Weighted O-Minimal Hybrid Systems

Patricia Bouyer

LSV - CNRS & ENS de Cachan
61, avenue du Président Wilson
94230 Cachan – France

Thomas Brihaye

Université de Mons
20, Place du Parc
7000 Mons – Belgium

Fabrice Chevalier

LSV - CNRS & ENS de Cachan
61, avenue du Président Wilson
94230 Cachan – France

Abstract

We consider weighted o-minimal hybrid systems, which extend classical o-minimal hybrid systems with cost functions. These cost functions are "observer variables" which increase while the system evolves but do not constrain the behaviour of the system. In this paper, we prove two main results: (*i*) optimal o-minimal hybrid games are decidable; (*ii*) the model-checking of WCTL, an extension of CTL which can constrain the cost variables, is decidable over that model. This has to be compared with the same problems in the framework of timed automata where both problems are undecidable in general, while they are decidable for the restricted class of one-clock timed automata.

1. Introduction

O-minimal hybrid systems. Hybrid systems are finite-state machines where each state is equipped with a continuous dynamics. In the last thirty years, formal verification of such systems has become a very active field of research in computer science. In this context, hybrid automata, an extension of timed automata [AD94], have been intensively studied [Hen95, Hen96], and decidable subclasses of hybrid systems have been drawn like initialized rectangular hybrid automata [Hen96] or o-minimal hybrid automata. This latter model has

Email addresses: `bouyer@lsv.ens-cachan.fr` (Patricia Bouyer), `thomas.brihaye@umons.ac.be` (Thomas Brihaye), `fabrice.chevalier@sfr.fr` (Fabrice Chevalier)

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been pointed out in [LPS00] as an interesting class of systems with very rich continuous dynamics, but limited discrete steps (at each discrete step, all variables have to be reset, independently from their initial values). Behaviours of such a system can be decoupled into continuous and discrete parts, thus properties of a global o-minimal system can be deduced directly from properties of the continuous parts of the system. This property and properties of o-minimal structures (see [vdD98] for an overview) are exploited in the word encoding techniques, which have been developed in [BMRT04] for (finitely) abstracting behaviours of the system. Using techniques based on this abstraction, reachability properties [BM05] and reachability control properties [BBC06] have been proved decidable for o-minimal hybrid systems. This technique was also used in order to compute a (tight) exponential bound on the size of the coarsest finite bisimulation of *Pfaffian hybrid systems* (see [KV06]).

Models for resource consumption. A research direction which has recently received substantial attention is the twist or extension of (decidable) models for representing more fairly interesting properties of embedded systems, for instance, resource consumption. In that context, timed automata [AD94] have been extended with cost information leading to the model of weighted timed automata [ALP01, BFH$^+$01]. A timed automaton is a finite automaton with clock variables (*i.e.* variables which increase as global time) that can be tested towards constants or reset. In the model of weighted timed automata, an extra cost variable is added which is used as an *observer* variable (it does not constrain the behaviour of the system), evolving linearly while time elapses, and subject to discrete jumps when discrete transitions are taken. This model was appealing for expressing quantitative properties of real-time systems, which was concretized by the decidability of the optimal reachability problem (find the best way — in terms of cost — of reaching a given state) [ALP01, BFH$^+$01, BBBR07] together with the development of the tool Uppaal Cora [cor06], and then by the computability of the optimal mean-cost (find the best way for the system to have a "cost per time unit" as low as possible) [BBL04]. However, more involved properties like cost-optimal reachability control (find the minimum cost that can be ensured for reaching a given state, regardless of the behavior of the environment in which the system is embedded) or WCTL model-checking (WCTL extends the branching-time temporal logic CTL with cost constraints on modalities [BBR04, BBR06]) have been proved undecidable for weighted timed automata with three clocks or more, see [BBR04, BBR05, BBM06]. Though both problems have recently been proved decidable for one-clock weighted timed automata [BLMR06, BLM07], these undecidability results are nevertheless disappointing, because the one-clock assumption is rather restrictive.

Our contributions. In this paper, we propose a natural extension of o-minimal hybrid systems with (definable) positive cost functions which increase while time progresses and which can be used in an optimization criterion, as in the case of weighted timed automata. It is worth noting here that though the underlying system is o-minimal, this extended model, called *weighted o-minimal hybrid automaton*, is not o-minimal as we absolutely do not require that the cost is reset when a discrete transition is taken. However, we prove in this paper that the cost-optimal reachability control problem and the WCTL model-checking problem are both decidable for this class of systems. Because of the existing results on weighted timed automata, this is really a surprise and makes o-minimal hybrid systems an analyzable, though powerful, model. The decidability results of course rely partly on the word encoding techniques that we mentioned earlier, but also require refinements and involved techniques, specific to each of the two problems.

Plan of the paper.. In Section 2, we recall the definition of the models of o-minimal hybrid systems and games. In Section 3, we extend the previously introduced models with cost functions leading to weighted o-minimal hybrid systems and games; we also introduce the optimal reachability control problem and the WCTL model-checking problem that we solved in Section 4 and Section 5, respectively.

A preliminary version of those results were presented in [BBC07], but for lack of space, no proofs were given.

2. General Background

Let $\mathcal{M}$ be a structure. In this paper, when we say that some relation, subset, or function is *definable*, we mean it is first-order definable (possibly with parameters) in the sense of the structure $\mathcal{M}$. A general reference for first-order logic is [Hod97]. We denote by $\mathsf{Th}(\mathcal{M})$ the theory of $\mathcal{M}$. In the sequel, we only consider structures $\mathcal{M}$ that are expansions of ordered groups containing two symbols of constants, *i.e.* $\mathcal{M} = \langle M, +, 0, 1, <, \ldots \rangle$.

2.1. O-Minimality

Let us recall the definition of o-minimal structures [PS86] and give some examples of such structures. The reader interested in o-minimality should refer to [vdD98] for further results and an extensive bibliography on this subject. In the sequel of the paper we focus on o-minimal structures with a decidable theory in order to obtain decidability and computability results.

Definition 1. An ordered structure $\mathcal{M} = \langle M, <, \ldots \rangle$ is *o-minimal* if every definable subset of M is a finite union of points and open intervals (possibly unbounded).

In other words the definable subsets of M are the simplest possible: the ones which are definable in $\langle M, < \rangle$. The following are examples of o-minimal structures.

Example 1. Examples of o-minimal structures are the ordered group of rationals $\langle \mathbb{Q}, <, +, 0, 1 \rangle$, the ordered field of reals $\langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$, and the field of reals with restricted pfaffian functions and the exponential function [Wil96].

An example of non o-minimal structure is the ordered field of reals with the sinus function $\langle \mathbb{R}, <, 0, 1, \sin\rangle$. For instance, the definable set $\{x \mid \sin x = 0\}$ is not a finite union of points and open intervals.

Remark 1. Let us recall that if o-minimal structures share a fine analysis of their definable sets, there is no general results about quantifier elimination in these structures or about the decidability of their theories. An old and celebrated result in this direction is Tarski's theorem which asserts that there exists a (Turing) effective quantifier elimination procedure for real closed fields. Another particularly spectacular and recent result is the model completeness of the theory of the field of real numbers expanded by the exponential function proved by Wilkie ([Wil96]), but it is not known whether the theory of $\langle \mathbb{R}, <, +, \cdot, 0, 1, e^x\rangle$ is decidable. The decidability of the latter structure is implied by *Schanuel's conjecture*, a famous unsolved problem in transcendental number theory (see [MW96, Wil97]).

2.2. O-Minimal Dynamical Systems

In this subsection, we define the notion of *o-minimal dynamical systems*.

Definition 2 (O-minimal dynamical system). An *o-minimal dynamical system* is a pair $(\mathcal{M}, \gamma)$ where:

- $\mathcal{M} = \langle M, +, 0, 1, <, \ldots\rangle$ is an o-minimal expansion of an ordered group,
- $\gamma : V_1 \times V \to V_2$ is a function definable in $\mathcal{M}$ (where $V_1 \subseteq M^{k_1}$, $V \subseteq M$, and $V_2 \subseteq M^{k_2}$).[1] The function γ is called the *dynamics* of the system.

Classically, when M is the set of real numbers, we see V as time, $V_1 \times V$ as space-time, V_2 as (output) space and V_1 as input space. We keep this terminology in the context of a more general structure $\mathcal{M}$.

Let us give a simple example of o-minimal dynamical system.

Example 2. We can view the continuous dynamics of *timed automata* [AD94] as an o-minimal dynamical system. In this case, we have that $\mathcal{M} = \langle \mathbb{R}, <, +, 0, 1\rangle$ and the dynamics $\gamma : (\mathbb{R}^+)^n \times \mathbb{R}^+ \to (\mathbb{R}^+)^n$ is defined by $\gamma(x_1, ..., x_n, t) = (x_1 + t, ..., x_n + t)$.

We define a transition system associated with the dynamical system; this definition is an adaptation to our context of the classical *continuous transition system* in the case of hybrid systems (see [LPS00] for example).

Definition 3. Given $(\mathcal{M}, \gamma)$, a dynamical system, we define a *transition system* $T_\gamma = (S_\gamma, \Sigma, \to_\gamma)$ *associated with* $(\mathcal{M}, \gamma)$ by:

- the set S_γ of states is V_2;

[1] We use this notation in the rest of the paper.

- the set Σ of events is $M^+ = \{m \in M \mid m \geq 0\}$;
- the transition relation $y_1 \xrightarrow{\tau}_\gamma y_2$ is defined by:

$$\exists x,\ \exists t_1, t_2 \text{ s.t. } t_1 \leq t_2,\ \gamma(x, t_1) = y_1,\ \gamma(x, t_2) = y_2,\ \text{and } \tau = t_2 - t_1\ .$$

2.3. O-Minimal Hybrid Systems and Games

In this subsection, we define o-minimal hybrid systems and games.

Definition 4 (O-minimal hybrid systems). Let $\mathcal{M} = (M, +, 0, 1, <, \cdots)$ be an o-minimal structure. An *$\mathcal{M}$-hybrid system*[2] $\mathcal{H}$ is a tuple $(Q, \Sigma, \delta, \gamma)$ where Q is a finite set of locations, Σ is a finite set of actions, δ consists in a finite number of edges $(q, g, a, R, q') \in Q \times 2^{V_2} \times \Sigma \times 2^{V_2} \times Q$ where the *guard* g and the *reset* R are definable in $\mathcal{M}$, and γ maps every location $q \in Q$ to a dynamics $\gamma_q : V_1 \times V \to V_2$ definable in $\mathcal{M}$.

An $\mathcal{M}$-hybrid system $\mathcal{H} = (Q, \Sigma, \delta, \gamma)$ defines a *mixed transition system* $T_\mathcal{H} = (S, \Gamma, \to)$ where:

- the set S of states is $Q \times V_2$;
- the set Γ of labels is $M^+ \cup \Sigma$;
- the transition relation $(q, y) \xrightarrow{e} (q', y')$ is defined when:
  - $e \in \Sigma$ and there exists $(q, g, e, R, q') \in \delta$ with $y \in g$ and $y' \in R(y)$, or
  - $e \in M^+$, $q = q'$, and $y \xrightarrow{e}_{\gamma_q} y'$ where γ_q is the dynamic in location q.

We will also need more precise notions of transitions. When $(q, y) \xrightarrow{\tau} (q, y')$ with $\tau \in M^+$, this is due to some choice of $(x, t) \in V_1 \times V$ such that $\gamma_q(x, t) = y$. We say that $(q, x, t, y) \xrightarrow{\tau} (q', x', t', y')$ if $q = q'$, $x = x'$, $t' = t + \tau$, $\gamma_q(x, t) = y$, and $\gamma_{q'}(x', t') = y'$. We say that an action $(\tau, a) \in M^+ \times \Sigma$ is enabled in a state (q, x, t, y) if there exists (q', x', t', y') and (q'', x'', t'', y'') such that $(q, x, t, y) \xrightarrow{\tau} (q', x', t', y') \xrightarrow{a} (q'', x'', t'', y'')$. We then write $(q, x, t, y) \xrightarrow{\tau, a} (q'', x'', t'', y'')$. We note $\mathsf{Enb}(q, x, t, y)$ the set of actions enabled in (q, x, t, y).

A *run* in $\mathcal{H}$ is a (finite[3] or infinite) sequence

$$\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, x_1, t_1, y_1) \xrightarrow{\tau_2, a_2} \dots .$$

A *position* p along ϱ is a pair $(i, \tau) \in \mathbb{N} \times M^+$ such that $\tau \leq \tau_{i+1}$. We define $\varrho[(i, \tau)] = (q_i, \gamma_{q_i}(x_i, t_i + \tau))$, $\varrho_{\leq (i,\tau)} = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1, a_1} \dots \xrightarrow{\tau_{i-1}, a_{i-1}} (q_i, x_i, t_i, y_i) \xrightarrow{\tau} (q_i, x_i, t_i + \tau_i, \gamma_{q_i}(x_i, t_i + \tau))$, and $\varrho_{\geq (i,0)} = (q_i, x_i, t_i, y_i) \xrightarrow{\tau_{i+1}, a_{i+1}}$

[2] In the following, we may simply say *o-minimal hybrid system* if the underlying structure is clear from the context.

[3] In the case of a finite run, the run may possibly end simply with a delay transition.

.... Let us notice that the positions of a given run are totally ordered in a natural way. If ϱ is finite we define $last(\varrho) = (q_n, x_n, t_n, y_n)$. We note $\mathsf{Runs}_f(\mathcal{H})$ the set of finite runs in $\mathcal{H}$, and $\mathsf{Runs}(\mathcal{H})$ the set of finite or infinite runs in $\mathcal{H}$.

Let us give an example of an o-minimal hybrid system directly inspired from the thermostat example of [ACH+95].

Example 3. The temperature of a room has to be maintained between m and M degrees. The room is equipped with a *thermostat* which senses the temperature and turns a heater *on* and *off*. The temperature is governed by differential equations. Let us denote the temperature by the variable y. When the heater is off, the temperature decreases according to the function $\gamma_{q_0}(\theta, t) = \theta e^{-Kt}$; when the heater is on, the temperature increases according to the function $\gamma_{q_1}(\theta, t) = \theta e^{-Kt} + h(1 - e^{-Kt})$, where t is the time, θ the initial temperature, h and K are parameters for the heater and the room respectively. This situation is described by the hybrid system of Fig. 1. The hybrid system of Fig. 1 is o-minimal as it is definable in $\langle \mathbb{R}, <, +, \cdot, 0, 1, e^x \rangle$.

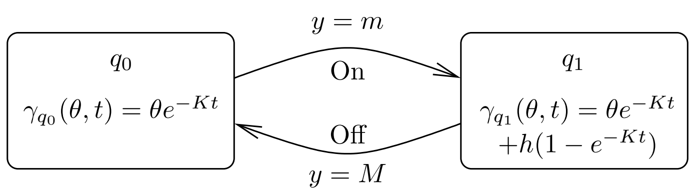


Figure 1: Thermostat

O-minimal hybrid systems are models for *closed systems*, where all transitions are controlled. If we want to consider *open systems* where we distinguish between the actions of a *controller* and the actions of an *environment*, we need to consider games on o-minimal hybrid systems. We are now going to define this notion.

