents either a possible local state belonging to the components observed in β, or a possible change of states involving only components observed in β.

3.2. Closure on Runs

A run of an occurrence net is an occurrence net itself; hence it defines its own lattice of closed sets. The connection between this local structure on each run and the global structure on the entire net is given by the following two propositions.

Let $N = (B, E, F)$ be an occurrence net, and R be a run of N. In the following, we will use R to denote either the net underlying a run, or the set of its elements. To simplify notation, we will use $(.)^\perp$ to denote the orthocomplementation in N, and $(.)^*$ to denote orthocomplementation in R.

Proposition 3.9. Let $A \in L(N)$ be a closed set in N, and $A_1 = A \cap R$. Then A_1 is closed in R, i.e.: $A_1 \in L(R)$

Proof:
Let $A_1 = A \cap R$. Define $A_3 = A_1^\perp \cap R$ and $A_2 = A_1^\perp \setminus R$. Then $A_3 = A_1^*$, since A_3 contains all the elements of R which are concurrent to all elements of A_1. Hence $A_3 \in L(R)$. Our aim is to show that $A_3^* = A_1$. We already know that $A_1 \subseteq A_3^*$. By way of contradiction, suppose there is $x \in A_3^*$ with $x \notin A_1$. Note that $x \in R$. Since $x \notin A_3$, there is $y \in A_1$ such that x **li** y. Either $x < y$ or $y < x$.

From $A_1 \subseteq A$, it follows $A_1^{\perp\perp} \subseteq A^{\perp\perp} = A$, hence $x \notin A_1^{\perp\perp}$. Then, there is z in $A_1^\perp$ such that x is not concurrent with z. The element z cannot be in A_3, since $x \in A_3^*$, implying that x **co** A_3; hence $z \in A_2$. Either z **li** x or $z \# x$.

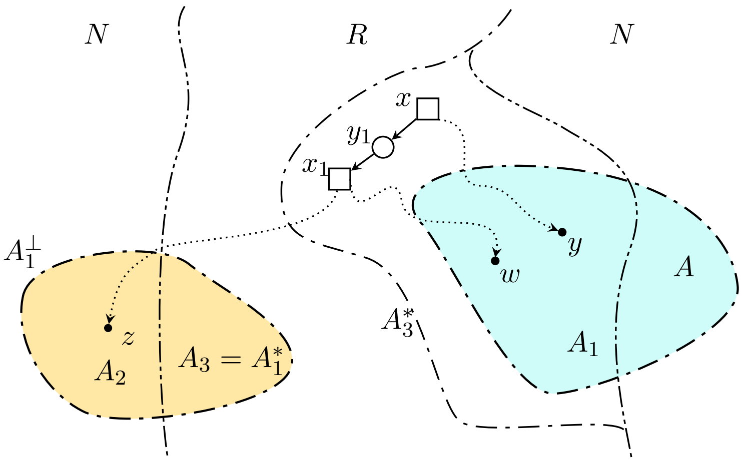
If $x < y$, then $z < x$ cannot hold, because in that case, z **li** y, while z **co** y. Similarly, if $y < x$, then $x < z$ cannot hold.

Four possible cases remain to be treated:

1. $x < y$ and $x \# z$
2. $x < y$ and $x < z$
3. $y < x$ and $z < x$
4. $y < x$ and $x \# z$

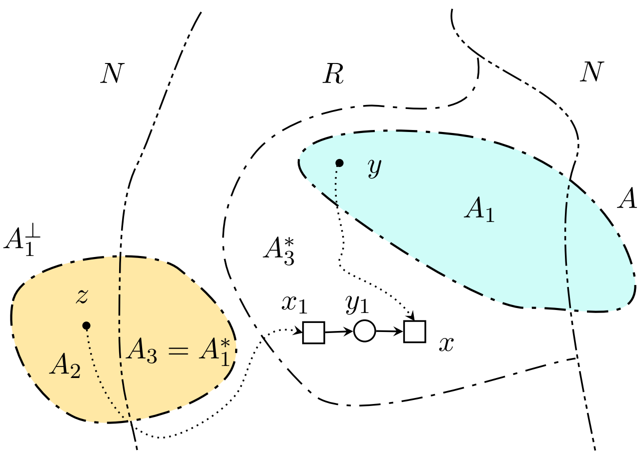
Case 1 is actually impossible, since y would inherit the conflict relation with z, while we know that z **co** y.

