Stochastic Bounds for Censored Markov Chains

Ana Bušić
INRIA / ENS
Paris, France
ana.busic@inria.fr

Hilal Djafri
LSV, ENS Cachan
Cachan, France
djafri@lsv.ens-cachan.fr

Jean-Michel Fourneau
PRiSM, Université de Versailles-St-Quentin
Versailles, France
jmf@prism.fr

***Abstract*—Censored Markov chains (CMC) allow to represent the conditional behavior of a system within a subset of observed states. They provide a theoretical framework to study the truncation of a discrete-time Markov chain when the generation of the state-space is too hard or when the number of states is too large. But the stochastic matrix of a CMC may be difficult to obtain. Dayar et al. (2006) have proposed an algorithm, called DPY, that computes a stochastic bounding matrix for a CMC with a smaller complexity with only a partial knowledge of the chain. We prove that this algorithm is optimal for the information they take into account. We also show how some additional knowledge on the chain can improve stochastic bounds for CMC.**

I. Introduction

Let $\{X_t\}_{t\geq 0}$ be a Discrete Time Markov Chain (DTMC) with a state space $\mathcal{X}$. Let $\mathcal{E} \subset \mathcal{X}$ be the set of observed states, $\mathcal{X} = \mathcal{E} \cup \mathcal{E}^c$, $\mathcal{E} \cap \mathcal{E}^c = \emptyset$. For simplicity of presentation, we assume here that the state space is finite and denote by $n = |\mathcal{E}|$ and $m = |\mathcal{E}^c|$. Assume that successive visits of $\{X_t\}_{t\geq 0}$ to $\mathcal{E}$ take place at time epochs $0 \leq t_0 < t_1 < \dots$ Then the chain $\{X_k^{\mathcal{E}}\}_{k\geq 0} = \{X_{t_k}\}_{k\geq 0}$ is called the censored chain with censoring set $\mathcal{E}$ [7]. Let P denote the transition probability matrix of chain $\{X_t\}_{t\geq 0}$. Consider the partition of the state space to obtain a block description of P:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Blocks A, B, C, and D contain respectively transitions from $\mathcal{E}$ to $\mathcal{E}$, from $\mathcal{E}$ to $\mathcal{E}^c$, from $\mathcal{E}^c$ to $\mathcal{E}$, and from $\mathcal{E}^c$ to $\mathcal{E}^c$. The censored chain only observes the states in $\mathcal{E}$. We assume that $\mathcal{E}^c$ does not contain any reducible classes (so that the matrix $Id-D$ is regular). Then the transition probability matrix of the censored chain, often also called the stochastic complement of matrix A, is equal to [7]:

$$S_A = A + B(Id - D)^{-1}C = A + B\left(\sum_{i=0}^{\infty} D^i\right)C. \quad (1)$$

The second term of the right-hand side represents the probabilities of paths that return to set $\mathcal{E}$ through states in $\mathcal{E}^c$.

In many problems, initial probability matrix P can be large or some transition probabilities may be unknown. Therefore, it is difficult or even impossible to compute $(Id - D)^{-1}$ to get S_A. Deriving bounds for S_A from block A of matrix P and from some information on the other blocks is thus an interesting alternative approach and several algorithms have been proposed in the literature. Truffet [6] considered the case when only block A is known. In that case, the stochastic bound is obtained by assuming that all the unknown returning transitions go to the last state of $\mathcal{E}$ (*i.e.* state n).

Dayar et al. [2] proposed an algorithm, called DPY, for the case when blocks A and C are known. We prove here that their algorithm is optimal when we do not have any information on blocks B (transitions between observed and non-observed states) and D (transitions between the non-observed states). We consider further how to improve bounds when some additional information is known on blocks B or D.

In Section II we introduce an alternative decomposition of stochastic complement S_A that turns out to be natural for deriving stochastic bounds. In Section III we describe algorithm DPY and prove its optimality using this decomposition. Then in Section IV we discuss how to use additional information on non-observed states to improve DPY bounds.

II. Decomposition of stochastic complement

Throughout the paper, all the vectors are row vectors, e is a vector with all components equal to 1, v^t denotes a transposed vector, $[x]^+ = \max\{x, 0\}$, for $x \in \mathbb{R}$, $\preceq_{el}$ is element-wise comparison of two vectors (or matrices), and $M[i,*]$ is row i of matrix M.