Definition 5 (O-minimal game). Let $\mathcal{M} = (M, +, 0, 1, <, \cdots)$ be an o-minimal structure. An $\mathcal{M}$-*game* (or simply an *o-minimal game*) is a tuple $(Q, \mathsf{Goal}, \Sigma, \delta, \gamma)$ where $(Q, \Sigma, \delta, \gamma)$ is an $\mathcal{M}$-hybrid system, $\mathsf{Goal} \subseteq Q$ is a subset of winning locations, and Σ is partitioned into two subsets Σ_c and Σ_u corresponding to *controllable* and *uncontrollable* actions.

Let $\mathcal{H}$ be an o-minimal game. The game is played by two players, the *controller* and the *environment*; the goal of the controller is to reach a winning state whatever the environment does. In every state s, the controller picks a delay τ and an action $a \in \Sigma_c$ such that there is a transition $s \xrightarrow{\tau,a} s'$. The environment has two choices:

- either it waits τ and executes a transition $s \xrightarrow{\tau,a} s'$ proposed by the controller, or

- it waits τ', $0 \leq \tau' \leq \tau$, and executes a transition $s \xrightarrow{\tau',u} s''$ with $u \in \Sigma_u$.

The game then evolves to a new state (according to the choice of the environment) and the two players proceed to play as before.

We will now formalize the semantics through the concept of *strategy*.

Definition 6 (Strategy). A (controller) *strategy*[4] is a partial function λ from $\mathsf{Runs}_f(\mathcal{H})$ to $M^+ \times \Sigma_c$ such that for all runs ϱ in $\mathsf{Runs}_f(\mathcal{H})$, if $\lambda(\varrho)$ is defined, then $\lambda(\varrho)$ is enabled in $last(\varrho)$.

Intuitively, the strategy tells what needs to be done for controlling the system: at each instant it tells how much time we need to wait and which controllable action needs to be performed after this delay. Note that even when the environment follows the controller's choice, it has to choose between several edges, each one labeled by the action given by the strategy (because the original game is not supposed to be deterministic).

Let $\varrho = (q_0, x_0, t'_0, y_0) \xrightarrow{\tau_1,a_1} \ldots$ be a run, and set for every i, ϱ_i the prefix of ϱ ending at position $(i, 0)$. The run ϱ is said *consistent with a strategy* λ when for all i, if $\lambda(\varrho_i) = (\tau, a)$, then either $\tau_{i+1} = \tau$ and $a_{i+1} = a$, or $\tau_{i+1} \leq \tau$ and $a_{i+1} \in \Sigma_u$. We denote by $\mathsf{Outcome}(s, \lambda)$ the set of runs starting from a state s of the (output) space consistent with the strategy λ. A run $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1,a_1} \ldots \xrightarrow{\tau_p,a_p} (q_p, x_p, t_p, y_p)$ is said to be *winning* if $q_i \in \mathsf{Goal}$ for some i. A run ϱ is said to be *maximal* with respect to a strategy λ if it is infinite or if $\lambda(\varrho)$ is not defined. A strategy λ is *winning from a state* (q, y) if for all (x, t) such that $\gamma_q(x, t) = y$, all maximal runs starting in (q, x, t, y) compatible with λ are winning.

2.4. Tools to Analyze O-Minimal Hybrid Systems

An interesting tool to study hybrid systems is the notion of *(time-abstract) bisimulation* (see [Hen95]). Let us recall its definition.

Definition 7. Let $\mathcal{H}$ be an o-minimal hybrid system and $\sim$ be an equivalence relation on $Q \times V_2$. We say that $\sim$ is a *time-abstract bisimulation* on $\mathcal{H}$ if the following condition holds:

$$\forall s_1, s'_1, s_2 \in Q \times V_2,\ \forall \tau \in M^+,\ \forall a \in \Sigma$$

$$\big(s_1 \sim s_2 \text{ and } s_1 \xrightarrow{\tau,a} s'_1\big) \Rightarrow \big(\exists \tau' \in M^+\ \exists s'_2 \in Q \times V_2\ s_2 \xrightarrow{\tau',a} s'_2 \text{ and } s'_1 \sim s'_2\big).$$

One of the main results concerning o-minimal hybrid systems is that they admit a finite time-abstract bisimulation. This result has been first proved in [LPS00][5]; it was reproved in [Dav99] in a more topological way. In [Bri06], the existence of finite time-abstract bisimulations for o-minimal hybrid systems is proved by means of the *suffix partition*, a technics initiated in [BMRT04, BM05, Bri07].

[4] In the context of control problems, a strategy is also called a *controller*.

[5] [LPS00] appeared as a preprint in 1998.

Let us briefly explain how the *suffix partition* is defined. We first associate words with trajectories. Given $(\mathcal{M}, \gamma)$, a dynamical system, $\mathcal{P}$ a finite partition of V_2, and $x \in V_1$, we associate a word with the trajectory $\Gamma_x = \{\gamma(x,t) \mid t \in V\}$ in the following way. We consider the sets $\{t \in V \mid \gamma(x,t) \in P\}$ for $P \in \mathcal{P}$. This gives a partition of the time V. In order to define a word on $\mathcal{P}$ associated with the trajectory determined by x, we need to define the set $\mathcal{F}_x$ composed of the time intervals I, which are maximal for the property *"there exists $P \in \mathcal{P}$ s.t. for all $t \in I$, $\gamma(x,t) \in P$."* For each x, the set $\mathcal{F}_x$ is totally ordered by the order induced from M. This allows us to define *the word on $\mathcal{P}$ associated with* Γ_x denoted ω_x.

Definition 8. Given $x \in V_1$, *the word associated with* Γ_x is given by the function $\omega_x : \mathcal{F}_x \to \mathcal{P}$ defined by $\omega_x(I) = P$, where $I \in \mathcal{F}_x$ is such that $\forall t \in I$, $\gamma(x,t) \in P$.

The set of words associated with $(\mathcal{M}, \gamma)$ over $\mathcal{P}$ gives in some sense a complete *static* description of the dynamical system $(\mathcal{M}, \gamma)$ through the partition $\mathcal{P}$. In order to recover the *dynamics*, we need further information.

Given a point x of the input space V_1, we have associated with x a trajectory Γ_x and a word ω_x. If we consider (x,t) a point of the space-time $V_1 \times V$, it corresponds to a point $\gamma(x,t)$ lying on Γ_x. To recover in some sense the position of $\gamma(x,t)$ on Γ_x from ω_x, we associate with (x,t) a suffix of the word ω_x denoted $\omega_{(x,t)}$. The construction of $\omega_{(x,t)}$ is similar to the construction of ω_x, we need only to consider the sets of intervals $\mathcal{F}_{(x,t)} = \big\{I \cap \{t' \in V \mid t' \geq t\} \bigm| I \in \mathcal{F}_x\big\}$.

Let us observe that given (x,t), a point of the space-time $V_1 \times V$, there is a unique suffix $\omega_{(x,t)}$ of ω_x associated with (x,t). Given a point $y \in V_2$, it may contain several (x,t) such that $\gamma(x,t) = y$, and so several suffixes are associated with y. In other words, given $y \in V_2$, the *future* of y is non-deterministic, and a single suffix $\omega_{(x,t)}$ is thus not sufficient to recover the dynamics of the transition system through the partition $\mathcal{P}$. To encode the dynamical behavior of a point y of the output space V_2 through the partition $\mathcal{P}$, we introduce the notion of *suffix dynamical type* of a point y w.r.t. $\mathcal{P}$.

Definition 9. Given a dynamical system $(\mathcal{M}, \gamma)$, a finite partition $\mathcal{P}$ of V_2, and a point $y \in V_2$, the *suffix dynamical type of y w.r.t.* $\mathcal{P}$ is denoted $\mathsf{Suf}_{\mathcal{P}}(y)$ and defined by $\mathsf{Suf}_{\mathcal{P}}(y) = \{\omega_{(x,t)} \mid \gamma(x,t) = y\}$.

This allows us to define an equivalence relation on V_2. Given y_1, $y_2 \in V_2$, we say that they are *suffix-equivalent* if and only if $\mathsf{Suf}_{\mathcal{P}}(y_1) = \mathsf{Suf}_{\mathcal{P}}(y_2)$.

We denote by $\mathsf{Suf}\,(\mathcal{P})$ the partition induced by this equivalence. We say that a partition $\mathcal{P}$ is *suffix-stable* if $\mathsf{Suf}(\mathcal{P}) = \mathcal{P}$ (it implies that if y_1 and y_2 belong to the same piece of $\mathcal{P}$ then $\mathsf{Suf}_{\mathcal{P}}(y_1) = \mathsf{Suf}_{\mathcal{P}}(y_2)$).

To understand the word encoding technique, we provide several examples.

Example 4. We first consider a two-dimensional timed automata dynamics (see Example 2). In this case, we have that $\gamma(x_1, x_2, t) = (x_1 + t, x_2 + t)$. We associate with this dynamics the partition $\mathcal{P} = \{A, B\}$ where $B = [1,2]^2$ and

$A = \mathbb{R}^2 \setminus B$. In this example, the suffix partition is made of three pieces, which are depicted in Figure 2.

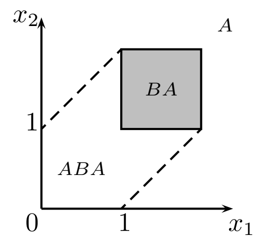


Figure 2: Suffixes for the timed automata dynamics

Example 5. We consider the dynamical system $(\mathcal{M}, \gamma)$ where $\mathcal{M} = \langle \mathbb{R}, +, \cdot, 0, 1, <, \sin_{|_{[0,2\pi]}}, \cos_{|_{[0,2\pi]}} \rangle$ [6] and $\gamma : \mathbb{R}^2 \times [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^2$ is defined as follows.

$$\gamma(x_1, x_2, \theta, t) = \begin{cases} (t.\cos(\theta), t.\sin(\theta)) & \text{if } (x_1, x_2) = (0, 0), \\ (x_1 + t.x_1, x_2 + t.x_2) & \text{if } (x_1, x_2) \neq (0, 0). \end{cases}$$

We associate with this dynamical system the partition $\mathcal{P} = \{A, B, C\}$ where $A = \{(0, 0)\}$, $B = \{\big(\theta\cos(\theta), \theta\sin(\theta)\big) \mid 0 < \theta \leq 2\pi\}$, and $C = \mathbb{R}^2 \setminus (A \cup C)$. Let us call piece B *the spiral* (see Figure 3). There are four dynamical types for this system: $\{ACBC\}$ for the central point $(0, 0)$, $\{CBC\}$ for the "interior" of the spiral, $\{BC\}$ for the spiral, and $\{C\}$ for the "exterior" of the spiral. Note that though the dynamical system is infinitely branching in $(0, 0)$, there is a unique suffix associated with each point y of the output space.

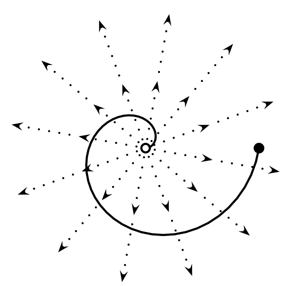
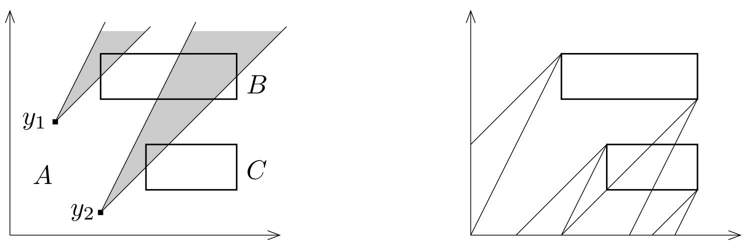


Figure 3: The dynamical system of the spiral

Dynamical systems and suffix dynamical types allow also to encode more sophisticated continuous dynamics. In the next example, we recover in some sense the continuous dynamics of *rectangular automata* [HKPV95], which requires the use of the suffix dynamical types (some of the points do not have a unique suffix).

[6] $\sin_{|_{[0,2\pi]}}$ and $\cos_{|_{[0,2\pi]}}$ correspond to the sinus and cosinus functions restricted to the segment $[0, 2\pi]$.

Example 6. Let us consider the structure $\mathcal{M} = \langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$, the dynamical system $(\mathcal{M}, \gamma)$ where the dynamics $\gamma : \mathbb{R}^2 \times [0,2] \times \mathbb{R}^+ \to \mathbb{R}^2$ is defined by $\gamma(x_1, x_2, p, t) = (x_1 + t, x_2 + p \cdot t)$. We associate with this dynamical system the partition $\mathcal{P} = \{A, B, C\}$ where $B = [2,5] \times [3,4]$, $C = [3,5] \times [1,2]$ and $A = \mathbb{R}^2 \setminus (B \cup C)$ (see Figure 4(a)). Let us focus on the suffix dynamical types of the two points $y_1 = (1, 2.5)$ and $y_2 = (2, 0.5)$. We have that $\mathrm{Suf}_{\mathcal{P}}(y_1) = \{A, ABA\}$ and $\mathrm{Suf}_{\mathcal{P}}(y_2) = \{ABA, ACABA\}$. Though several points have several possible suffixes, the partition induced by the suffix dynamical type is finite and illustrated in Figure 4(b).



(a) The dynamics

(b) The suffix partition

Figure 4: A rectangular dynamics

The main result concering this suffix partition is the following.

Theorem 1. ([Bri06, Theorem 12.6.14])
Let $\mathcal{H}$ be an o-minimal hybrid system and $\mathcal{P}_{\mathcal{H}}$ be a finite and definable partition respecting the guards and the resets of $\mathcal{H}$. The partition $\mathsf{Suf}\,(\mathcal{P}_{\mathcal{H}})$ is a finite, definable partition and induces a bisimulation on $\mathcal{H}$.

3. Weighted O-Minimal Hybrid Systems and Games

In this section, we define the weighted o-minimal hybrid systems and games, which extend the two models of the previous section with cost functions. These cost functions give quantitative information on the behaviours of the systems, which allows the giving of a measure of the performance of the system. These models are respectively inspired by the model of weighted (priced) timed automata [ALP01, BFH$^+$01] and the model of weighted (priced) timed games [ABM04, BCFL04].

3.1. Definitions

In the sequel, we will only consider cost functions which are **non-negative** and **time-non-decreasing**. Note that cost functions in weighted timed automata [ALP01, BFH$^+$01] satisfy these hypotheses.

A *non-negative and time-non-decreasing cost function* is a definable function $\mathsf{Cost} : Q \times V_1 \times V \times M^+ \to M^+$ such that for all $q \in Q$, $x \in V_1$, $t \in V$ and $\tau_1,\ \tau_2 \in M^+$ with $\tau_1 \leq \tau_2$ we have that $\mathsf{Cost}(q, x, t, \tau_1) \leq \mathsf{Cost}(q, x, t, \tau_2)$.

Definition 10 (Weighted o-minimal hybrid system and game). An $\mathcal{M}$-*weighted hybrid system (resp. game)* is an $\mathcal{M}$-hybrid system (resp. game) with a definable non-negative and time-non-decreasing cost function Cost.