We prove the remaining cases by contradiction, showing that, from the assumptions, it is possible to build infinite intervals in N.



Case 2 (see picture above): we can assume that x is an event, since, if it were a condition, then it would have a post-event satisfying the same relations with respect to z (z is concurrent with y). Consider a path π from x to z; we will show that this path is infinite, which leads to a contradiction. Let $y_1 \in x^\bullet$ be the first element on π; since A_3^* is closed in R, $y_1 \in A_3^*$ (and, also, $y_1 \in R$). From $y_1 \notin A_1$, we deduce that there is $w \in A_1$ with $y_1 < w$. Since w **co** z, the paths from y_1 to z and from y_1 to w must pass through x_1, chosen as the unique event which belongs both to R and to $y_1^\bullet$. Apply to x_1 the same argument previously applied to x, and find $y_2 \in A_3^* \setminus A_1$. By iterating, we build an infinite chain, formed by the y_i's and the x_i's, all lying between x and z; but N satisfies the property of finite intervals, and we have a contradiction, so there can be no $x \in A_3^* \setminus A_1$.

Case 3 (see picture below): like in Case 2, we can assume that x is an event. Let y_1 be a pre-condition of x along a path from z to x. Since A_3^* is closed, $y_1 \in A_3^*$; from $z < y_1$, it follows that $y_1 \notin A_1$. Let x_1 be the unique event in ${}^\bullet y_1$; this x_1 can not be in relation **li** with any element of A_3 (otherwise there would be a path from A_3 to A_3^*), hence $x_1 \in A_3^*$; on the other hand, $x_1 \notin A_1$ (otherwise there would be a path from $A_1^\perp$ to A_1).



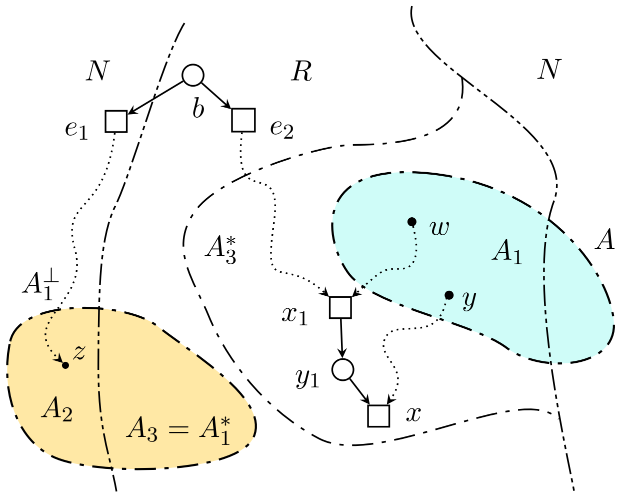
By applying to x_1 the same argument applied to x, we find $y_2 \in {}^\bullet x_1$ such that $z < y_2$ and $y_2 \in A_3^* \setminus A_1$. If we iterate the procedure, we build an infinite path between z and x, but this contradicts the hypothesis of finite intervals in N.

Case 4: we can assume that x is an event, because, if x is a condition, then the unique pre-event of x bears the same relations as x with y and z.

By the definition of conflict, there are $b \in B$, $e_1, e_2 \in E$ such that $(b, e_1), (b, e_2) \in F$, $e_1 \leq z$, $e_2 \leq x$. Let $y_1 \in {}^\bullet x$ lie along a path from b to x. Since A_3^* is closed, $y_1 \in A_3^*$ (hence $y_1 \in R$ and $y_1 \notin A_3$).

Assume $b = y_1$; then $b \notin A_1$ (otherwise there would be a path from A_1 to $A_1^\perp$). But then, being $b \notin A_3$ and $A_3 = A_1^*$, there is $w \in A_1$ such that b **li** w (since we are in a run, it cannot be $b \# w$). If $b < w$, then $z \# w$, which is impossible, since z **co** A_1. Hence $w < b$; but this implies $w < z$, which again is impossible, for z **co** A_1.