Let us denote by W the diagonal matrix $diag(1-s_1, 1-s_2, \dots, 1-s_m)$, where $s_i = \sum_{j=1}^{m} D[i,j]$, for all $1 \leq i \leq m$. If for some $1 \leq i \leq m$, $s_i = 1$, then row i of W is equal to $\mathbf{0}$ and matrix W is singular. Therefore, we define a new matrix

$$\widetilde{W} = W + W',\ W' = diag(\mathbf{1}_{\{s_1=1\}}, \dots, \mathbf{1}_{\{s_m=1\}}).$$

Matrix $\widetilde{W}$ is always regular, so $\widetilde{W}^{-1}$ is well defined. Now matrix S_A can be decomposed as:

$$S_A = A + B(Id - D)^{-1}\widetilde{W}\widetilde{W}^{-1}C. \quad (2)$$

It is straightforward to check that:

Lemma 1. *Matrices W' and $\widetilde{W}$ satisfy:*

$$(Id - D)^{-1}W'\widetilde{W}^{-1}C = \mathbf{0}.$$

The following proposition gives an alternative decomposition of stochastic complement. To the best of our knowledge such a representation was not previously stated even if it appears quite simple. Using this new representation we can derive new arguments to prove stochastic bounds based on comparison of stochastic vectors. Such an approach was harder with the usual representation in Eq. 1. as $(Id - D)^{-1}$ is a matrix of expectations.

Proposition 1 (Decomposition of stochastic complement)**.**

1) *Matrix* $(Id - D)^{-1}W$ *is stochastic.*
2) *Matrix* $\widetilde{W}^{-1}C$ *has rows that are either stochastic or equal to* $\mathbf{0}$.
3) *Matrix* S_A *can be decomposed as:*

$$S_A = A + B(Id - D)^{-1}W\widetilde{W}^{-1}C. \tag{3}$$

Proof: 1) We know that row i of matrix $(Id - D)^{-1}C$ is equal to the conditional probability vector of entering the set $\mathcal{E}$, knowing that we initially start in $i \in \mathcal{E}^c$. Let $G = (Id - D)^{-1}$. Therefore, for all i, $\sum_k (GC)[i,k] = 1$ and:

$$\sum_j (GW)[i,j] = \sum_j G[i,j](1 - s_j) = \sum_j G[i,j] \sum_k C[j,k]$$
$$= \sum_k \sum_j G[i,j]C[j,k] = \sum_k (GC)[i,k] = 1.$$

Thus matrix $(Id - D)^{-1}W$ is stochastic.

2) For each row i of matrix $\widetilde{W}^{-1}C$, we have two cases: If $s_i = 1$, then row i of matrix C is equal to $\mathbf{0}$ and so is row i of matrix $\widetilde{W}^{-1}C$. If $s_i < 1$, then $1 - s_i = C[i,*]e^t$, so row i of matrix $\widetilde{W}^{-1}C$ is stochastic.

3) Lemma 1 and Eq. 2. imply Eq. 3. ∎

III. DPY

Prior to stating the algorithm DPY and proving its optimality, we recall first the definition of strong stochastic ordering of random variables on a finite state space $\{1, \ldots, n\}$ (see [5] for more details on stochastic orders).

A. Some Fundamental Results on Stochastic Bounds

We will define operators r and v as in [1] and s for any $m \times n$ matrix M:

$$r(M)[i,j] = \sum_{k=j}^{n} M[i,k], \ \forall i,j, \tag{4}$$

$$v(M)[i,j] = \max_{k \le i} \{r(M)[k,j]\}, \ \forall i,j. \tag{5}$$

$$s(M)[j] = \max_i \{r(M)[i,j]\}, \ \forall j. \tag{6}$$

Let X and Y be two random variables with probability vectors p and q ($p[k] = P(X = k), q[k] = P(Y = k), \ \forall k$).

Definition 1. $X \preceq_{st} Y$ if $\sum_{k=j}^n p[k] \le \sum_{k=j}^n q[k], \ \forall j$ (*i.e.* $r(p) \preceq_{el} r(q)$).

Let $\{X_t\}_{t \ge 0}$ and $\{Y_t\}_{t \ge 0}$ be two DTMC with transition probability matrices P and Q. Then we say that $\{X_t\}_{t\ge 0} \preceq_{st} \{Y_t\}_{t \ge 0}$ if $X_t \preceq_{st} Y_t$ for all $t \ge 0$.