The semantics of an o-minimal weighted hybrid system (resp. game) is that of the underlying o-minimal hybrid system (resp. game). Hence, the cost function does not interfere with the behaviours of the system; it gives for every single step of the system a non-negative value, which represents the cost of evolving following that step. It naturally extends to a run in the system: let $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1, a_1} \dots \xrightarrow{\tau_p, a_p} (q_p, x_p, t_p, y_p)$ be a finite run in $\mathcal{H}$. The *cost*[7] *of* ϱ, denoted $\mathsf{Cost}(\varrho)$, is defined as follows:

$$\mathsf{Cost}(\varrho) \;=\; \sum_{i=1}^{p} \mathsf{Cost}(q_{i-1}, x_{i-1}, t_{i-1}, \tau_i).$$

Let us give an example of weighted o-minimal hybrid system.

Example 7 ([BLM07]). The weighted o-minimal hybrid system of Fig. 5 models a never-ending process of repairing problems. The repair of a problem has a certain cost, captured in the model by the cost function Cost. As soon as a problem occurs (modeled by the *pb* transition), the value of the cost grows with rate 1, until actual repair takes place by one of the transitions rp_1 (cheap but long repair) or rp_2 (expensive but quick repair).

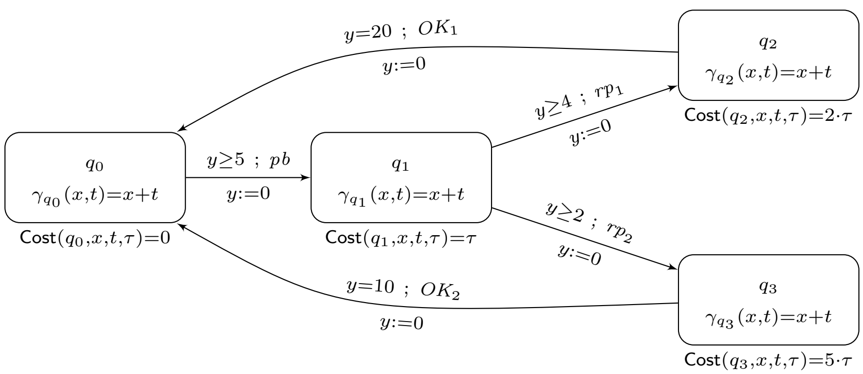


Figure 5: A repair problem example

Remark 2. Let us notice that a weighted o-minimal hybrid system $(\mathcal{H}, \mathsf{Cost})$ can be seen as an hybrid system $\mathcal{H}_c$ where the cost is a regular variable of the

[7] In the case when ϱ ends with a delay transition, *i.e.* there is an additional transition $(q_p, x_p, t_p, y_p) \xrightarrow{\tau_{p+1}} (q_{p+1}, x_{p+1}, t_{p+1}, y_{p+1})$, there is an additional term $\mathsf{Cost}(q_p, x_p, t_p, \tau_{p+1})$ in $\mathsf{Cost}(\varrho)$.

system. However $\mathcal{H}_c$ is not an o-minimal hybrid system, as the cost variable is never reset. One can also check that $\mathcal{H}_c$ is not an *extended o-minimal hybrid systems* as defined in [Gen05]. In particular, weighted o-minimal hybrid systems (seen as hybrid systems with an extra variable) do not admit finite (time-abstract) bisimulations. Let us give a simple example of this fact. Let us consider the single location weighted o-minimal hybrid system depicted in Fig. 6 where the dynamics $\gamma : \mathbb{R}^+ \times \{1,2\} \times \mathbb{R}^+$ is given by $\gamma(x,t) = x + p \cdot t$ and the cost function is given by $\mathsf{Cost}(q,x,t,\tau) = \tau$. Regarding the above discussion, we consider the hybrid system $\mathcal{H}_c$ with two variables x and z, where x is the variable of the o-minimal hybrid system whose dynamics is given by γ and z is the cost variable. Roughly speaking, the variable x can evolve like a clock or twice as fast, and the variable z always evolves as a clock. The states space to consider is thus $(\mathbb{R}^+)^2$. Let us assume that in the initial partition, the point $\{(1,1)\}$ is isolated and the set $x = 0$ has also to be isolated (because of the reset $x := 0$).

We prove that $\mathcal{H}_c$ does not admit a finite bisimulation by showing that the classical bisimulation algorithm [PT87, BFH90, KS90, Hen95, HMR05] does not terminate. Roughly speaking, the bisimulation algorithm works as follows: given two pieces P_1, P_2 of the initial partition, the algorithm computes $P_1 \cap \mathsf{Pre}(P_2)$ (where $\mathsf{Pre}(P_2)$ is the set of predecessors[8] of P_2). If the new set $P_1 \cap \mathsf{Pre}(P_2)$ refines piece P_1, then we create a new partition where piece P_1 is replaced by two pieces $P_1 \cap \mathsf{Pre}(P_2)$ and $P_1 \setminus \mathsf{Pre}(P_2)$. The bisimulation algorithm iterates this operation until the partition is stable w.r.t. $\mathsf{Pre}(\cdot)$ operator. If the computation terminates, the obtained partition is in fact a finite time-abstract bisimulation. However, if the bisimulation algorithm does not terminate, it means that no finite time-abstract bisimulation exists.

Let us now apply the bisimulation algorithm to the example of Fig. 6. We start with the initial partition $\mathcal{P} = \{P_1, P_2, P_3\}$ where $P_1 = \{(1,1)\}$ and $P_2 = \{(0,z) \mid z \in \mathbb{R}^+\}$. From Fig. 7, one can be convinced that $\mathsf{Pre}_t(P_1) \cap P_2$ isolates point $(0, \frac{1}{2})$. Again in Fig. 7, we notice that $\mathsf{Pre}_t(P_1) \cap P_3$ also refines the partition $\mathcal{P}$; let us denote this new piece P_4 (represented by the two plain diagonal lines in Fig. 7). In Fig. 8, we have that $\mathsf{Pre}_a(0, \frac{1}{2}) \cap P_4$ isolates the point $(\frac{1}{2}, \frac{1}{2})$. In Fig. 8, we iterate the same two steps, starting from $(\frac{1}{2}, \frac{1}{2})$ instead of $(1,1)$, to isolate the point $(\frac{1}{4}, \frac{1}{4})$ (see Fig. 9). This process does not terminate, proving the lack of finite bisimulation for this system (see Fig. 10).

3.2. Related Problems

In this subsection, we define the two problems we are interested in: the *cost-optimal control problem* and the WCTL *model-checking problem.*

[8] $\mathsf{Pre}(P)$ is either $\mathsf{Pre}_\tau = \{y \in V_2 \mid \exists y' \in P \text{ s.t. } y \xrightarrow{\tau} y'\}$ or $\mathsf{Pre}_a = \{y \in V_2 \mid \exists y' \in P \text{ s.t. } y \xrightarrow{a} y'\}$.

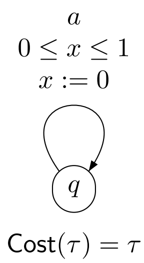


Figure 6:

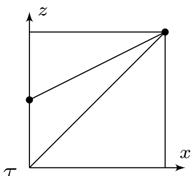


Figure 7: $\mathsf{Pre}(\cdot)$

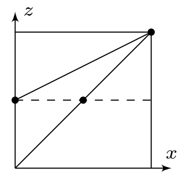


Figure 8: $\mathsf{Pre}^2(\cdot)$

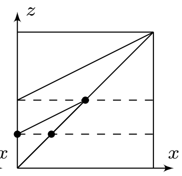


Figure 9: $\mathsf{Pre}^4(\cdot)$

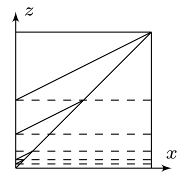


Figure 10: $\mathsf{Pre}^i(\cdot)$

3.2.1. The cost-optimal control problem.

The cost-optimal reachability control problem was first considered in the context of weighted timed automata in [ABM04, BCFL04]. However, it has be shown in [BBR05, BBM06] that the cost-optimal reachability control problem for weighted timed automata is undecidable.

In our context of weighted o-minimal games, the *cost-optimal control problem* asks what is the optimal cost for the controller to reach Goal regardless of what the environment does. In order to take the cost function into account, we now need to define the *cost of a strategy from a state* and the *optimal cost from a state.*

Definition 11 (Cost of a strategy from a state). Let $(\mathcal{H}, \mathsf{Cost})$ be a weighted o-minimal game, s be a state of the (output) space V_2, and λ be a strategy. The *cost* $\mathsf{Cost}(s, \lambda)$ *of* λ *from* s is defined by:

$$\mathsf{Cost}(s, \lambda) = \sup \left\{ \mathsf{Cost}(\varrho) \ \middle| \ \varrho \in \mathsf{Outcome}(s, \lambda) \right\}.$$

Intuitively, the presence of the supremum is explained by the fact that the environment tries to maximize the cost.

Definition 12 (Optimal cost from a state). Let $(\mathcal{H}, \mathsf{Cost})$ be a weighted o-minimal game, and s be a state of the (output) space V_2. The *optimal cost* $\mathsf{OptCost}(q)$ *associated with* s is defined by:

$$\mathsf{OptCost}(s) = \inf \left\{ \mathsf{Cost}(s, \lambda) \ \middle| \ \lambda \text{ is a winning strategy} \right\}.$$

A winning strategy from s is called *optimal* whenever $\mathsf{Cost}(s, \lambda) = \mathsf{OptCost}(s)$.

Problem 1 (Cost-optimal control problem). Given a weighted o-minimal game $\mathcal{H}$, a definable state s, and a definable constant c, decide if there exists a winning strategy λ from s such that $\mathsf{Cost}(s, \lambda) \leq c$.

Problem 2 (Computation of the optimal cost). Given a weighted o-minimal game $\mathcal{H}$ and a definable state s, compute the optimal cost $\mathsf{OptCost}(s)$.

Remark 3. There is an optimal winning strategy from state s iff the infimum can be replaced by a minimum in the definition of $\mathsf{OptCost}(s)$. If we solve problems 1 and 2, we can also determine if there is an optimal winning strategy by asking if there is a strategy λ with $\mathsf{Cost}(s, \lambda) \leq \mathsf{OptCost}(s)$. In [BBR05, BBM06], it has been shown that the variant of Problem 1 for weighted timed automata is undecidable.

3.2.2. The WCTL *model-checking problem.*

The logic Weighted CTL logic, denoted WCTL has been proposed in the context of (weighted) timed systems as an extension of CTL with cost constraints on modalities [BBR04, BBM06, BLM07]. In our context, we define for every structure $\mathcal{M}$ the logic $\mathsf{WCTL}_{\mathcal{M}}$ over Σ inductively as follows:

$$\mathsf{WCTL}_{\mathcal{M}} \ni \varphi \ ::= \ a \ | \ \varphi \vee \varphi \ | \ \neg\varphi \ | \ \mathsf{E}\,\varphi \mathsf{U}_{\sim c}\varphi \ | \ \mathsf{A}\,\varphi \mathsf{U}_{\sim c}\varphi$$

where $a \in \Sigma$, $\sim \in \{<, \leq, =, \geq, >\}$ and c is an $\mathcal{M}$-definable constant.

Let $(\mathcal{H}, \mathsf{Cost})$ be an $\mathcal{M}$-hybrid system. The semantics of $\mathsf{WCTL}_{\mathcal{M}}$ is defined for every state $(q, y) \in Q \times V_2$ of $(\mathcal{H}, \mathsf{Cost})$ as follows:

$$\begin{aligned}
(q,y) \models a \ &\Leftrightarrow\ (q,y) \xrightarrow{a} (q',y') \text{ for some } (q',y') \in Q \times V_2 \text{ [9]} \\
(q,y) \models \neg\varphi \ &\Leftrightarrow\ (q,y) \not\models \varphi \\
(q,y) \models \varphi_1 \vee \varphi_2 \ &\Leftrightarrow\ (q,y) \models \varphi_1 \text{ or } (q,y) \models \varphi_2 \\
(q,y) \models \mathsf{E}\,\varphi_1 \mathsf{U}_{\sim c}\varphi_2 \ &\Leftrightarrow\ \text{there is an infinite run } \varrho \text{ in } \mathcal{H} \text{ from } (q,y) \\
&\qquad \text{s.t. } \varrho \models \varphi_1 \mathsf{U}_{\sim c}\varphi_2 \\
(q,y) \models \mathsf{A}\,\varphi_1 \mathsf{U}_{\sim c}\varphi_2 \ &\Leftrightarrow\ \text{every infinite run } \varrho \text{ in } \mathcal{H} \text{ from } (q,y) \\
&\qquad \text{satisfies } \varrho \models \varphi_1 \mathsf{U}_{\sim c}\varphi_2 \\
\varrho \models \varphi_1 \mathsf{U}_{\sim c}\varphi_2 \ &\Leftrightarrow\ \text{there exists } p \geq 0 \text{ position along } \varrho \text{ s.t.} \\
&\qquad \varrho[p] \models \varphi_2, \text{ for all positions } 0 \leq p' < p \text{ on } \varrho, \\
&\qquad \varrho[p'] \models \varphi_1 \vee \varphi_2, \text{ and } \mathsf{Cost}(\varrho_{\leq p}) \sim c \text{ [10]}.
\end{aligned}$$

We use $\top$ for $a \vee \neg a$, and classical "eventually" and "always" operators: $\mathsf{E}\,\mathsf{F}_{\sim c}\varphi$ (resp. $\mathsf{A}\,\mathsf{F}_{\sim c}\varphi$) stands for $\mathsf{E}\,\top\mathsf{U}_{\sim c}\varphi$ (resp. $\mathsf{A}\,\top\mathsf{U}_{\sim c}\varphi$) and $\mathsf{A}\,\mathsf{G}_{\sim c}\varphi$ (resp. $\mathsf{E}\,\mathsf{G}_{\sim c}\varphi$) stands for $\neg\mathsf{E}\,\mathsf{F}_{\sim c}\neg\varphi$ (resp. $\neg\mathsf{A}\,\mathsf{F}_{\sim c}\neg\varphi$).

Let us give an example of WCTL formulae on the repair problem of Example 7.

Example 8. [BLM07] An example of property that can be expressed with WCTL is "*Whenever a problem occurs, it can be repaired within a total cost of* 55." It can be expressed with the following formula:

$$\mathsf{A}\,\mathsf{G}\big(pb \implies \mathsf{E}\,\mathsf{F}_{\leq 55}(OK_1 \vee OK_2)\big).$$

One can easily check that this formula holds for every state of the weighted o-minimal hybrid system of Fig. 5.

Problem 3 (Model-checking of WCTL). Given $(\mathcal{H}, \mathsf{Cost})$, an $\mathcal{M}$-weighted hybrid system, φ a $\mathsf{WCTL}_{\mathcal{M}}$-formula, and $(q, y) \in Q \times V_2$ a definable state of $\mathcal{H}$, decide whether $(q, y) \models \varphi$.

Remark 4. Note that classical results on o-minimal hybrid systems cannot be used to solve the problems presented above (as we have already seen, weighted o-minimal hybrid systems are not o-minimal hybrid systems, and even have no finite bisimulation), and hence *ad-hoc* proofs have to be developed.

[9] This can be viewed equivalently as atomic propositions.

[10] Following [Ras99], we use $\varphi_1 \vee \varphi_2$ to handle open intervals in timed models.