Assume now $b < y_1$ (see picture below).



If $b < y_1$, then $y_1 \in R$ and $y_1 \# z$, so that $y_1 \in A_3^* \setminus A_1$; hence as above, since we are in a run, there is $w \in A_1$, with $w < y_1$ (if $y_1 < w$, then also $x < w$ since y_1 is an immediate predecessor of x, and there would be a path from y to w through the element $x \notin A$, contradicting the hypothesis that A is closed, hence convex). Let x_1 be the unique pre-event of y_1. Then $w < x_1$ and $x_1 \in A_3^*$ by convexity of closed sets. Since $z \# x_1$, x_1 cannot belong to A_1 and we can apply to x_1 the same argument applied to x, and find $y_2 \in {}^\bullet x_1$, with $b \leq y_2$, and $y_2 \in A_3^*$. By iterating, we build an infinite path between b and x, contradicting the property of finite intervals in N. □

In the next proposition we show that any set which is closed in a run can be obtained as the intersection of the run with a closed set of the whole net.

Proposition 3.10. For every $A_1 \in L(R)$, there is $A \in L(N)$ such that $A_1 = A \cap R$.

Proof:
Take a closed set in R: $A_1 \in L(R)$. We shall prove that there is a closed set in N such that its intersection with R coincides with A_1:

$$\exists S \in L(N) : S \cap R = A_1$$

We shall in fact prove that the closure of A_1 in N is the required set: $A_1^{\perp\perp} \cap R = A_1$.

To this end, define $A_3 = A_1^\perp \cap R$, $A_2 = A_1^\perp \setminus A_3$. Since $A_1^\perp$ contains all the elements of N which are concurrent with A_1, we deduce that $A_3 = A_1^*$. Since A_1 is closed in R, we have $A_1^{**} = A_1$; hence, taken an element $x \in A_1^{\perp\perp}$, if x belongs to R, then it must belong to A_1. We can then deduce that $A_1^{\perp\perp} \cap R = A_1$. □

In Proposition 3.10, one can choose $A = A_1^{\perp\perp}$, with the closure computed in N.

4. Dynamically Closed Sets

In this section we introduce dynamically closed sets, and show that they form a complete algebraic lattice. Then, in section 4.1, we prove that, if an occurrence net is B-dense, then the two notions of closed sets coincide and then also the lattice $L(N)$ and the lattice of dynamically closed sets coincide.

Dynamically closed sets were introduced, for occurrence nets without conflicts, in [3]. Here, we extend the idea. Informally, A is a dynamically closed set in an occurrence net N if A is a sub-occurrence net of N, and it is closed with respect to immediate causes and effects: if A contains an event e, then it contains also ${}^\bullet e^\bullet$; if it contains ${}^\bullet e$ or $e^\bullet$, then it contains e.

Definition 4.1. Let $N = (B, E, F)$ be an occurrence net, and $A \subseteq B \cup E$. A is *dynamically closed* if, for all $e \in E$:

i. $\min(A) \subseteq B$ is a B-coset in N;

ii. $e \in A \Rightarrow {}^\bullet e^\bullet \subseteq A$;

iii. ${}^\bullet e \subseteq A \Rightarrow e \in A$;

iv. $e^\bullet \subseteq A \Rightarrow e \in A$.

Denote the set of dynamically closed sets of N by $D(N)$. The empty set and $B \cup E$ are easily seen to be dynamically closed. From the definition and the fact that N is condition bordered, it follows immediately that the border of a dynamically closed set does not contain events.

We will show that $D(N)$ is a complete lattice, however first we need a couple of lemmas. The first lemma states that a dynamically closed set is convex in the partial order associated to N.

Lemma 4.2. Let $A \in D(N)$, $x, y \in A$ and $x \leq y$. Then $[x, y] \subseteq A$.