Definition 2. For two probability matrices P and Q, $P \preceq_{st} Q$ if $r(P) \preceq_{el} r(Q)$.

Definition 3. A probability matrix P is $\preceq_{st}$-monotone if for any two probability vectors p and q:

$$p \preceq_{st} q \ \Rightarrow \ pP \preceq_{st} qP.$$

We will use the following characterization of monotonicity (see [5] for the proof):

Proposition 2. *A probability matrix* P *is* $\preceq_{st}$*-monotone iff:*

$$P[i-1,*] \preceq_{st} P[i,*], \ \forall i > 1. \tag{7}$$

i.e. iff $v(P) = P$.

Sufficient conditions for comparison of two DTMC based on stochastic comparison and monotonicity can be found in [5]. These conditions can be easily checked algorithmically and it is also possible to construct a monotone upper bound for an arbitrary stochastic matrix P [1].

Proposition 3. *(Vincent's algorithm [1]) Let* P *be any stochastic matrix. Then the Vincent's bound is given by* $Q = r^{-1}v(P)$*, where* r^{-1} *denotes the inverse of* r*. Then* Q *is* $\preceq_{st}$*-monotone and* $P \preceq_{st} Q$*, therefore* Q *is a transition probability matrix of an upper bounding DTMC. Furthermore, if* $P_1 \preceq_{st} P_2$*, then* $r^{-1}v(P_1) \preceq_{st} r^{-1}v(P_2)$.

Corollary 1. *(Optimality [1, Lemma 3.1]) Let* P *be any stochastic matrix and* $Q = r^{-1}v(P)$*. Then* Q *is the smallest* $\preceq_{st}$*-monotone upper bound for* P*, i.e. if* R *is any stochastic matrix such that* R *is* $\preceq_{st}$*-monotone and* $P \preceq_{st} R$*, then* $Q \preceq_{st} R$.

B. Stochastic bounds for CMC

Now we can formally state the problems we consider:

1) Given only block A, compute a matrix Q such that $S_A \preceq_{st} Q$. Is there an optimal bound (in the sense of Definition 4), knowing only block A?
2) Given blocks A and C, compute a matrix Q such that $S_A \preceq_{st} Q$. Is this bound better than the one obtained knowing only block A? Is there an optimal bound knowing only blocks A and C?
3) Can some additional information on blocks B and D improve stochastic bounds for CMC?

The first question was already answered by Truffet [6]. Denote by $\beta = e - Ae^t$ the slack of probability mass for matrix A. Then the bound in Truffet [6] is given by:

$$T(A) = A + \beta^t(0, \ldots, 0, 1). \tag{8}$$

It is straightforward to see that $S_A \preceq_{st} T(A)$. Furthermore, this is the best bound one can obtain knowing only block A. More formally:

Definition 4. Let $\mathcal{M}$ be a family of stochastic matrices. A stochastic matrix Q is an $\preceq_{st}$-upper bound for family $\mathcal{M}$ if:

$$P \preceq_{st} Q, \ \forall P \in \mathcal{M}.$$

An $\preceq_{st}$-upper bound Q of $\mathcal{M}$ is optimal if:

$$Q \preceq_{st} R, \text{ for any } \preceq_{st}\text{-upper bound } R \text{ of } \mathcal{M}.$$

Let $\mathcal{R}$ be the set of all stochastic matrices such that $\mathcal{E}^c$ does not contain any reducible class (so that for any matrix $Z \in \mathcal{R}$, the stochastic complement S_Z is well defined by Eq. 1). Then the Truffet's bound $T(A)$ in Eq. 8 is the optimal $\preceq_{st}$-upper bound for family:

$$\mathcal{M}(A) = \{S_Z \ : \ Z \in \mathcal{R}, \ Z_{\mathcal{E},\mathcal{E}} = A\}.$$

The proof is straightforward.

The second question was partially answered by Dayar et al. [2]: they derived an algorithm DPY that computes a stochastic bound for S_A when blocks A and C are known. In Algorithm 1 we give the algorithm DPY in its original form in [2]. We show later in this section that DPY bound is optimal, which then fully answers the second question. The third question will be discussed in Section IV.