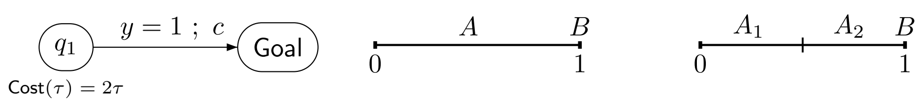
In the following, to avoid critical situations where the system could be blocked, we assume that all the o-minimal hybrid systems (or games) we consider are *non-blocking*, *i.e.*, that from every state (q, y) there exists $\tau \in M^+$ and an action a such that $(q, y) \xrightarrow{\tau,a} (q', y')$ for some (q', y').

4. Solving the Cost-Optimal Control Problem

In this section, we prove the decidability of Problem 1.

Remark 5. In [BBC06], in order to prove the decidability of the reachability problem, we rely on the fact that the suffix equivalence is enough to distinguish between losing and winning states. In the context of Problem 1, this is no longer true. Let us consider the very simple weighted o-minimal game $\mathcal{H}$ depicted on Figure 11(a), where the dynamics in q_1 is given by $\gamma(x, t) = x + t$ and the cost function in q_1 is given by $\mathsf{Cost}(q_1, x, t, \tau) = 2\tau$. In this example, the partition of $[0, 1]$ respecting the guards and resets consists of isolating the point $\{1\}$, *i.e.* $\mathcal{P}_{\mathcal{H}} = \{A, B\}$ where $A = [0, 1[$ and $B = \{1\}$ (see Figure 11(b)). One can easily be convinced that $\mathsf{Suf}(\mathcal{P}_{\mathcal{H}}) = \mathcal{P}_{\mathcal{H}}$.

Assume we want to reach Goal with a cost no greater than 1. It is fairly easy to see that point $s = (q_1, 0)$ cannot achieve this goal although the point $s' = (q_1, 0.6)$ can. However, the two points s and s' have the same suffix w.r.t. $\mathcal{P}_{\mathcal{H}}$. The correct partition for this game is given in Figure 11(c), which refines the suffix partition.



(a) The $\mathcal{M}$-game $\mathcal{A}$ (b) The suffix partition (c) The correct partition

Figure 11: The suffix partition is not sufficient

We are now going to prove a key proposition: we can restrict to winning strategies that cross each edge at most once. In order to prove this proposition, we use the fact that the cost functions have non-negative values.

Definition 13. Let $(\mathcal{H}, \mathsf{Cost})$ be a weighted o-minimal game. We say that a run $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1,a_1} (q_1, x_1, t_1, y_1) \xrightarrow{\tau_2,a_2} \dots$ crosses the edge $e = (q, g, a, R, q')$ at step i if $q = q_{i-1}$, $q' = q_i$, $a = a_i$, $\gamma_{q_{i-1}}(x_{i-1}, t_{i-1} + \tau_i) \in g$, and $y_i \in R$.

Definition 14. Let $(\mathcal{H}, \mathsf{Cost})$ be a weighted o-minimal game. We say that a run $\varrho' = s'_0 \to s'_1 \to \dots \to s'_k$ extends a run $\varrho = s_0 \to s_1 \to \dots \to s_l$ if $l \leq k$, and for every $0 \leq i \leq l$, $s'_i = s_i$.

Proposition 1. *Let $(\mathcal{H}, \mathsf{Cost})$ be a weighted o-minimal game, and (q_0, y_0) be a state of $\mathcal{H}$. If there exists a winning strategy λ from (q_0, y_0), then there exists a winning strategy λ_{new} with $\mathsf{Cost}((q_0, y_0), \lambda_{new}) \leq \mathsf{Cost}((q_0, y_0), \lambda)$ such that every run starting in (q_0, y_0) and compatible with λ_{new} crosses each edge of $\mathcal{H}$ at most once.*

Proof. The idea of the proof is the following: iteratively for each edge e, we ensure that there is a winning strategy (of cost $c \leq \mathsf{Cost}((q, y), \lambda)$) crossing e at most once.

We see $\mathsf{Outcome}((q, y), \lambda)$ as a(n infinitely branching) tree: every path in this tree is finite and reaches Goal, but it may cross e several times. We construct a new strategy λ_{new} which shortcuts λ in the following sense: it simulates λ until e is crossed for the first time and then switches to a descendant in the tree from which all pathes no longer cross e anymore (this will be possible, as λ is winning). Such a strategy crosses e at most once, its cost is smaller than that of λ as its compatible runs are "shorter" than the ones of λ and the cost function is non-negative. As we will see, this *'shortcut'* procedure only works thanks to the strong reset assumptions.

Without loss of generality, we assume that there are no outgoing edges from states of Goal. We fix a state (q_0, y_0), a winning strategy λ from (q_0, y_0), and an edge $e = (q, g, a, R, q')$. We will show that there exists a winning strategy λ_{new} such that all runs compatible with λ_{new} cross e at most once.

The construction of λ_{new} is based on the following lemma:

Lemma 1. *Let ϱ be a finite run from (q_0, y_0), compatible with λ and crossing e at the last step. Let $(q', x', t', y') = last(\varrho)$ (in particular, $(q', x', t', y') \in R(e)$). There exists a run ϱ^{ext} extending ϱ, compatible with λ, such that $last(\varrho^{ext}) = (q', x', t', y')$, and such that for every $\widetilde{\varrho}$ extending ϱ^{ext} and compatible with λ, $\widetilde{\varrho}$ does not cross e anymore after the prefix ϱ^{ext}.*

Proof (of Lemma 1). Towards a contradiction, assume that there exists ϱ not satisfying the lemma. Then for every run ϱ^{ext} extending ϱ, compatible with λ, such that $last(\varrho^{ext}) = (q', x', t', y')$, there exists $\widetilde{\varrho}$ extending ϱ^{ext} compatible with λ, and crossing e strictly after ϱ^{ext}.

We set $\varrho_1 = \varrho$. This run satisfies the above condition for ϱ^{ext}. We can extend ϱ_1 into ϱ_2 such that ϱ_2 is compatible with λ and crosses e strictly after ϱ_1. Without loss of generality, we can assume that ϱ_2 ends with a move corresponding to e, and also without loss of generality we assume that $last(\varrho_2) = (q', x', t', y')$ (this is without loss of generality because of the strong reset condition). The run ϱ_2 then satisfies the above conditions for ϱ^{ext}, and we can thus iteratively construct an increasing sequence $(\varrho_i)_{i \geq 1}$ that all end in (q', x', t', y') after a last move corresponding to edge e. At the limit, we obtain an infinite run which is compatible with λ, and which crosses e infinitely often. This contradicts the assumptions that λ is a winning strategy and that Goal has no outgoing edges.

□

Remark 6. Note that the set $\mathsf{Outcome}((q_0, y_0), \lambda)$ forms an *a priori* infinitely branching tree. That is the reason why we cannot directly apply König's lemma stating that every finitely branching tree with infinitely many nodes has an infinite branch. If the tree had been finitely branching, then we could have said that if e appears infinitely often in that tree, there is a branch with infinitely many occurrences of e, hence a losing branch. However, in our framework we need to use a stronger property corresponding to Lemma 1.

We define the new strategy λ_{new} on runs crossing at most once edge e. We will see that it is sufficient as λ_{new} will only generate runs crossing edge e at most once. Let $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, x_1, t_1, y_1) \xrightarrow{\tau_2, a_2} \dots \xrightarrow{\tau_p, a_p} (q_p, x_p, t_p, y_p)$ be a finite run compatible with λ_{new}, $\lambda_{new}(\varrho)$ is then defined as follows:

- If ϱ does not cross e then $\lambda_{new}(\varrho) = \lambda(\varrho)$, otherwise
- Assume that the step $(q_{i-1}, x_{i-1}, t_{i-1}, y_{i-1}) \xrightarrow{\tau_i, a_i} (q_i, x_i, t_i, y_i)$ corresponds to the last edge e. The run $\varrho_{\leq(i,0)}$ satisfies the hypothesis of Lemma 1. We call ϱ^{ext} the run which is constructed in this lemma, in particular ϱ^{ext} also ends in (q_i, x_i, t_i, y_i). We have that $last(\varrho^{ext}) = (q_i, x_i, t_i, y_i)$, and we can define $\lambda_{new}(\varrho)$ as $\lambda(\varrho^{ext} \cdot \varrho_{\geq(i,0)})$, where $\varrho^{ext} \cdot \varrho_{\geq(i,0)}$ denotes the concatenation of ϱ^{ext} and $\varrho_{\geq(i,0)}$. Note that by construction, $\varrho^{ext} \cdot \varrho_{\geq(i,0)}$ is compatible with λ, hence $\varrho_{\geq(i,0)}$ does not cross e (Lemma 1).

Informally, λ_{new} acts like λ until e is crossed. Then it behaves like λ *assuming* that ϱ^{ext} has been read. By construction, λ_{new} immediately satisfies the following properties.

Lemma 2. *Let ϱ be a run compatible with λ_{new}:*

- *ϱ crosses e at most once,*
- *if ϱ is maximal w.r.t. λ_{new} then it is winning,*
- *there exists a run ϱ_0 compatible with λ such that $\mathsf{Cost}(\varrho) \leq \mathsf{Cost}(\varrho_0)$,*
- *if e' is an edge and ϱ crosses e' k times, then there exist ϱ_1 compatible with λ which crosses e' $k' \geq k$ times.*

Point 4 of Lemma 2 ensures that we can apply this procedure successively to all transitions and find a winning strategy which crosses every edge at most once. This concludes the proof of Proposition 1. □

Remark 7. The proof of Proposition 1 heavily relies on the strong reset property of o-minimal hybrid systems and does not hold for control problems over timed automata (for instance). In Figure 12, we give an example of a timed game (with two clocks x and y) not satisfying Proposition 1. Let λ be the following (memoryless) strategy:[11]

$$\lambda(\ell_1,(0,0)) = (0,b)\ ;\ \lambda(\ell_2,(0,0)) = (1,a)\ ;\ \lambda(\ell_1,(1,1)) = (0,b)\ ;$$
$$\lambda(\ell_2,(1,0)) = (1,a)\ ;\ \lambda(\ell_1,(2,1)) = (0,b).$$

Informally, the strategy is to move imediately to location ℓ_2, to avoid that an uncontrollable u be taken. Then one time unit later, we move back to ℓ_1 (also to avoid an uncontrollable move), and then immediately back to ℓ_2, and one time unit later to ℓ_1 so that the edge leading to the winning state can be taken. The strategy λ is winning from $(\ell_1,(0,0))$ and requires twice crossing the transition (ℓ_1,ℓ_2). However, one can easily be convinced that no winning strategy from $(\ell_1,(0,0))$ can cross the transition (ℓ_1,ℓ_2) only once.

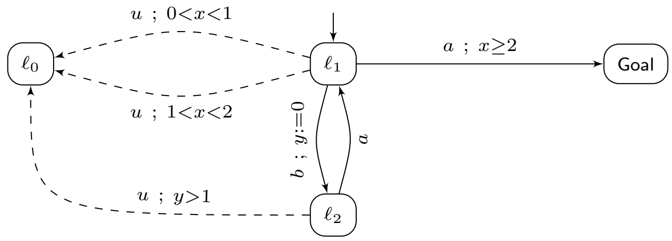


Figure 12: A timed game not satisfying Proposition 1.

We now give a backward algorithm which computes the optimal cost, based on the formulation of the problem given in [LMM02, ABM04]. The termination of the algorithm will rely on Proposition 1. For this, given $(q,y) \in Q \times V_2$ and $n \in \mathbb{N}$, we define $c_n(q,y)$, the optimal cost of reaching Goal from (q,y) in at most n steps. Formally we define:

$$\begin{cases} c_0(q,y) = 0 & \text{if } q \in \mathsf{Goal} \\ c_0(q,y) = +\infty & \text{if } q \notin \mathsf{Goal} \end{cases}$$

and for every $n \in \mathbb{N}$,

$$c_{n+1}(q,y) = \sup_{\gamma_q(x,t)=y}\ \inf_{(\tau,a)\in\mathsf{Enb}(q,x,t,y)}\ \max\begin{cases} \mathsf{Cost}^{\tau,a}_{c_n}(q,x,t,y) \\ \sup\limits_{\substack{(\tau',u)\in\mathsf{Enb}(q,x,t,y)\\ \tau'\leq\tau}} \mathsf{Cost}^{\tau',u}_{c_n}(q,x,t,y) \end{cases}$$

[11] A *memoryless* strategy is a strategy which only depend on the last state of the run. It is thus sufficient to define $\lambda(s)$ for a state s, meaning that $\lambda(\varrho) = \lambda(last(\varrho))$.

where $(\tau, a) \in \mathsf{Enb}(q, x, t, y)$ iff $a \in \mathsf{Enb}(q, x, t+\tau, \gamma_q(x, t+\tau))$, and

$$\mathsf{Cost}_{c_n}^{\tau,e}(q, x, t, y) = \mathsf{Cost}(q, x, t, y, \tau) + \sup\{c_n(q', y') \mid (q, x, t, y) \xrightarrow{\tau,a} (q', x', t', y')\}.$$

The intuition behind the formula is rather simple: when arriving in a state q, the environment chooses the pair (x, t) such that $\gamma_q(x, t) = y$ from which the game will evolve, hence we need to be as pessimistic as possible, *i.e.* to choose the pair (x, t) which gives a cost as big as possible, hence the first sup operator. Then the controller can choose the next controllable move he wants to make (and at which date); this is why there is an inf operator; Finally, the environment can let the controller play his action, or can take an uncontrollable action before. Of course, we have to consider the worst case for the controller, *i.e.* when the resulting cost will be as high as possible, hence the next max and sup operators.

We will now show that these intuitive formulas are indeed correct and, together with Proposition 1, this will lead to an algorithm for computing the optimal cost in an o-minimal game (if the underlying structure is decidable).

Definition 15. Let $(\mathcal{H}, \mathsf{Cost})$ be a weighted o-minimal game. A strategy λ is said to be *n-bounded* from (q, y) if every run compatible with λ starting from (q, y) has length at most n.

Let us notice that given a definable, n-bounded strategy λ, and a definable state (q, y), the set $\mathsf{Outcome}((q, y), \lambda)$ is definable.

Lemma 3. *For every $\varepsilon > 0$ and for every $(q, y) \in Q \times V_2$ s.t. $c_n(q, y) < +\infty$, there exists a definable n-bounded strategy λ from (q, y) such that $\mathsf{Cost}((q, y), \lambda) \leq c_n(q, y) + \varepsilon$.*

Proof. We show the lemma by induction on n. For $n = 0$, $c_0(q, y) < +\infty$ implies that $(q, y) \in \mathsf{Goal}$. Hence the strategy λ which is undefined satisfies the condition.

Suppose the lemma is true for $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $(q_0, y_0) \in Q \times V_2$ such that $c_{n+1}(q_0, y_0) < +\infty$. We will construct an $(n+1)$-bounded strategy λ from (q_0, y_0) such that $\mathsf{Cost}((q_0, y_0), \lambda) \leq c_{n+1}(q_0, y_0) + \varepsilon$. As we suppose the lemma is true for n, for every (q', y') such that $c_n(q', y') < +\infty$ there is an n-bounded strategy from (q', y'), denoted $\lambda^{q',y'}$, such that $\mathsf{Cost}((q', y'), \lambda^{q',y'}) \leq c_n(q', y') + \frac{\varepsilon}{2}$.