Proof:
Let π be a path in N from x to y. Suppose that π is not entirely contained in A. Then, going backwards from y to x along π, we will find an arc $(v, w) \in F$, with $v \notin A$ and $w \in A$. Since the border of A can only contain conditions of N, $w \in B$. Now, starting from x and going backwards along any path in N, we eventually reach a node $z \in B$ such that $z \in A$, and either ${}^\bullet z = \emptyset$ or ${}^\bullet z \cap A = \emptyset$ (z might coincide with x). Then $z \in \min(A)$ and $w \in \min(A)$, but $z \leq w$, contradicting the assumption that A is dynamically closed. □

The following lemma says that, if a dynamically closed set contains two conflicting elements, then it also contains their past history, up to the origin of the conflict.

Lemma 4.3. Let $A \in D(N)$, $x, y \in A$, and $x \# y$. Then $\mathrm{cfs}(x, y) \subseteq A$.

Proof:
Since the minimal elements of A form a B-coset, x and y cannot be both minimal. Assume that x is not minimal; then there must be z along a path from $\mathrm{cfs}(x, y)$ to x which is in A. If $z \in \mathrm{cfs}(x, y)$ the proof is complete, since A is convex and contains ${}^\bullet e^\bullet$ for all events $e \in A$. Otherwise, $z \# y$, so z and y cannot be both minimal in A, and we can iterate the argument. Since any interval is finite in N, we eventually reach some element in $\mathrm{cfs}(x, y)$. □

Now we can prove that the intersection of an arbitrary family of dynamically closed sets is dynamically closed. Together with the remark that the emptyset and $B \cup E$ are dynamically closed, this allows us to deduce that $D(N)$ is a complete lattice.

Theorem 4.4. Let A_i be a collection of dynamically closed sets for the occurrence net N. Then, $\bigcap_i A_i$ is a dynamically closed set.

Proof:
Property (ii) of Definition 4.1 holds since $e \in \bigcap_i A_i$ implies that for all i, $e \in A_i$. Consequently ${}^\bullet e^\bullet \subseteq A_i$ for all i and ${}^\bullet e^\bullet \subseteq \bigcap_i A_i$. A similar reasoning shows that properties (iii) and (iv) of Definition 4.1 hold as well. Concerning property (i) of Definition 4.1, $\min(\bigcap_i A_i)$ is composed of elements of B by property (ii). Let us suppose $\min(\bigcap_i A_i)$ is not a clique of **co** in N. Then there exist two elements b, c in $\min(\bigcap_i A_i)$, $b \neq c$, such that either b **li** c or $b \# c$ in N. Let b **li** c in N. Then b and c should belong to each one of the A_i (while not necessarily being minimal elements in them). By convexity of the closed sets, the chain between b and c should belong to all of the A_i and consequently to $\bigcap_i A_i$ and either b or c is not minimal. Let $b \# c$ in N. Then b and c should belong to each one of the A_i as above. b and c cannot be immediate conflict events by definition and, by Lemma 4.3, each of the A_i should contain the past history of b and c until the set P of the immediate conflict places leading to b and c. Consequently, either b or c is not a minimal element in $\bigcap_i A_i$. □

Now we can define the complete lattice of dynamically closed sets of the occurrence net $N = (B, E, F)$ as: $\langle D(N), \subseteq, \emptyset, B \cup E \rangle$ where the meet operation coincides with intersection, and the join operation is defined as the dynamic closure of the set union. This lattice is associated to a closure operator: given $H \subseteq B \cup E$, define its closure, denoted $\Delta(H)$, as the smallest dynamically closed set containing H:

$$\Delta(H) = \bigcap_{\{A \in D(N) \mid H \subseteq A\}} A.$$

In general, the union of dynamically closed sets is not dynamically closed. In the next lemma, we show that for some special families of sets, this is instead the case. A family of sets is said to be *directed* if, for any pair of sets in it, it also contains a set larger than their union.