Algorithm 1: DPY(A, C) [2]

Data: Blocks A and C.
Result: Matrix Q such that $S_A \preceq_{st} Q$.
begin
  $\beta = e - Ae^t$;
  for $j = n$ **downto** 1 **do**
    $H[j] = \max_{k \in \mathcal{E}^c} \left(\frac{\sum_{l=j}^n C[k,l]}{\sum_{l=1}^n C[k,l]} \right)$;
    for $i = 1$ **to** n **do**
      $F[i,j] = (\beta[i]H[j] - \sum_{l=j+1}^n F[i,l])^+$;
  $Q = A + F$;
  return Q;
end

Using operators r and s as defined in Eq. 4 and 6, DPY can be rewritten as:

1) $H = s(\widetilde{W}^{-1}C)$,
2) $F = \beta^t r^{-1}(H)$,
3) $Q = A + F$.

Thus:

$$DPY(A, C) = A + \beta^t r^{-1}(s(\widetilde{W}^{-1}C)). \tag{9}$$

From the above equation it obviously follows that $DPY(A, C) \preceq_{st} T(A)$. The following example shows that DPY can improve the Truffet's bounds.

Example 1. Let

$$A = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.0 \\ 0.2 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.0 & 0.3 \\ 0.0 & 0.0 & 0.4 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.2 & 0.0 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 \\ 0.3 & 0.2 & 0.0 & 0.0 \end{bmatrix}.$$

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.1 & 0.0 \\ 0.1 & 0.3 & 0.0 & 0.0 \\ 0.4 & 0.1 & 0.2 & 0.1 \\ 0.0 & 0.1 & 0.0 & 0.0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 \\ 0.3 & 0.0 & 0.3 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.1 \\ 0.3 & 0.3 & 0.3 & 0.0 \end{bmatrix}.$$

Then the stochastic complement of A is equal to:

$$S_A = \begin{bmatrix} 0.521 & 0.214 & 0.253 & 0.012 \\ 0.334 & 0.185 & 0.371 & 0.110 \\ 0.352 & 0.272 & 0.059 & 0.317 \\ 0.210 & 0.171 & 0.500 & 0.119 \end{bmatrix}.$$

The Truffet's bound $T(A)$ and the DPY bound $DPY(A, C)$ are:

$$T(A) = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.3 \\ 0.2 & 0.1 & 0.3 & 0.4 \\ 0.2 & 0.1 & 0.0 & 0.7 \\ 0.0 & 0.0 & 0.4 & 0.6 \end{bmatrix},$$

$$DPY(A, C) = \begin{bmatrix} 0.4 & 0.287 & 0.275 & 0.038 \\ 0.2 & 0.287 & 0.375 & 0.138 \\ 0.2 & 0.35 & 0.10 & 0.35 \\ 0.0 & 0.312 & 0.525 & 0.163 \end{bmatrix}.$$

C. Optimality of DPY

We will show here the optimality of DPY for family

$$\mathcal{M}(A, C) = \{S_Z \ : \ Z \in \mathcal{R}, \ Z_{\mathcal{E},\mathcal{E}} = A, \ Z_{\mathcal{E}^c,\mathcal{E}} = C\}$$

of all transition probability matrices in $\mathcal{R}$ with given blocks A and C.

Theorem 1 (Optimality of DPY)**.** *Matrix $DPY(A, C)$ is the optimal $\preceq_{st}$-upper bound for family $\mathcal{M}(A, C)$.*

Proof: The proof that $DPY(A, C)$ is an $\preceq_{st}$-upper bound for family $\mathcal{M}(A, C)$ was already given in [2].

We prove here that $DPY(A, C)$ is the optimal bound for $\mathcal{M}(A, C)$. Consider any non-zero row j of matrix C (i.e. such that $s_j < 1$) and denote by B_j the matrix such that for all i, $B_j[i, k] = 0$, if $k \neq j$, and $B_j[i, j] = \beta[i]$. Let D_j be a matrix such that for all i, $D_j[i, k] = 0$, if $k \neq j$, and $D_j[i, j] = s_i$. Let Z_j be the matrix composed of blocks A, B_j, C, and D_j. Then clearly $Z_j \in \mathcal{M}(A, C)$. For Z_j all the returning paths from $\mathcal{E}^c$ to $\mathcal{E}$ go by state $j \in \mathcal{E}^c$. Denote $\widetilde{C} = \widetilde{W}^{-1}C$ to ease the notation. Any $\preceq_{st}$-upper bound R for family $\mathcal{M}(A, C)$ satisfies in particular:

$$S_{Z_j} = A + \beta^t \widetilde{C}[j, *] \preceq_{st} R,$$

i.e. $r(A + \beta^t \widetilde{C}[j, *]) \preceq_{el} r(R)$. This is valid for all j such that $\widetilde{C}[j, *] \neq 0$. Thus:

$$\max_j r(A + \beta^t \widetilde{C}[j, *]) = \max_{\{j \,:\, \widetilde{C}[j,*] \neq 0\}} r(A + \beta^t \widetilde{C}[j, *]) \preceq_{el} r(R).$$

And

$$\begin{aligned} \max_j r(A + \beta^t \widetilde{C}[j, *]) &= r(A) + \beta^t s(\widetilde{C}) \\ &= r(A) + \beta^t H = r(A + F), \end{aligned}$$

so $DPY(A, C) = A + F \preceq_{st} R$. ∎

Similarly, let $\mathcal{M}^e(A, C)$ be the family of all ergodic matrices in $\mathcal{M}(A, C)$.

Theorem 2 (Optimality of DPY for the ergodic matrices)**.** *Matrix $DPY(A, C)$ is the optimal $\preceq_{st}$-upper bound for family $\mathcal{M}^e(A, C)$.*

Proof: The main step of the proof is to show that family $\mathcal{M}^e(A, C)$ is dense within $\mathcal{M}(A, C)$, *i.e.* that for any $U \in \mathcal{M}(A, C)$ and for any $\epsilon > 0$ there exists $V \in \mathcal{M}^e(A, C)$ such that $||U - V||_1 \leq \epsilon$. ∎

In order to obtain an upper bound for the chain $\{X_k^{\mathcal{E}}\}_{k \geq 0}$, we can now apply Proposition 3 and Corollary 1:

Corollary 2. *The smallest $\preceq_{st}$-monotone upper bound for $\{X_k^{\mathcal{E}}\}_{k \geq 0}$ is given by the transition probability matrix:*

$$r^{-1}(v(DPY(A, C))).$$

Remark 1 (Lower bounds)**.** Similar algorithm to compute lower bounds can be obtained using the symmetry of $\preceq_{st}$ order. The proof is omitted here for the sake of conciseness.

We proved the optimality of DPY for the case when only blocks A and C are known. In the following section we consider the case when we have some additional information about block B.

IV. Using Additional Information

The bounds consist in two parts: 1) find a deterministic part we can obtain from A, C and all the additional information on the model; and 2) apply DPY to the unknown part. Thus the optimality of DPY is not sufficient in general to imply the optimality of these bounds. Let us first assume that we also know block B.

Proposition 4. *Assume that A, B, and C are known. Then:*

$$S_A \preceq_{st} DPY(A + BC, C).$$

Proof: The proof is based on two steps. First we build a new expression for the stochastic complement associated with a new matrix. Then we prove that the matrix we have built is stochastic and we use DPY to obtain a bound of the stochastic complement of that matrix. Let us remember Eq. 3 and remember that as D does not contain any recurrent class we have: $(Id-D)^{-1} = \sum_{i=0}^{\infty} D^i = Id+D(Id-D)^{-1}$. After substitution we get: $S_A = A+B(Id+D(Id-D)^{-1})W\widetilde{W}^{-1}C$. After simplification we obtain:

$$S_A = A + BC + BD(Id - D)^{-1}W\widetilde{W}^{-1}C. \qquad (10)$$

Therefore we obtain S_A as the complement of matrix $\left(\begin{array}{c|c} A+BC & BD \\ \hline C & D \end{array}\right)$. Simple algebraic manipulations allow to prove that this matrix is stochastic. Thus S_A is upper bounded by $DPY(A + BC, C)$. ■

Using as example the same blocks A, B, and C already defined, we obtained a new upper bound of the stochastic complement S_A denoted as $H0$. Clearly the bound is better than the one obtained with DPY using only A and C:

$$H0 = \begin{bmatrix} 0.440 & 0.282 & 0.255 & 0.023 \\ 0.260 & 0.242 & 0.375 & 0.123 \\ 0.240 & 0.370 & 0.060 & 0.330 \\ 0.080 & 0.277 & 0.505 & 0.138 \end{bmatrix}.$$

Now assume that we also know D, but we cannot compute $(Id - D)^{-1}$ because of the computational complexity. This assumption is similar to the one developed in [4] where graph theoretical arguments were used to obtain bounds.