Next, fix (x_0, t_0) such that $\gamma_{q_0}(x_0, t_0) = y_0$, and if $q_0 \notin \mathsf{Goal}$, let (τ, a) be such that

$$\max \left(\begin{array}{c} \mathsf{Cost}_{c_n}^{\tau,a}(q_0, x_0, t_0, y_0), \\ \sup\limits_{\substack{(\tau', u) \in \mathsf{Enb}(q_0, x_0, t_0, y_0) \\ \tau' \leq \tau}} \mathsf{Cost}_{c_n}^{\tau',u}(q_0, x_0, t_0, y_0) \end{array} \right) \leq c_{n+1}(q, y) + \frac{\varepsilon}{2} \ .$$

We construct the strategy λ from this (τ, a) and from the strategies $\lambda^{q',y'}$ defined above, for runs starting from (q_0, x_0, t_0, y_0). Let $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1, a_1}$

$(q_1, x_1, t_1, y_1) \dots \xrightarrow{\tau_m, a_m} (q_m, x_m, t_m, y_m)$ be a finite run. We define $\lambda(\varrho)$ as follows:

- if $|\varrho| = 0$ (*i.e.* the run ϱ reduces to the state (q_0, x_0, t_0, y_0)) and if $q_0 \notin \mathsf{Goal}$, then set $\lambda(\varrho) = (\tau, a)$,
- if $|\varrho| \geq 1$, then set $\lambda(\varrho) = \lambda^{q_1,y_1}(\varrho_{\geq(1,0)})$.

We now show that λ is an $(n+1)$-bounded strategy from (q_0, y_0) and that $\mathsf{Cost}((q_0, y_0), \lambda) \leq c_{n+1}(q_0, y_0) + \varepsilon$. Let $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, x_1, t_1, y_1) \dots$ be a maximal run compatible with λ. If $|\varrho| \geq 1$, the suffix $\varrho_{\geq(1,0)}$ is compatible with λ^{q_1,y_1}, so by induction hypothesis, it is winning, hence finite, and

$$\mathsf{Cost}(\varrho_{\geq(1,0)}) \leq c_n(q_1, y_1) + \frac{\varepsilon}{2} \ .$$

The first move of ϱ is compatible with λ, hence we deduce that

$$\mathsf{Cost}(q_0, x_0, t_0, \tau_0) + c_n(q_1, y_1) \leq c_{n+1}(q, y) + \frac{\varepsilon}{2} \ .$$

Hence we deduce that

$$\mathsf{Cost}(\varrho) = \mathsf{Cost}(q_0, x_0, t_0, \tau_0) + \mathsf{Cost}(\varrho_{\geq(1,0)}) \leq c_{n+1}(q, y) + \varepsilon$$

If $|\varrho| = 0$, as ϱ is maximal, it means that $q_0 \in \mathsf{Goal}$, which immediately yields $0 = \mathsf{Cost}(\varrho) \leq c_{n+1}(q, y) + \varepsilon$.

Finally, note that λ is definable. Indeed λ^{q_1,y_1} is definable by induction hypothesis. Moreover, if $q_0 \notin \mathsf{Goal}$, there is an $a \in \Sigma_c$ such that the set $\mathit{Time}_{q_0,x_0,t_0,y_0,\varepsilon,a}$ defined by

$$\left\{ \tau \;\middle|\; \max \left(\begin{array}{c} \mathsf{Cost}_{c_n}^{\tau,a}(q_0, x_0, t_0, y_0), \\ \sup\limits_{\substack{(\tau',u) \in \mathsf{Enb}(q_0,x_0,t_0,y_0) \\ \tau' \leq \tau}} \mathsf{Cost}_{c_n}^{\tau',u}(q_0, x_0, t_0, y_0) \end{array} \right) \leq c_n(q, y) + \frac{\varepsilon}{2} \right\}.$$

is not empty. Thus, using the *curve selection* of o-minimal expansions of ordered groups (see [vdD98, chap.6]), we can definably pick a τ in this set. □

Lemma 4. *If λ is an n-bounded winning strategy from (q, y), then* $\mathsf{Cost}((q, y), \lambda) \geq c_n(q, y)$.

Proof. We show the lemma by induction on n. It is immediate for $n = 0$.

Suppose the lemma is true for $n \in \mathbb{N}$, and let λ be an $(n+1)$-bounded winning strategy from (q_0, y_0). We show that $\mathsf{Cost}((q_0, y_0), \lambda) \geq c_{n+1}(q_0, y_0)$. Let (x_0, t_0) be such that $\gamma_{q_0}(x_0, t_0) = y_0$. We want to find (τ, a) such that

$$\mathsf{Cost}((q_0, y_0), \lambda) \geq \max \left(\begin{array}{c} \mathsf{Cost}_{c_n}^{\tau,a}(q_0, x_0, t_0, y_0), \\ \sup\limits_{\substack{u \in \mathsf{Enb}(q_0,x_0,t_0,y_0) \\ \tau' \leq \tau}} \mathsf{Cost}_{c_n}^{\tau',u}(q_0, x_0, t_0, y_0) \end{array} \right). \quad (1)$$

Define $(\tau, a) = \lambda((q_0, x_0, t_0, y_0))$ and let (q', y') be such that $(q_0, x_0, t_0, y_0) \xrightarrow{\tau,a} (q', y')$. We define the strategy λ' from (q', y') by $\lambda'(\varrho') = \lambda((q_0, y_0) \xrightarrow{\tau,a} \cdot\varrho')$ if ϱ' is a run starting in (q', y'). As λ is an $(n+1)$-bounded winning strategy from (q_0, y_0), λ' is an n-bounded winning strategy from (q', y'). By induction hypothesis, we have that

$$\mathsf{Cost}((q', y'), \lambda') \geq c_n(q', y').$$

It follows that for every $\varepsilon > 0$, there exists a winning run ϱ'_ε compatible with λ' such that $\mathsf{Cost}(\varrho'_\varepsilon) \geq c_n(q', y') - \varepsilon$. The run $\varrho = (q_0, x_0, t_0, y_0) \xrightarrow{\tau,a} \varrho'_\varepsilon$ is a winning run compatible with λ. Thus,

$$\mathsf{Cost}((q_0, y_0), \lambda) \geq \mathsf{Cost}(q_0, x_0, t_0, \tau) + \mathsf{Cost}(\varrho'_\varepsilon) \geq \mathsf{Cost}_{c_n}^{\tau,a}(q_0, x_0, t_0, y_0) - \varepsilon.$$

As this is true for every $\varepsilon > 0$, we get that $\mathsf{Cost}((q_0, y_0), \lambda) \geq \mathsf{Cost}_{c_n}^{\tau,a}(q_0, x_0, t_0, y_0)$.

We can do the same reasoning for every $(\tau', u) \in \mathsf{Enb}(q_0, x_0, t_0, y_0)$ such that $\tau' \leq \tau$, and we similarly get that $\mathsf{Cost}((q_0, y_0), \lambda) \geq \mathsf{Cost}_{c_n}^{\tau',u}(q_0, x_0, t_0, y_0))$.

Taking the maximum of the two expressions, we get the expected inequality (1). □

Theorem 2. *Let $\mathcal{M} = \langle M, +, 0, 1, \ldots \rangle$ be an o-minimal structure such that $\mathsf{Th}(\mathcal{M})$ is decidable. The optimal cost is computable over weighted $\mathcal{M}$-games. Moreover, the cost-optimal control problem over $\mathcal{M}$-games is decidable.*

Proof. Let $\mathcal{H}$ be a weighted $\mathcal{M}$-game, and s a state of $\mathcal{H}$. Lemmas 3 and 4 imply that for every state s of $\mathcal{H}$:

$$c_k(s) = \inf \big\{ \mathsf{Cost}(s, \lambda) \ \mid \ \lambda \text{ is a } k\text{-bounded winning strategy} \big\}.$$

Let n be the number of transitions of $\mathcal{H}$. Proposition 1 shows that for every winning strategy from s there is an $(n+1)$-bounded winning strategy from s with a smaller cost. Thus, for every state s of $\mathcal{H}$,

$$\begin{aligned} \mathsf{OptCost}(s) &= \inf \big\{ \mathsf{Cost}(s, \lambda) \ \mid \ \lambda \text{ is an } (n+1)\text{-bounded winning strategy} \big\} \\ &= c_{n+1}(s). \end{aligned}$$

Note that $c_{n+1}(s)$ is computable since $\mathsf{Th}(\mathcal{M})$ is decidable.

We can moreover decide if the optimal cost can be achieved by a strategy: by Proposition 1 it is sufficient to enumerate all $(n+1)$-bounded strategies using τ_i's as parameters and check if the cost of one of them is equal to $c_{n+1}(s)$. Thus, the cost-optimal control problem is decidable over $\mathcal{M}$-games. □

Remark 8. Note that Theorem 2 encompasses the decidability of the time-bounded reachability problem considered in [Gen05], because we have a computability and decidability result in a larger framework and for two players.

Remark 9. Note that there exist examples of weighted o-minimal games where (i) there do not exists optimal strategies or (ii) the optimal cost can only be achieved with memory. The same holds for weighted timed automata with one clock studied in [BLMR06]. The following examples are directly inspired from those given in [BLMR06]. The game in Fig. 13 has optimal cost 0, but no strategy can achieve this cost (c is a controllable action). The game in Fig. 14 has optimal cost 2. This value can be achieved, but memory is needed to achieve that value.

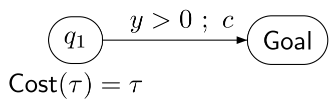


Figure 13: No optimal stategy

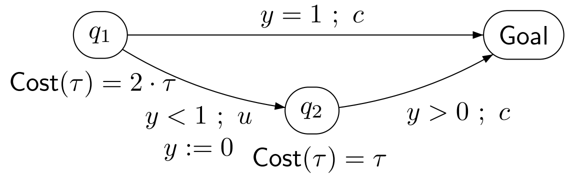


Figure 14: Optimal cost needs memory

5. Solving the WCTL Model-Checking Problem

The aim of this section is to prove the decidability of the WCTL model-checking problem. Techniques that we will develop for proving this result are partly inspired by the recent decidability proof for WCTL over one-clock weighted timed automata [BLM07].

Theorem 3. *Let $\mathcal{M} = \langle M, +, 0, 1, <, \ldots\rangle$ be an o-minimal structure such that $\mathcal{M}$ is Archimedean*[12] *and $\mathsf{Th}(\mathcal{M})$ is decidable. The model-checking of $\mathsf{WCTL}_{\mathcal{M}}$ over $\mathcal{M}$-weighted hybrid systems is decidable.*

If $\mathcal{P}$ and $\mathcal{P}'$ are two partitions of $Q \times V_2$, we write $\mathcal{P} \sqcap \mathcal{P}'$ for the joint partition,*i.e.* the smallest partition that refines both $\mathcal{P}$ and $\mathcal{P}'$. Note that if $\mathcal{P}$ and $\mathcal{P}'$ are both finite and definable, the joint partition $\mathcal{P} \sqcap \mathcal{P}'$ is also finite and definable.

Let $\mathcal{H} = (Q, \Sigma, \delta, \gamma)$ be an $\mathcal{M}$-weighted hybrid system. Let $\mathcal{P}$ be a partition of the state-space $S = Q \times V_2$ and φ a formula of $\mathsf{WCTL}_{\mathcal{M}}$-formula. We say that $\mathcal{P}$ is a *partition for* φ if for all $P \in \mathcal{P}$, either all states $(q, y) \in P$ satisfy φ or all states $(q, y) \in P$ do not satisfy φ.

Let $\mathcal{H} = (Q, \mathsf{Goal}, \Sigma, \delta, \gamma)$ be an $\mathcal{M}$-weighted hybrid system. On each location $q \in Q$, we denote by $\mathcal{P}_q$ the partition induced by the guards and the resets associated with location q. We denote by $\mathcal{P}_{\mathcal{H}}$ the partition of the state-space $S = Q \times V_2$ induced by the $\mathcal{P}_q$'s. $\mathcal{P}_{\mathcal{H}}$ is a finite definable partition of S.

[12]A structure is *Archimedean* whenever for every pair $(a, b) \in M^2$ such that $a > 0$, there exists some integer n such that $n \cdot a > b$.

Two states of the same piece P of $\mathcal{P}_{\mathcal{H}}$ agree on all atomic formulae $a \in \Sigma$. We will now inductively construct for every $\mathsf{WCTL}_{\mathcal{M}}$-formula φ a refined (finite and definable) partition $\mathcal{P}_\varphi$ of $\mathcal{P}_{\mathcal{H}}$ such that two states of a piece of $\mathcal{P}_\varphi$ agree on formula φ. We will proceed in three steps: from partitions for φ and ψ we will successively construct a partition for $\mathsf{E}\,\varphi\mathsf{U}\psi$ and $\mathsf{A}\,\varphi\mathsf{U}\psi$, then for $\mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi$ and finally for $\mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi$.

For the rest of this section, let $\mathcal{M} = \langle M, +, 0, 1, <, \ldots\rangle$ be an o-minimal structure such that $\mathcal{M}$ is Archimedean and $\mathsf{Th}(\mathcal{M})$ is decidable. We let $\mathcal{H} = (Q, \mathsf{Goal}, \Sigma, \delta, \gamma)$ be an $\mathcal{M}$-weighted hybrid system, and φ and ψ be two WCTL formulas. We furthermore assume that we have built two finite and definable partitions $\mathcal{P}_\varphi$ and $\mathcal{P}_\psi$ for φ and ψ respectively. We will now construct partitions for formulas $\mathsf{E}\,\varphi\mathsf{U}\psi$ and $\mathsf{A}\,\varphi\mathsf{U}\psi$ (Subsection 5.1), then for formula $\mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi$ (Subsection 5.2), and finally for $\mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi$ (Subsection 5.3).

5.1. Partition for $\mathsf{E}\,\varphi\mathsf{U}\psi$ and $\mathsf{A}\,\varphi\mathsf{U}\psi$

The first step is achieved using the following lemma, which applies the construction of a finite and definable partition (the so-called suffix-partition) correct w.r.t. bisimulation (see Theorem 1), and thus for CTL-formulas [AHLP00, HMR05].

Lemma 5. *We can compute a definable finite partition for the formulas $\mathsf{E}\,\varphi\mathsf{U}\psi$ and $\mathsf{A}\,\varphi\mathsf{U}\psi$.*

PROOF. $\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi$ is a partition for both the formula φ and the formula ψ. By its definition, the suffix partition $\mathsf{Suf}(\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi)$ refines the partition $\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi$. By using Theorem 1, we know that $\mathsf{Suf}(\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi)$ is a time-abstract bisimulation on $\mathcal{H}$ respecting both $\mathcal{P}_\varphi$ and $\mathcal{P}_\psi$. Thus this is a partition for all CTL formula where φ and ψ are used as atomic proposition (see for instance [AHLP00, HMR05]) and in particular, a partition for $\mathsf{E}\,\varphi\mathsf{U}\psi$ and for $\mathsf{A}\,\varphi\mathsf{U}\psi$. $\square$

5.2. Partition for $\mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi$

The second step, the construction of a (finite and definable) partition for $\mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi$, is more involved. We denote by $\mathcal{P}$ the partition $\mathsf{Suf}(\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi)$. As we have seen, this partition is sufficient for the formula $\mathsf{E}\,\varphi\mathsf{U}\psi$, but not for the additional 'quantitative' constraint which says that the witness for ψ must be within a cost satisfying the constraint $\sim c$. Let q and q' be two locations of $\mathcal{H}$. Let P and P' be two pieces of $\mathcal{P}$. We give formulae which define the set of possible costs of paths from a state (or a piece of $\mathcal{P}$) to another one:

- $\theta^{\varphi\vee\psi}_{P\to P'}(c, (q, y))$ expresses that it is possible to go from some $(q, y) \in P$ to some $(q', y') \in P'$ by a continuous step followed by a discrete action, with cost c, and always satisfying $\varphi \vee \psi$ before arriving in (q', y').
- $\theta^{\varphi\vee\psi}_{P\rightsquigarrow P'}(c, (q, y))$ expresses that it is possible to go from $(q, y) \in P$ to some $(q, y') \in P'$ by a continuous step (and no discrete action), with cost c, and always satisfying $\varphi \vee \psi$ before arriving in (q, y').