Lemma 4.5. Let $\{A_i\}_{i \in I}$ be a directed family of subsets of $B \cup E$. Then

$$A = \bigcup_{i \in I} \Delta(A_i) \in D(N)$$

Proof:
Let $e \in A$ be an event. Then, there is $i \in I$ such that $e \in \Delta(A_i)$, whence ${}^\bullet e^\bullet \subseteq \Delta(A_i) \subseteq A$. Let now $t \in E$ be an event such that ${}^\bullet t \subseteq A$. Since ${}^\bullet t$ is a finite set, and $\{A_i\}$ is directed, there is $h \in I$ such that ${}^\bullet t \subseteq \Delta(A_h)$, whence $t \in \Delta(A_h)$ and $t \in A$. The case $t^\bullet \subseteq A$ is treated in the same way.

Let $x, y \in \min(A)$. Then, being $\{A_i\}$ directed, there is $h \in I$ such that $x, y \in \Delta(A_h)$. The elements x and y must be minimal in $\Delta(A_h)$, otherwise they would not be minimal in A. Since $\Delta(A_h)$ is dynamically closed, they must be concurrent local states: $x, y \in B$, x **co** y. Hence $\min(A)$ is a B-coset. □

From this lemma, we deduce that $D(N)$ is an algebraic lattice (see, for instance, [9]).

4.1. Relations between Closed Sets and Dynamically Closed Sets

We now have two different notions of closed set, and two corresponding closure operators on the set of elements of an occurrence net. In this section, we explore their relationship.

Proposition 4.6. Every closed set is dynamically closed.

Proof:
Let $A \in L(N)$. Take two distinct elements $x, y \in \min(A)$.

If $x \# y$, then, by Prop. 3.2(v), $\mathrm{cfs}(x, y) \subseteq A$, so x and y would not be minimal in A.

If xF^*y, then, by Prop. 3.2(iv), $[x, y] \subseteq A$, and y would not be minimal in A (similarly for yF^*x).

If x **co** y, then, by Prop. 3.2(i), x cannot be in E since in this case ${}^\bullet x \subseteq A$ and x would not be minimal in A. The same holds for y, so $x, y \in B$.

Let $e \in E$ be an element of A. Then, by Proposition 3.2, ${}^\bullet e^\bullet \subseteq A$.

Finally, let $e \in E$ be such that ${}^\bullet e \subseteq A$. Let z **co** b for all $b \in {}^\bullet e$; then z cannot be in conflict with e otherwise $z \# b$ for at least one $b \in {}^\bullet e$. Similarly, z cannot be in relation **li** with e. So z **co** e and $e \in A$ by definition of $L(N)$. A similar reasoning holds for $e^\bullet \subseteq A$. □

There are dynamically closed sets which are not closed: take for example the set $\Gamma = \{b_i \mid i \geq 1\}$ in the net shown in Figure 5. This set is dynamically closed, but it is not closed: it is actually a cut, so $\Gamma^\perp = \emptyset$, and its closure is the set of all elements of the net.

The dynamical closure of a B-coset can be defined by means of an iterative procedure, which justifies the view of dynamically closed sets as causally closed subprocesses. Starting from the considered B-coset, the procedure adds all events whose pre-conditions or post-conditions are already in the set (see Figure 6).

More formally, let A be a B-coset of N. Starting with $A_0 = A$, define

$$A_{i+1} = A_i \cup \bigcup_{{}^\bullet e \subseteq A_i \vee e^\bullet \subseteq A_i} \left({}^\bullet e \cup \{e\} \cup e^\bullet\right).$$

Define now $\widehat{A} = \bigcup_{i=0}^{\infty} A_i$. The following proposition shows that this construction yields indeed $\Delta(A)$.

Proposition 4.7. Let $\widehat{A}$ be defined as above. Then $\widehat{A} = \Delta(A)$.