Proposition 5. *For any* $K \geq 0$, $S_A \preceq_{st} DPY(A + B\sum_{i=0}^{K} D^i C, C)$.

The proof relies on the same technique as the latter proposition and is omitted here for the sake of conciseness. Let us turn back now to the example for the same blocks and for $K = 1$ (bound $H1$) and $K = 2$ (bound $H2$).

$$H1 = \begin{bmatrix} 0.484 & 0.244 & 0.254 & 0.018 \\ 0.284 & 0.225 & 0.374 & 0.117 \\ 0.312 & 0.304 & 0.060 & 0.324 \\ 0.140 & 0.227 & 0.504 & 0.129 \end{bmatrix},$$

$$H2 = \begin{bmatrix} 0.499 & 0.232 & 0.254 & 0.015 \\ 0.310 & 0.204 & 0.372 & 0.114 \\ 0.324 & 0.295 & 0.060 & 0.321 \\ 0.172 & 0.201 & 0.502 & 0.125 \end{bmatrix}.$$

For the same blocks and the same values of K we have also computed the bound obtained with algorithm based on breadth first search algorithm [4], which are clearly less accurate than the bounds we obtain with the last proposition:

$$FPY1 = \begin{bmatrix} 0.484 & 0.192 & 0.231 & 0.093 \\ 0.284 & 0.160 & 0.339 & 0.217 \\ 0.312 & 0.244 & 0.036 & 0.408 \\ 0.140 & 0.132 & 0.457 & 0.271 \end{bmatrix},$$

$$FPY2 = \begin{bmatrix} 0.499 & 0.205 & 0.237 & 0.059 \\ 0.310 & 0.176 & 0.351 & 0.163 \\ 0.324 & 0.258 & 0.041 & 0.377 \\ 0.172 & 0.155 & 0.471 & 0.202 \end{bmatrix}.$$

V. Conclusions and final remarks

This approach gives a theoretical framework for the partial generation of the state-space and the transition matrix of a really large Markov chain. Partial generation is often performed heuristically by software tools without any control on the accuracy of the results. If the chain is designed using an initial state and the successor function, when we stop the generation, we obtain blocks A and B. Similarly, using an initial state and the predecessor function we get blocks A and C when the partial generation is achieved. Tensor based representation [3] allows to build all blocks, but it is also possible to take advantage of a partial representation to reduce the complexity of the computational algorithms. Clearly, the more information (i.e. blocks) we put in the model, the more accurate are the bounds. Similarly, when we increase K, we also increase the tightness of the bounds. We also want to emphasize the importance of DPY algorithm, which is optimal when only A and C are known and which allows to derive better bounds when we add further useful information. More algorithms and results will be presented in the full version of this paper.

Acknowledgements. This research is partially supported by the European Network of Excellence EURO-NF and CheckBound project from ANR SETIN 2006.

References

[1] T. Dayar, J.-M. Fourneau, N. Pekergin, and J.-M. Vincent. Polynomials of a stochastic matrix and strong stochastic bounds. In *Markov Anniversary Meeting*, pages 211–228. Boson Books, 2006.

[2] T. Dayar, N. Pekergin, and S. Younès. Conditional steady-state bounds for a subset of states in Markov chains. In *SMCtools '06*. ACM Press, 2006.

[3] P. Fernandes, B. Plateau, and W. J. Stewart. Efficient descriptor-vector multiplications in Stochastic Automata Networks. *J. ACM*, 45(3):381–414, 1998.

[4] J.-M. Fourneau, N. Pekergin, and S. Younès. Censoring Markov chains and stochastic bounds. In *EPEW 2007*, LNCS 4748, pages 213–227. Springer, 2007.

[5] A. Muller and D. Stoyan. *Comparison Methods for Stochastic Models and Risks*. Wiley, New York, NY, 2002.

[6] L. Truffet. Near complete decomposability: Bounding the error by a stochastic comparison method. *Ad. in App. Prob.*, 29:830–855, 1997.

[7] Y. Zhao and D. Liu. The censored Markov chain and the best augmentation. *Jour. of App. Prob.*, 33:623–629, 1996.