$\theta^{\varphi\vee\psi}_{P\to P'}(c,(q,y))$ (resp. $\theta^{\varphi\vee\psi}_{P\rightsquigarrow P''}(c,(q,y))$) can be formally defined as follows:

$$\begin{aligned}\theta^{\varphi\vee\psi}_{P\to P'}(c,(q,y)) \;\equiv\; & \exists x\in V_1,\ \exists t\in V,\ \exists\tau\in M^+, \exists a\in\Sigma,\ \exists q'\in Q,\ \exists y'\in V_2\\ & \gamma_q(x,t)=y \ \wedge\ (q,y)\xrightarrow{\tau,a}(q',y')\ \wedge\ (q,y)\in P\ \wedge\ (q',y')\in P'\ \wedge\\ & \forall\tau'<\tau,\ (q,\gamma_q(x,t+\tau'))\models\varphi\vee\psi\ \ \wedge\ \mathsf{Cost}((q,x,t,y,\tau))=c.\end{aligned}$$

$$\begin{aligned}\theta^{\varphi\vee\psi}_{P\rightsquigarrow P'}(c,(q,y)) \;\equiv\; & \exists x\in V_1,\ \exists t\in V,\ \exists\tau\in M^+,\ \exists y'\in V_2\\ & \gamma_q(x,t)=y \ \wedge\ (q,y)\xrightarrow{\tau}(q,y')\ \wedge\ (q,y)\in P\ \wedge\ (q,y')\in P'\ \wedge\\ & \forall\tau'<\tau,\ (q,\gamma_q(x,t+\tau'))\models\varphi\vee\psi\ \ \wedge\ \mathsf{Cost}((q,x,t,y,\tau))=c.\end{aligned}$$

From the two previous formulas, we define the following definable sets:

$$\begin{aligned}\kappa^{\varphi\vee\psi}_{P\to P'} &= \{c\in M^+ \mid \exists(q,y)\in P \text{ s.t. } \theta^{\varphi\vee\psi}_{P\to P'}(c,(q,y))\}\\ \kappa^{\varphi\vee\psi}_{P\rightsquigarrow P'} &= \{c\in M^+ \mid \exists(q,y)\in P \text{ s.t. } \theta^{\varphi\vee\psi}_{P\rightsquigarrow P'}(c,(q,y))\}\\ \lambda^{\varphi\vee\psi}_{P\to P'}(y) &= \{c\in M^+ \mid \exists q\in Q,\ \theta^{\varphi\vee\psi}_{P\to P'}(c,(q,y))\}\\ \lambda^{\varphi\vee\psi}_{P\rightsquigarrow P'}(y) &= \{c\in M^+ \mid \exists q\in Q,\ \theta^{\varphi\vee\psi}_{P\rightsquigarrow P'}(c,(q,y))\}.\end{aligned}$$

Note that for every piece P, there is a single state $q\in Q$ for which there exists $y\in V_2$ with $(q,y)\in P$.

We now construct a weighted finite graph which abstracts away all dynamical parts of $\mathcal{H}$ and which will be restricted to parts of $\mathcal{H}$ which satisfy formula $\mathsf{E}\,\varphi\mathsf{U}\psi$. Each edge of this graph will be labeled with a weight (indeed a definable set) which will represent the set of costs of all paths in $\mathcal{H}$ witnessing formula $\mathsf{E}\,\varphi\mathsf{U}\psi$. More formally, we construct a *(definable) weighted finite graph* $\mathcal{G}_{\varphi,\psi}=(V,E)$ as follows:

- its set of vertices V is

$$\{P,(P,\mathsf{init}) \mid P\in\mathcal{P}, \text{ and } P\models\varphi\vee\psi\}\cup\{(P,\mathsf{final}) \mid P\in\mathcal{P}, \text{ and } P\models\psi\}^{13}$$

- its set of edges E is

$$\begin{aligned}& \{(P,\mathsf{init})\xrightarrow{\lambda^{\varphi\vee\psi}_{P\to P'}(y)}P'\}\cup\{(P,\mathsf{init})\xrightarrow{\lambda^{\varphi\vee\psi}_{P\rightsquigarrow P''}(y)}(P'',\mathsf{final})\}\\ \cup\ & \{(P,\mathsf{init})\xrightarrow{[0]}(P,\mathsf{final})\}\cup\{P\xrightarrow{\kappa^{\varphi\vee\psi}_{P\to P'}}P'\}\\ \cup\ & \{P\xrightarrow{\kappa^{\varphi\vee\psi}_{P\to P'}}(P',\mathsf{final})\}\cup\{P\xrightarrow{\kappa^{\varphi\vee\psi}_{P\rightsquigarrow P''}}(P'',\mathsf{final})\}.\end{aligned}$$

[13] This is a misuse of notation, but $(q,P)\models\varphi\vee\psi$ means that for all $y\in P$, $(q,y)\models\varphi\vee\psi$, which is equivalent to the property that $(q,y)\models\varphi\vee\psi$ for some $y\in P$, by definition of $\mathcal{P}$.

Let $\xi = (P_0, \mathsf{init}) \xrightarrow{\lambda(y)} P_1 \xrightarrow{\kappa_1} \dots \xrightarrow{\kappa_n} (P_n, \mathsf{final})$ be a finite path of $\mathcal{G}_{\varphi,\psi}$. Then we define $K_\xi(y)$ as the set of all possible costs of the path ξ:

$$K_\xi(y) = \{\lambda(y) + c_2 + \cdots + c_n \mid c_i \in \kappa_i\}.$$

The following propositions show that the graph $\mathcal{G}_{\varphi,\psi}$ can be used to model-check the formula $\mathsf{E}\,\varphi \mathsf{U}_{\sim c} \psi$.

Proposition 2. *Let $\xi = (P_0, \mathit{init}) \xrightarrow{\lambda(y)} P_1 \dots \xrightarrow{\kappa_{n-1}} P_{n-1} \xrightarrow{\kappa_n} (P_n, \mathit{final})$ be a finite path of $\mathcal{G}_{\varphi,\psi}$, and let $(q_0, y_0) \in P_0$. Then, for every $\zeta \in K_\xi(y_0)$, there exists a real path $\varrho = (q_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, y_1) \dots \xrightarrow{\tau_n (,a_n)} (q_n, y_n)$[14] in $\mathcal{H}$ such that $(q_i, y_i) \in P_i$ for every $1 \le i \le n$, $\mathsf{Cost}(\varrho) = \zeta$, for all positions $p < (n, 0)$ along ϱ, $\varrho[p] \models \varphi \vee \psi$, and $(q_n, y_n) \models \psi$. The converse also holds.*

Proof. Let us first consider the particular case of paths of length one. Assume $\xi = (P_0, \mathsf{init}) \xrightarrow{\lambda(y)} (P_1, \mathsf{final})$. In particular, P_0 and P_1 correspond to the same discrete location of $\mathcal{H}$. Two cases have to be distinguished: either $\lambda(y) = [0]$, or $\lambda(y) = \lambda^{\varphi \vee \psi}_{P_0 \rightsquigarrow P_1}(y)$. In the first case, the real path ϱ is given by (q_0, y_0) (or $(q_0, y_0) \xrightarrow{0} (q_0, y_0)$), which clearly satifies the desired properties. In the second situation, the desired real path $(q_0, y_0) \xrightarrow{\tau} (q_1, y_1)$ can easily be obtained using the definition of $\lambda^{\varphi \vee \psi}_{P_0 \rightsquigarrow P_1}(y)$.

Let us now consider the general case. Let ξ be the following path:

$$\xi = (P_0, \mathsf{init}) \xrightarrow{\lambda(y)} P_1 \dots \xrightarrow{\kappa_{n-1}} P_{n-1} \xrightarrow{\kappa_n} (P_n, \mathsf{final})$$

and let ζ be a point in $K_\xi(y_0)$. Since $\zeta \in K_\xi(y_0)$, there exists $c_2, \dots, c_n \in M^+$ such that $c_i \in \kappa_i$ and $\zeta = \lambda(y_0) + c_2 + \cdots + c_n$.

Since $(P_0, \mathsf{init}) \xrightarrow{\lambda(y_0)} P_1$, there exists a time $\tau_1 \in M^+$ and $(q_1, y_1) \in P_1$ such that $(q_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, y_1)$, and $\mathsf{Cost}((q_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, y_1)) = \lambda(y_0)$. Moreover, due to the strong-reset assumption made on $\mathcal{H}$, the above property holds for every $(q_1, y_1) \in P_1$. Let us furthermore notice that $\lambda(y_0)$ ensures that every intermediate position leading to P_1 satisfies $\varphi \vee \psi$.

Then, since $P_1 \xrightarrow{\kappa_2} P_2$ and $c_2 \in \kappa_2$, there exists a point $(q_1, y_1) \in P_1$ and a time τ_2 such that for every $(q_2, y_2) \in P_2$ (we can use a universal quantification here, again due to the strong-reset assumption), $(q_1, y_1) \xrightarrow{\tau_2, a_2} (q_2, y_2)$, and $\mathsf{Cost}((q_1, y_1) \xrightarrow{\tau_2, a_2} (q_2, y_2)) = c_2$. Again, all intermediate positions leading to P_2 are guaranteed to satisfy $\varphi \vee \psi$.

Until now, for every $(q_2, y_2) \in P_2$, we have built a path $(q_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, y_1) \xrightarrow{\tau_2, a_2} (q_2, y_2)$ whose cost is $\lambda(y_0) + c_2$. Iterating the same construction, for every $(q_{n-1}, y_{n-1}) \in P_{n-1}$, we obtain a real path $(q_0, y_0) \xrightarrow{\tau_1, a_1} (q_1, y_1) \xrightarrow{\tau_2, a_2}$

[14]This means that the last move can either be a delay move (thus without the action a_n), or a normal combined move.

$\dots \xrightarrow{\tau_{n-1},a_{n-1}} (q_{n-1},y_{n-1})$ where all the positions before (q_{n-1},y_{n-1}) satisfy $\varphi \vee \psi$. It remains to consider the final transition $P_{n-1} \xrightarrow{\kappa_n} (P_n, \mathsf{final})$. For this final transition, we distinguish between two cases: either κ_n is of the form $\kappa^{\varphi\vee\psi}_{P\to P'}$ or it is of the form $\kappa^{\varphi\vee\psi}_{P\rightsquigarrow P'}$. In the first case, since $P_{n-1} \xrightarrow{\kappa_n} (P_n, \mathsf{final})$ and $c_n \in \kappa_n$, by definition of $\kappa_n = \kappa^{\varphi\vee\psi}_{P_{n-1}\to P_n}$, one can find $(q_{n-1},y_{n-1}) \in P_{n-1}$, $(q_n,y_n) \in P_n$ and a time τ_n such that $(q_{n-1},y_{n-1}) \xrightarrow{\tau_n,a_n} (q_n,y_n)$, $\mathsf{Cost}((q_{n-1},y_{n-1}) \xrightarrow{\tau_n,a_n} (q_n,y_n)) = c_n$, (q_n,y_n) satifies ψ, and all intermediate positions leading to (q_n,y_n) satisfy $\varphi \vee \psi$. In the second case, by using the definition of $\kappa_n = \kappa^{\varphi\vee\psi}_{P_{n-1}\rightsquigarrow P_n}$, one can build a transition $(q_{n-1},y_{n-1}) \xrightarrow{\tau_n} (q_n,y_n)$, with the desired properties. This concludes the proof of the first implication.

The converse of the proposition is proved using similar arguments. □

The previous proposition immediately implies the following proposition.

Proposition 3. *Let $P_0 \in \mathcal{P}$ and $(q_0,y_0) \in P_0$. Then $(q_0,y_0) \models \mathsf{E}\,\varphi \mathsf{U}_{\sim c}\psi$ iff there exists a path $\xi = (P_0, \mathit{init}) \xrightarrow{\lambda(y)} P_1 \xrightarrow{\kappa_2} \dots \xrightarrow{\kappa_n} (P_n, \mathit{final})$ in $\mathcal{G}_{\varphi,\psi}$, there exists $\zeta \in M$ such that $\zeta \in K_\xi(y_0)$ and $\zeta \sim c$.*

We prove that, under the Archimedian hypothesis, we can bound the length of witnessing paths in the graph $\mathcal{G}_{\varphi,\psi}$. This hypothesis has not been used yet, as everything holds without it, but it is required by the next proposition. For every $(m,c) \in M^2$ such that $m > 0$, we denote $\left[\frac{c}{m}\right]$ the smallest integer k such that $k \cdot m > c$.

Proposition 4. *We assume that $(q_0,y_0) \models \mathsf{E}\,\varphi \mathsf{U}_{\sim c}\psi$, and we define $N = 2 + (|\mathcal{P}|+1)\cdot\left(\left[\frac{c}{m}\right]+2\right)$ where m is the smallest positive constant defining an interval*[15] *of some weight κ labelling the transitions of $\mathcal{G}_{\varphi,\psi}$. Then there exists a path $\xi = (P_0, \mathit{init}) \xrightarrow{\lambda(y)} P_1 \xrightarrow{\kappa_2} \dots \xrightarrow{\kappa_n} (P_n, \mathit{final})$ in $\mathcal{G}_{\varphi,\psi}$ such that $(q_0,y_0) \in P_0$, $n \le N$, and there exists $\zeta \in M$ such that $\zeta \in K_\xi(y_0)$ and $\zeta \sim c$.*

Proof. First of all, we introduce the notion of *realization* of a run. Let $\xi = (P_0, \mathsf{init}) \xrightarrow{\lambda(y)} P_1 \xrightarrow{\kappa_2} \dots \xrightarrow{\kappa_n} (P_n, \mathsf{final})$ be a path in $\mathcal{G}_{\varphi,\psi}$. Each value κ_i is definable, hence is a finite union of points and open intervals (as the underlying structure is o-minimal). Similarly, if we fix a definable value y_0, $\lambda(y_0)$ is also a finite union of points and open intervals. A *realization* of ξ is a choice, for each step along ξ, of an interval (or point) of the cost labelling the transition (we write $\mathcal{I}$ for the choice function and $\xi[\mathcal{I}]$ for the realization). A *reasonable realization* of ξ is such that no cycle is labeled only by the point $\{0\}$. Any non-reasonable realization is not interesting as all $\{0\}$-cycles can be removed. Finally, we will say that a realization $\varrho[\mathcal{I}]$ contains $\zeta \in M^+$ whenever ζ belongs to the sum of all intervals labelling the edges of the realization $\varrho[\mathcal{I}]$.