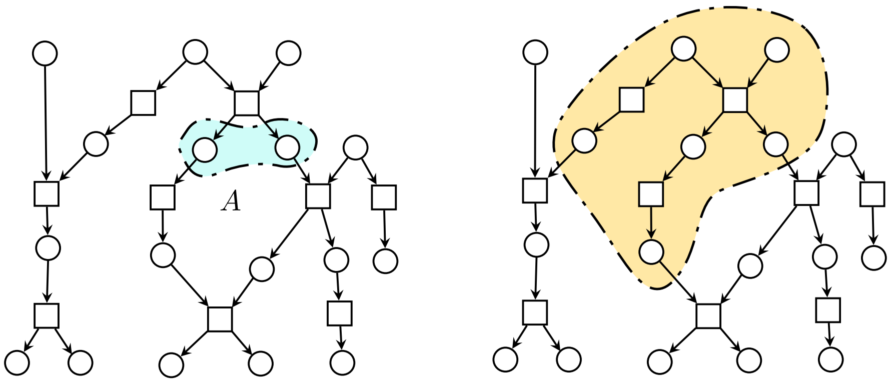


Figure 6. The dynamic closure (right) of the B-coset A on the left

Proof:
We will show that $\widehat{A}$ is dynamically closed, and that it is the smallest dynamically closed set containing A, hence $\widehat{A} = \Delta(A)$.

We show that $\min(\widehat{A})$ is a B-coset. The property holds for A_0. Assume it holds for A_i. The B-elements added in A_{i+1} belong either to ${}^\bullet e$ or $e^\bullet$ for some e. Assume that there are $b_1, b_2 \in \min(A_{i+1})$ with $b_1 \# b_2$. Then there must be $b_1', b_2' \in \min(A)$ such that $b_i F^* b_i'$ and $b_1' \# b_2'$, which is a contradiction.

We show that, for all events $e \in \widehat{A}$, ${}^\bullet e^\bullet \subseteq A$. If $e \in A$, then $e \in A_i$ for some i. Then ${}^\bullet e^\bullet \subseteq A_i \subseteq \widehat{A}$.

We show that, for all events $e \in E$, if ${}^\bullet e \subseteq \widehat{A}$, then $e \in \widehat{A}$ (and similarly for $e^\bullet$). Since ${}^\bullet e$ is a finite set, from ${}^\bullet e \subseteq \widehat{A}$ it follows that ${}^\bullet e \subseteq A_i$ for some i; then $e \in A_i$, hence $e \in \widehat{A}$.

To show that $\widehat{A}$ is the smallest dynamically closed set containing A, observe first that $A_0 = A \subseteq \Delta(A)$. Assume now that $A_i \subseteq \Delta(A)$ for some i. From the definition of A_{i+1} it follows immediately that $A_{i+1} \subseteq \Delta(A)$. So $\widehat{A} \subseteq \Delta(A)$. The thesis is proved since $\Delta(A)$ is by definition the smallest dynamically closed set containing A. $\square$

Our aim is to prove that, in B-dense nets, closed sets as defined in Section 3 and dynamically closed sets coincide. A first step towards that end consists in showing that the closure operators $(.)^{\perp\perp}$ and Δ coincide in B-dense nets, when applied to B-cosets.

Proposition 4.8. Let $N = (B, E, F)$ be a B-dense occurrence net. For every B-coset A, $\Delta(A) = A^{\perp\perp}$.

Proof:
We first show that $\Delta(A) \subseteq A^{\perp\perp}$. Put $A_0 = A$ and apply the construction defined above. Clearly, $A = A_0$ **co** $A^\perp$. By Lemma 2.7, if A_i **co** $A^\perp$, then A_{i+1} **co** $A^\perp$, hence $\widehat{A}$ **co** $A^\perp$, which implies $\Delta(A) = \widehat{A} \subseteq A^{\perp\perp}$.

To show that $\Delta(A) \supseteq A^{\perp\perp}$, assume, by contradiction, that there is $x \in A^{\perp\perp} \setminus \Delta(A)$. From $x \notin \Delta(A)$ it follows $x \notin A$. There are four possible cases.

Case 1: x **co** A. This is in contradiction with $x \in A^{\perp\perp}$.

Case 2: there exists $y \in A$ such that xF^+y. Then there is a path in $A^{\perp\perp}$ from x to y and along this path there is an event $e_0 \notin \Delta(A)$ with a post-condition which belongs both to the path and to the border of $\Delta(A)$, and with at least another post-condition $b_0 \notin \Delta(A)$. Since $e_0 \in A^{\perp\perp}$, then necessarily also $b_0 \in A^{\perp\perp}$. Clearly, b_0 can not be concurrent with all elements of A, so there must be $y_0 \in A$ such that one of the following cases apply.

2.a) $b_0 \; \# \; y_0$; then, by Proposition 3.2(v), there is a condition $b \in A^{\perp\perp}$, $b \in \mathrm{cfs}(b_0, y_0)$. Since $bF^+e_0F^+y$, this would imply $y \; \# \; y_0$, contradicting the hypothesis that A is a B-coset.

2.b) $y_0F^+b_0$; any path leading from y_0 to b_0 must contain e_0, which would imply y_0F^+y, whereas, belonging both to A, y_0 and y should be concurrent.

2.c) $b_0F^+y_0$; then, by repeating the argument at the beginning of Case 2, we find, along a path from b_0 to y_0, an event $e_1 \notin \Delta(A)$, $e_1 \in A^{\perp\perp}$ with a post-condition which belongs both to the path and to the border of $\Delta(A)$, and with at least another post-condition $b_1 \notin \Delta(A)$. This case can not apply indefinitely, otherwise we would find a line, passing through the e_i and b_i, forming, together with the y_i, the pattern shown in Fig. 2, and N would not be B-dense.

Case 3: there exists $y \in A$ such that $x\#y$. Then there is $b \in A^{\perp\perp}$, $b \in \mathrm{cfs}(x, y)$. If $b \notin \Delta(A)$ then apply Case 2 with $x = b$. If $b \in \Delta(A)$ then there is an event e_1 in a path bF^+x such that $e_1 \in A^{\perp\perp} \setminus \Delta(A)$, a precondition of e_1 belongs to $\Delta(A)$, and at least an other precondition, say b_1, is such that $b_1 \in A^{\perp\perp} \setminus \Delta(A)$. b_1 cannot be concurrent to A because it is in $A^{\perp\perp}$, then there should be in A an element y_1 such that one of the following three cases is true:

3.a) $b_1F^+y_1$, then apply Case 2 with $x = b_1$ and $y = y_1$.

3.b) $y_1F^+b_1$, then in a path from y_1 to b_1 crossing the border find $e_2 \in A^{\perp\perp} \setminus \Delta(A)$ with at least a precondition $b_2 \in A^{\perp\perp} \setminus \Delta(A)$. This argument cannot be iterated indefinitely often backwards from b_2 because otherwise there should be an infinite chain backwards.

3.c) $b_1 \; \# \; y_1$, then again there exists $b' \in A^{\perp\perp}$, $b' \in \mathrm{cfs}(b_1, y_1)$. If $b' \notin \Delta(A)$ then apply Case 2 with $x = b'$. If $b' \in \Delta(A)$ then there is an event e_2 in a path $b'F^+b_1$ such that: $e_2 \in A^{\perp\perp} \setminus \Delta(A)$ a precondition of e_2 is in $\Delta(A)$ and at least an other precondition $b_2 \in A^{\perp\perp} \setminus \Delta(A)$. Apply to b_2 the same arguments as to b_1 above.

Case 4: there exists $y \in A$ such that yF^+x. Then there is a path from y to x crossing the border of $\Delta(A)$ with an event $e_0 \notin \Delta(A)$ having a precondition both on the path and in $\Delta(A)$ and at least a precondition $b_0 \in A^{\perp\perp} \setminus \Delta(A)$. Being b_0 in $A^{\perp\perp}$ it cannot be concurrent with A; hence there exists $y_0 \in A$ such that one of the following cases apply:

4.a) $y_0F^+b_0$, then in this path there is an event e_1 such that $e_1 \in A^{\perp\perp} \setminus \Delta(A)$ a precondition of e_1 belongs to $\Delta(A)$ and at least an other precondition, say b_1, is such that $b_1 \in A^{\perp\perp} \setminus \Delta(A)$. Starting again from b_1, we can apply the same argument, but this case can not apply indefinitely, since otherwise we would build a chain infinitely extending into the past of e_0, and this is impossible in the occurrence nets as we have defined.