[15] Note that labels of edges are definable and hence correspond to a finite union of points and open intervals. For the definition of m, we ignore the first steps labelled with functions $\lambda(y)$.

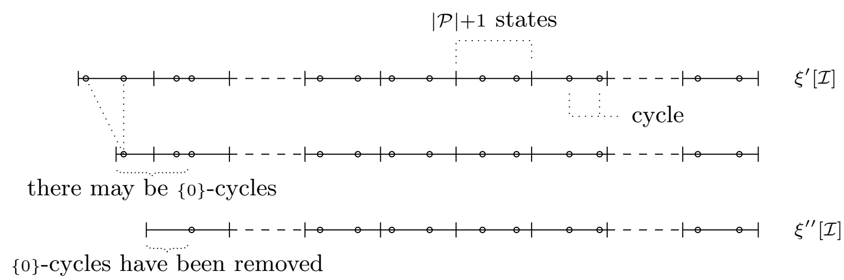


Figure 15: Construction of the witness

We now prove the proposition. Let $P_0 \in \mathcal{P}$, $(q_0, y_0) \in P_0$, and suppose that $(q_0, y_0) \models \mathsf{E}\,\varphi \mathsf{U}_{\sim c}\psi$. By Proposition 3, there exists a run $\xi' = (P'_0, \mathsf{init}) \xrightarrow{\lambda'(y)} P'_1 \xrightarrow{\kappa'_1} \dots \xrightarrow{\kappa'_k} (P'_k, \mathsf{final})$ in $\mathcal{G}_{\varphi,\psi}$, there exists $\zeta' \in M$ such that $\zeta' \in K_{\xi'}(y_0)$ and $\zeta' \sim c$. From ξ', we will construct a run $\xi = (P_0, \mathsf{init}) \xrightarrow{\lambda(y)} P_1 \xrightarrow{\kappa_1} \dots \xrightarrow{\kappa_n} (P_n, \mathsf{final})$ with $n \leq N$, $P_0 = P'_0$ and $P_n = P'_k$ such that there exists $\zeta \in M$ with $\zeta \in K_\xi(y_0)$ and $\zeta \sim c$.

We suppose that there exists a reasonable realization of ξ' which contains ζ': if it is not the case, there are cycles in ξ' in which all κ'_i's are reduced to $\{0\}$. We remove all these cycles, obtaining a run ξ'_1 which has the same accumulated cost as ξ', and we complete the proof with ξ'_1 instead of ξ.

If the length of $\xi'[\mathcal{I}]$ is shorter than N then we are done. Otherwise ξ' contains at least $\left(\left[\frac{c}{m}\right] + 2\right)$ simple cycles[16] (see Fig. 15), each of which is labeled by an (accumulated) interval whose right bound is larger than, or equal to m. Thus, $K_{\xi'[\mathcal{I}]}(y_0) = \langle a, b\rangle$ [17] with $b > c + 2m$ (because $\left[\frac{c}{m}\right] \cdot m > c$). By hypothesis, there exists $\zeta' \in \langle a, b\rangle$ such that $\zeta' \sim c$.

We then remove the first cycle of the run and remove [0]-cycles that can have been potentially created (see Fig. 15). We obtain $\xi''[\mathcal{I}]$, a realization of a strictly smaller run. Note that $K_{\xi''[\mathcal{I}]}(y_0)$ might be different from $K_{\xi[\mathcal{I}]}(y_0)$; indeed as we removed some cycles, $K_{\xi''[\mathcal{I}]}(y_0) = \langle a', b'\rangle$ with $a' \leq a$ and $b' \leq b$. Nevertheless $\xi''[\mathcal{I}]$ still has at least $\left[\frac{c}{m}\right]$ simple cycles so $b' > c$ (as previously). As noted, there exists $\zeta' \in \langle a, b\rangle$ such that $\zeta' \sim c$, hence there also exists $\zeta'' \in K_{\xi''[\mathcal{I}]}(y_0)$ with $\zeta'' \sim c$. We iterate this process until we find a run ξ of length no more than N satisfying this property. □

Applying the previous proposition, we can build a first-order formula which checks if a given state (q, y) satisfies the WCTL formula $\mathsf{E}\,\varphi \mathsf{U}_{\sim c}\psi$, and we thus get the following corollary.

[16] The first term 2 in the definition of N is for removing the initial and final transitions, respectively starting in some (q, P, init) and leading to some (q', P', final).

[17] $\langle a, b\rangle$ stands for either $[a, b]$ or $[a, b)$ or $(a, b]$ or (a, b).

Corollary 1. *We can compute a definable finite partition for the formula* $\mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi$*.*

Proof. From Propositions 3 and 4, one can easily deduce that a state (q,y) of $\mathcal{H}$ satisfies the formula $\mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi$ if and only if there exists a real path of length at most N, starting from (q,y), which satisfies the formula $\varphi\mathsf{U}_{\sim c}\psi$.

Given $n \in \mathbb{N}$, it is possible to write a first-order formula $\alpha_n(\cdot)$ such that $\mathcal{M} \models \alpha_n(q,y)$ if and only if there exists a real path of length exactly n, starting from (q,y), which satisfies the formula $\varphi\mathsf{U}_{\sim c}\psi$. We thus obtain the following equivalence:

$$(q,y) \models \mathsf{E}\,\varphi\mathsf{U}_{\sim c}\psi \;\text{ if and only if }\; \mathcal{M} \models \bigvee_{n=0}^{N} \alpha_n(q,y).$$

The desired partition is the one obtained by refining the finite and definable partition $\mathrm{Suf}(\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi)$ with the formula $\bigvee_{n=0}^{N} \alpha_n(q,y)$. This new partition is clearly definable and contains at most $2 \cdot |\mathrm{Suf}(\mathcal{P}_\varphi \sqcap \mathcal{P}_\psi)|$ pieces. □

5.3. Partition for $\mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi$

In this section we explain how to construct a partition for the formula $\mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi$.

In the sequel, it will be useful to isolate (in a definable way) the points from where there is an infinite path of cost always smaller than a definable constant c; we thus state the following lemma.

Lemma 6. *Let c a definable constant. The set of states (q,y) such that there is an infinite run ϱ starting from (q,y) whose cost $\mathsf{Cost}(\varrho)$ satisfies $\mathsf{Cost}(\varrho) \leq c$ (resp. $\mathsf{Cost}(\varrho) < c$) is definable.*[18] *Moreover, if such a run does not exist, there is a computable index n_0, not depending on (q,y), such that every run ϱ of length greater than n_0 is such that $\mathsf{Cost}(\varrho) > c$ (resp. $\mathsf{Cost}(\varrho) \geq c$).*

Proof. We will consider the case '$< c$.' The case '$\leq c$' being a slight extension. We first characterize the set of states from which there is an infinite run ϱ such that $\mathsf{Cost}(\varrho) < c$, and then use it to prove the lemma.

We consider the graph $\mathcal{G}_{\top,\top}$ (simply denoted $\mathcal{G}$ in the following part of the proof) constructed in Subsection 5.2, where $\varphi = \psi = \top$. We call $\langle 0,\cdot\rangle$-cycle a cycle of $\mathcal{G}$ where all labels of edges either contains 0, or contains an interval whose left bound is 0.

We will first prove the following characterization:

Lemma 7. *There is an infinite run ϱ from (q,y) such that $\mathsf{Cost}(\varrho) < c$ iff there exist*

$$\begin{array}{ll} & \xi = (P_0, \mathit{init}) \xrightarrow{\lambda(y)} P_1 \ldots \xrightarrow{\kappa_{n-1}} P_{n-1} \xrightarrow{\kappa_n} P_n \\ & \textit{a run of } \mathcal{G} \textit{ with } (q,y) \in P_0, \textit{ and } K_\xi(y) \cap [0,c) \neq \emptyset \\ \textit{and} & \xi' = P'_0 \xrightarrow{\kappa'_1} P'_1 \ldots \xrightarrow{\kappa'_{k-1}} P'_{k-1} \xrightarrow{\kappa'_k} P'_k \\ & \textit{a } \langle 0,\cdot\rangle\textit{-cycle with } P'_0 = P'_k = P_n. \end{array} \tag{2}$$

[18] If ϱ is an infinite run, $\mathsf{Cost}(\varrho) = \lim_{n\to+\infty} \mathsf{Cost}(\varrho_{\leq(n,0)})$.

Proof. Let $m > 0$ be the smallest positive constant defining an interval of some weight κ labelling the transitions of $\mathcal{G}$. We define $n_0 = 1 + (|\mathcal{P}|+1) \cdot \left[\frac{c}{m}\right]$.

We assume that $\varrho = (q_0, y_0) \xrightarrow{\tau_1, a_1} \cdots \xrightarrow{\tau_n, a_n} (q_n, y_n) \dots$ is an infinite run from $(q, y) = (q_0, y_0)$ such that $\mathsf{Cost}(\varrho) < c$. Extending straightforwardly Proposition 2 to infinite runs, there is a corresponding run $\xi = (P_0, \mathsf{init}) \xrightarrow{\lambda(y)} P_1 \dots \xrightarrow{\kappa_n} P_n \dots$ such that for every $n \in \mathbb{N}$, $\mathsf{Cost}(\varrho_{\leq(n,0)}) \in K_{\xi_{\leq n}}(y)$, where $\xi_{\leq n}$ is the prefix of length n of ξ.

In particular, we must have $\mathsf{Cost}(\varrho_{\leq(n_0,0)}) < c$, hence $K_{\xi_{\leq n_0}}(y) \cap [0, c) \neq \emptyset$. We assume that there is no $\langle 0, \cdot\rangle$-cycle in $\xi_{\leq n_0}$. This means that along each cycle of $\xi_{\leq n_0}$, the cost has increased by at least m. As there are at least $\left[\frac{c}{m}\right]$ cycles along $\xi_{\leq n_0}$, it means that any cost of a run read along $\xi_{\leq n_0}$ is at least $\left[\frac{c}{m}\right] \cdot m > c$. This contradicts the assumption that $\mathsf{Cost}(\varrho_{\leq(n_0,0)}) < c$. Hence, there is at least a $\langle 0, \cdot\rangle$-cycle in $\xi_{\leq n_0}$. Moreover, the cost of the prefix of $\varrho_{\leq(n_0,0)}$ leading to that $\langle 0, \cdot\rangle$-cycle must contain a value strictly smaller than c.

Let ξ and ξ' be as in (2). By the definition of ξ and Proposition 2, there exists $\varrho = (q_0, y_0) \xrightarrow{\tau_1, a_1} \dots \xrightarrow{\tau_n, a_n} (q_n, y_n)$ such that $(q_i, y_i) \in P_i$ for every i, and $\mathsf{Cost}(\varrho) < c$. Moreover, due to the strong-reset assumption, the choice of y_n does not really matter for the global cost (like in the proof of Proposition 2).

By the definition of ξ' and Proposition 2, for every $\varepsilon > 0$, there exists $\varrho'_\varepsilon = (q'_0, y'_0) \xrightarrow{\tau'_1, a'_1} \cdots \xrightarrow{\tau'_k, a'_k} (q'_k, y'_k)$ such that $(q'_i, y'_i) \in P'_i$ for every i, and $\mathsf{Cost}(\varrho'_\varepsilon) < \varepsilon$. Moreover, the choice of y'_k does not matter (this will allow us to glue together those runs).

Let $\varepsilon' = c - \mathsf{Cost}(\varrho)$. Concatenating ϱ and $\varrho_{\frac{\varepsilon'}{2^{n+1}}}$ for every $n \geq 1$, we get an infinite run $\widetilde{\varrho}$ from $(q, y) = (q_0, y_0)$ such that $\mathsf{Cost}(\widetilde{\varrho}) < c$.

This concludes the proof of the characterization (2). □

Given c a definable constant, we want to prove that the following set is definable:

$$\mathsf{Cost}_{<c} = \{(q, y) \mid \text{there exists } \varrho \text{ starting from } (q, y) \text{ such that } \mathsf{Cost}(\varrho) < c\}.$$

The previous lemma allows us to rely on the the finite graph $\mathcal{G}$ in order to check whether a given point (q, y) belongs to $\mathsf{Cost}_{<c}$. In order to prove the definability of $\mathsf{Cost}_{<c}$, we will show that if $(q, y) \notin \mathsf{Cost}_{<c}$ then every run ϱ of length greater than n_0 (as defined in the previous lemma) is such that $\mathsf{Cost}(\varrho) > c$. Note that n_0 is clearly computable.

Assume $(q, y) \notin \mathsf{Cost}_{<c}$ and let ϱ be a run starting from (q, y) of length larger than n_0. Let ξ be the corresponding run in $\mathcal{G}$ obtained by Proposition 2. Either ξ contains a $\langle 0, \cdot\rangle$-cycle, in which case it must be the case that $K_\xi \subseteq [c, +\infty)$ (otherwise, by (2), there would exist an infinite run of cost strictly less than c, contradicting that $(q, y) \notin \mathsf{Cost}_{<c}$), or ξ does not contain a $\langle 0, \cdot\rangle$-cycle, in which case, it is easy to prove, as in previous proofs, that $K_\xi \subseteq \left[\left[\frac{c}{m}\right] \cdot m, +\infty\right) \subseteq (c, +\infty)$ (because there are at least $\left[\frac{c}{m}\right]$ segments whose cost increases by at least m).

Using (2) and the uniform bound n_0 independent of (q, y), we can easily provide a first-order formula $\theta(q, y)$ expressing the existence of an infinite run ϱ from (q, y) such that $\mathsf{Cost}(\varrho) < c$. $\square$

It is equivalent to build a partition for $\mathsf{A}\, \varphi \mathsf{U}_{\sim c} \psi$, or for its negation. We first explain the simple case of $\mathsf{E}\,\mathsf{G}_{\sim c}\varphi$ (negation of $\mathsf{A}\,\mathsf{F}_{\sim c}\neg\varphi$), which will be useful for the general case.

Lemma 8. *We can compute a definable finite partition for the formula* $\mathsf{E}\,\mathsf{G}_{\sim c}\varphi$*.*

Proof. *Case of formula* $\mathsf{E}\,\mathsf{G}_{>c}\varphi$*.* Let n_0 be as in Lemma 6. We will use the following characterization:

$$(q, y) \models \mathsf{E}\,\mathsf{G}_{>c}\varphi \quad \text{iff} \\ \begin{cases} \text{either there is an infinite run } \varrho \text{ from } (q, y) \text{ s.t. } \mathsf{Cost}(\varrho) \leq c \\ \quad \text{or there is a run } \varrho \text{ from } (q, y) \text{ of length } n_0 + |\mathcal{P}| + 1 \\ \quad\quad \text{s.t. for all position } p,\ \mathsf{Cost}(\varrho_{\leq p}) > c \text{ implies } \varrho[p] \models \varphi. \end{cases} \tag{3}$$

The left-to-right implication is obvious.

It is also obvious that if there is an infinite run ϱ from (q, y) such that $\mathsf{Cost}(\varrho) \leq c$, then $(q, y) \models \mathsf{E}\,\mathsf{G}_{>c}\varphi$.