4.b) $b_0F^+y_0$, then apply the same argument as in Case 2 with $x = b_0$ and $y = y_0$.

4.c) $b_0 \# y_0$, then apply the same argument as in Case 3 with $x = b_0$ and $y = y_0$.

□

We will now show that any dynamically closed set can be obtained by applying the closure operator Δ to the set of its minimal elements. To this end, the following lemma will be useful.

Lemma 4.9. Let $A \in D(N)$. Then, for all $x \in A$, for all $y \in \min(A)$, x and y are not in conflict.

Proof:
Take $x \in A$ and $y \in \min(A)$. If $x \# y$, then there is $z \in B$ such that zF^+x, zF^+y, and z explains the conflict between x and y. By Lemma 4.3, $z \in A$, which is a contradiction, since $y \in \min(A)$. □

Proposition 4.10. Let N be a B-dense occurrence net, and $A \in D(N)$. Then $A = \Delta(\min(A))$.

Proof:
Let $\Gamma = (\min(A))^{\perp\perp}$. To prove that $\Gamma \subseteq A$, observe that, by Prop. 4.8, $\Gamma = \Delta(\min(A))$, so Γ is the smallest dynamically closed set containing $\min(A)$, and $\min(A) \subseteq A$.

To prove that $A \subseteq \Gamma$, proceed by contradiction. Let $x \in A \setminus \Gamma$. Since $\min(A) \subseteq \Gamma$, there is $a_0 \in A$ such that a_0F^+x.

The border of Γ contains only conditions, so, along a path from a_0 to x there is an event e_0 and an arc (b, e_0), with $b \in \Gamma$, $e_0 \notin \Gamma$. By Prop. 3.2 applied to Γ, e_0 has a precondition $b_0 \notin \Gamma$. By the definition of dynamically closed set, $e_0, b_0 \in A$. By Lemma 4.9, b_0 cannot be in conflict with any element in $\min(A)$, so there is $a_1 \in \min(A)$ such that $a_1F^+b_0$. By applying the same argument as before, we find an event $e_1 \in A \setminus \Gamma$, and $b_1 \in {}^\bullet e_1$ such that $b_1 \in A \setminus \Gamma$. This leads to a chain extending indefinitely in the past of e_0, but this is impossible in an occurrence net. □

By Prop. 4.8, $\Delta(\min(A)) = (\min(A))^{\perp\perp}$, hence $A \in L(N)$. And then we can state the following

Corollary 4.11. Let $N = (B, E, F)$ be a B-dense occurrence net, and $A \subseteq B \cup E$. Then $A \in D(N)$ if, and only if, $A \in L(N)$.

As a consequence of the previous corollary and of Lemma 4.5, we can state that in the case of B-dense occurrence nets also $L(N)$ is an algebraic lattice.

5. Conclusion

In this paper we studied concurrency-based closure operations in occurrence nets with forward branching. The algebraic structure associated to the closed sets has been shown to be a complete and orthomodular lattice.

We have introduced dynamically closed sets, which represent subprocesses closed with respect to the firing rule of Petri nets. It is established that closed sets are *dynamically closed*, while the converse does not hold in general.

The B-density property - which is the natural *branching* counterpart of K-density - implies the coincidence of closed and dynamically closed sets, and coincides with the property that every trail crosses either a closed set or its orthocomplement. This fact has two main consequences: 1) it gives further evidence to the fundamental importance of density properties in net structures, also in relation to axiomatizations of the relativity theory [6]; and 2) it opens the way to a new study of occurrence nets as logical models [10] by exploiting the properties of negation in orthomodular structures and truth-value assignments, as consequence of Theorem 3.6. This latter fact is corroborated by the traditional use of orthomodular structures as algebraic models of quantum logics [1].

Our work will continue by defining closure operators based on other relations (for example the conflict relation $\#$), by further exploring points 1) and 2) above and by studying the relations between the orthomodular structures of occurrence nets and net systems, following in this way a line of research on structural representation of concurrency initiated with [2].

Among the applications of our results, we envisage the definition of operators of abstraction for occurrence nets by exploiting the fact that *dynamically closed* sets are *causally closed* subprocesses.

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