Assume now that there is no such infinite run, and assume that there is a finite run ϱ from (q, y) of length $n_0 + |\mathcal{P}| + 1$ such that $\varrho \models \mathsf{G}_{>c}\varphi$. We have to show that $(q, y) \models \mathsf{E}\,\mathsf{G}_{>c}\varphi$. For every i, we write P_i for the piece containing (q_i, y_i). Applying Lemma 6, we have that $\mathsf{Cost}(\varrho_{\leq (n_0, 0)}) > c$. Hence, for every position p after $(n_0, 0)$, we have that $\varrho[p] \models \varphi$ (because the cost is non-negative and time-non-decreasing, hence along a given run, the accumulated cost cannot decrease). We can find two indices $n_0 < i_1 < i_2 \leq n_0 + |\mathcal{P}| + 1$ such that $P_{i_1} = P_{i_2}$. We can thus repeat the part between $\varrho[(i_1, 0)]$ and $\varrho[(i_2, 0)]$ infinitely often, and build an infinite path which will satisfy the expected property $\mathsf{G}_{>c}\varphi$. This concludes the proof of the characterization (3).

Lemma 6 and characterization (3) show that there is a first-order formula to define the set of states (q, y) which satisfy $\mathsf{E}\,\mathsf{G}_{>c}\varphi$, hence we can define a finite partition for the formula $\mathsf{E}\,\mathsf{G}_{>c}\varphi$, which is finite (same argument as in Proposition 3).

The cases of other formulas can be done using a very similar reasoning, we thus omit the details. $\square$

Remark 10. In the proof of Lemma 8, we strongly use the hypothesis that the cost function is non-negative and time-non-decreasing. In particular, the characterization (3) only holds under that assumption. We do not know if Lemma 8 holds without those hypotheses.

We are now ready to construct a partition for the formula $\mathsf{A}\, \varphi \mathsf{U}_{\sim c} \psi$. In fact, we will consider the formula $\neg\mathsf{A}\, \varphi \mathsf{U}_{\sim c} \psi$, whose partition is the same. We will decompose $\neg \mathsf{A}\,\varphi \mathsf{U}_{\sim c}\psi$ into three path predicates and show that these predicates admit witnesses of finite (computable) length, which will prove the existence and definability of the partition $\mathcal{P}_{\neg \mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi}$.

Definition 16. A *path predicate* Q is a function $Q : \mathsf{Runs}(\mathcal{H}) \to \{\top, \bot\}$. We write $\varrho \models Q$ when $Q(\varrho) = \top$.

We define the two following predicates which will be used to characterize the negation of $\varphi \mathsf{U}_{\sim c}\psi$:

- $\varrho \models Q_1^{\sim c}$ iff $\varrho \models \mathsf{G}_{\sim c}\neg\psi$;
- $\varrho \models Q_2^{\sim c}$ iff there is a position p such that $\varrho[p] \models \neg\varphi \wedge \neg\psi$, and for every position $p' \leq p$, $\mathsf{Cost}(\varrho_{\leq p'}) \sim c$ implies and $\varrho[p'] \models \neg\psi$.

For $i \in \{1, 2\}$, we say that $(q, y) \models \varphi_{Q_i^{\sim c}}$ if there is an infinite run ϱ from (q, y) such that $\varrho \models Q_i^{\sim c}$. Note that $\varphi_{Q_i^{\sim c}}$ is not necessarily a WCTL formula, it is just a Boolean proposition depending on (q, y).

Definition 17. Let Q be a path predicate. We say that it has *witnesses* of length $n \in \mathbb{N}$ from (q, y) whenever the existence of an infinite run ϱ from (q, y) such that $\varrho \models Q$ is equivalent to the existence of a finite run from (q, y), of length no more than n, and such that $\varrho \models Q$.

Remark 11. If Q is definable and admits witnesses of finite computable length, then we can compute a definable finite partition for the property $(q, y) \models Q$. Indeed, we can construct a first-order formula enumerating all the witnesses as in the proof of Corollary 1.

Proposition 5. *$(q, y) \models \neg(\mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi)$ iff $(q, y) \models \varphi_{Q_1^{\sim c}}$ or $(q, y) \models \varphi_{Q_2^{\sim c}}$.*

Proof. $(q, y) \models \neg\mathsf{A}\,\varphi\mathsf{U}_{\sim c}\psi$ means that there exists an infinite run ϱ from (q, y) such that $\varrho \not\models \varphi\mathsf{U}_{\sim c}\psi$. Hence, the proposition will follow if we prove that:

$$\varrho \not\models \varphi\mathsf{U}_{\sim c}\psi \quad \text{iff} \quad \varrho \models Q_1^{\sim c} \text{ or } \varrho \models Q_2^{\sim c}. \tag{4}$$

We first remark that $\varrho \not\models \varphi\mathsf{U}_{\sim c}\psi$ is equivalent to

$$\forall p, (\varrho[p] \models \psi \text{ and } \mathsf{Cost}(\varrho_{\leq p}) \sim c) \;\Rightarrow \exists p^{\text{wit}} < p \text{ s.t. } \varrho[p^{\text{wit}}] \models \neg\varphi \wedge \neg\psi. \tag{5}$$

In other words, not satisfying formula $\varphi\mathsf{U}_{\sim c}\psi$ means that for every position p where ψ holds and $\mathsf{Cost}(\varrho_{\leq p}) \sim c$, there is a witness for $\neg\varphi \wedge \neg\psi$ at a position $p^{\text{wit}} < p$.

We now prove (4) using (5). We first assume that $\varrho \models Q_i^{\sim c}$ for some $i \in \{1, 2\}$.

- If $i = 1$, it means that $\varrho \models \mathsf{G}_{\sim c}\neg\psi$. Thus, it is never the case that $\varrho[p] \models \psi$ and $\mathsf{Cost}(\varrho_p) \sim c$. Thus, by (5), $\varrho \not\models \varphi\mathsf{U}_{\sim c}\psi$.

- If $i = 2$, it means that there is a position p such that $\varrho[p] \models \neg\varphi \wedge \neg\psi$, and for every position $p' \leq p$, $\mathsf{Cost}(\varrho_{\leq p'}) \sim c$ implies $\varrho[p'] \models \neg\psi$. For every position $p' \leq p$, there is nothing to verify if we want to check implication (5). For a position $p' > p$, position p is a witness for property (5). Thus, $\varrho \not\models \varphi \mathsf{U}_{\sim c} \psi$.

Conversely, we assume that $\varrho \not\models \varphi \mathsf{U}_{\sim c} \psi$. If for every position p, the conditions ($\varrho[p] \models \psi$) and $\mathsf{Cost}(\varrho_{\leq p} \sim c)$ don't hold, then obviously, $\varrho \models Q_1^{\sim c}$.

Otherwise we can define $p_0 = \inf\{p \mid \varrho[p] \models \psi \text{ and } \mathsf{Cost}(\varrho_{\leq p}) \sim c\}$.

- If this infimum is a minimum, that is $\varrho[p_0] \models \psi$ and $\mathsf{Cost}(\varrho_{\leq p_0}) \sim c$, then by hypothesis there is a witness p for $\neg\varphi \wedge \neg\psi$ with $p < p_0$. Then for every position $p' \leq p$, $\mathsf{Cost}(\varrho_{\leq p'} \sim c)$ implies $\varrho[p'] \models \neg\psi$, so $\varrho \models Q_2^{\sim c}$.
- We assume that the infimum is not a minimum and we set $p_0 = (n_0, t_0)$. There are positions (n_0, t) with $t > t_0$ arbitrarily close to t_0 verifying $\varrho[(n_0, t)] \models \psi$ and $\mathsf{Cost}(\varrho_{\leq (n_0,t)}) \sim c$. Moreover, we can define (using a first-order formula) the set $E = \{t > t_0 \mid \varrho[(n_0, t)] \models \psi \text{ and } \mathsf{Cost}(\varrho_{\leq (n_0,t)}) \sim c\}$. By o-minimality of the structure, E is a finite union of points and open intervals. As t_0 is not in E, E must contain an interval of the form (t_0, t') with $t' > t_0$. For every $t \in (t_0, t')$, there exists a position $p < (n_0, t)$ such that $\varrho[p] \models \neg\varphi \wedge \neg\psi$. Hence, that position p must be such that $p \leq p_0$. This position can be used to witness the fact that $\varrho \models Q_2^{\sim c}$.

□

Remark 12. Proposition 5 is, to our knowledge, the first time a characterization of $\neg\mathsf{A}\,\varphi \mathsf{U}_{\sim c} \psi$ has been rigorously proved.

Note that we used the hypothesis of o-minimality. Indeed without this assumption the characterization is not correct: if the truth of ψ can vary infinitely often at the neighbourhood of a point, $\neg\mathsf{A}\,\varphi \mathsf{U}_{\sim c} \psi$ may hold but the characterization not. Consider the run ϱ depicted in Fig. 16: φ holds for $x \leq 1$ and ψ holds at times $x = 1 + \frac{1}{2^n}$ for $n \in \mathbb{N}$. Thus, $\varrho \models \neg(\varphi \mathsf{U} \psi)$, but it does not satisfy the characterization. Note that even the classical LTL-characterization $\neg(\varphi \mathsf{U} \psi) \equiv (\neg\psi \mathsf{U}(\neg\varphi \wedge \neg\psi)) \vee \mathsf{G}\neg\psi$ is not satisfied on this model either.

Proposition 5 is, however, robust as it is verified in most of timed logics in which models cannot vary infinitely often at the neighbourhood of a point.

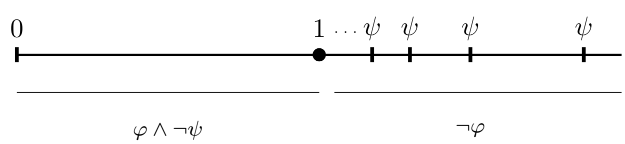


Figure 16: An infinitely varying model

Proposition 6. *We can compute a definable finite partition for the formula* $\mathsf{A}\,\varphi \mathsf{U}_{\sim c} \psi$.

Proof. We have seen that for every state (q, y), $(q, y) \not\models \mathsf{A}\,\varphi \mathsf{U}_{\sim c} \psi$ iff $(q, y) \models \varphi_{Q_1^{\sim c}}$ or $(q, y) \models \varphi_{Q_2^{\sim c}}$. Moreover, $(q, y) \models \varphi_{Q_1^{\sim c}}$ iff $(q, y) \models \mathsf{E}\,\mathsf{G}_{\sim c} \neg\psi$, and Lemma 8 has already explained how to build a partition for that formula.

It remains to build a partition for predicate $\varphi_{Q_2^{\sim c}}$, and for that we will prove that we can compute a bound n such that $\varphi_{Q_2^{\sim c}}$ admits witnesses of length no more than n. As previously stated earlier, this is then sufficient to get a partition for the formula.

We first consider the case of $\varphi_{Q_2^{=c}}$. We define $\alpha^{<c}$ the path predicate such that $\varrho \models \alpha^{<c}$ iff $\mathsf{Cost}(\varrho) < c$. Obviously, if $\varrho \models \alpha^{<c}$, then $\varrho \models \mathsf{F}(\neg\varphi \wedge \neg\psi)$ iff $\varrho \models Q_2^{=c}$. Thus, it is sufficient to prove the following lemma:

Lemma 9. *We assume that* $(q, y) \not\models \varphi_{\alpha^{<c}}$. *Then,* $Q_2^{=c}$ *admits witnesses of finite and definable length from* (q, y).

Proof. We define n_0 as in Lemma 6 and we set $n = 2n_0 + |\mathcal{P}| + 1$. We want to prove that the existence of an infinite run ϱ from (q, y) such that $\varrho \models Q_2^{=c}$ is equivalent to the existence of a finite run ϱ from (q, y), of length no more than n, and such that $\varrho \models Q_2^{=c}$ under the assumption that $(q, y) \not\models \varphi_{\alpha^{<c}}$.

As the hybrid system $\mathcal{H}$ is supposed to be non-blocking, the implication from left-to-right is obvious.

Assume now that $(q, y) \not\models \varphi_{\alpha^{<c}}$ and that there exists an infinite run ϱ from (q, y) such that $\varrho \models Q_2^{=c}$. We know that there is a position p such that $\varrho[p] \models \neg\varphi \wedge \neg\psi$, and for every $p' \leq p$, $\mathsf{Cost}(\varrho_{\leq p'}) \sim c$ implies $\varrho[p'] \models \neg\psi$.

- If $p \leq (n, 0)$, then we are done; the finite run $\varrho_{\leq p}$ is a witness of the right length.

- Assume that $p > (n, 0)$. As $p > (n_0, 0)$, applying Lemma 6 we get that $\mathsf{Cost}(\varrho_{\leq p}) > c$. Applying the same lemma, we get that $p_1 \leq (n_0, 0)$ and also that $p_2 \leq (n_0, 0)$ (because the cost accumulated along the portions up to p_1, and from p_1 to p_2 is repectively equal to c and to 0). Now, the precise value of the cost between p_2 and p has no real importance. Hence, we can pump in that portion (remove all cycles, as we have done in the proof of Lemma 7). Hence, we get a finite run satisfying the expected properties. □

We have seen in Lemma 6 how to build a partition for predicate $\varphi_{\alpha^{<c}}$. Thus, as a consequence, we get a definable finite partition for $\varphi_{Q_2^{=c}}$, hence for $\neg(\mathsf{A}\,\varphi \mathsf{U}_{=c} \psi)$.

We then consider the case of $\varphi_{Q_2^{>c}}$. We do a very similar reasoning, also refining with the predicate $\alpha^{\leq c}$. Obviously, if $\varrho \models \alpha^{\leq c}$, then $\varrho \models \mathsf{F}(\neg\varphi \wedge \neg\psi)$ iff $\varrho \models Q_2^{>c}$. It is then sufficient to analyze the predicate $Q_2^{>c}$ under the assumption that $(q, y) \not\models \varphi_{\alpha^{\leq c}}$. That time, we set $n = n_0 + |\mathcal{P}| + 1$.

If ϱ is an infinite run such that $\varrho \models Q_2^{>c}$, and if p is the witnessing position, then either $p \leq (n, 0)$, or every position p' between $(n_0, 0)$ and p is such that $\mathsf{Cost}(\varrho_{\leq p'}) > c$. Hence, we can cut the cycles which are in between $(n_0, 0)$ and p (as in the proof of Lemma 7), to get a short run witnessing the predicate $Q_2^{>c}$.

The case of $\varphi_{Q_2^{\geq c}}$ *is similar.*

We then consider the case of $\varphi_{Q_2^{<c}}$. It is easy to check that $\varphi_{Q_2^{<c}}$ is equivalent to the WCTL formula

$$\mathsf{E}\,(\neg\psi \mathsf{U}_{<c}(\neg\varphi \wedge \neg\psi)) \vee \mathsf{E}\,(\neg\psi \mathsf{U}_{=c}(\mathsf{E}\,\mathsf{F}(\neg\varphi \wedge \neg\psi))).$$

The case of $\varphi_{Q_2^{\leq c}}$ *is similar.* □

6. Conclusion

In this paper, we have studied two problems: the cost-optimal control reachability problem and the WCTL model-checking problem. We have proved that both problems are decidable when considered on subclasses of $\mathcal{M}$-weighted hybrid games (resp. systems). It is worth recalling that both problems are undecidable when considered on timed automata.

There are several interesting directions for further research: we would like to relax the assumptions on the cost function in order to study decreasing and/or negative cost functions. Maybe these extensions could lead to undecidability. Also, it would be nice to consider the WCTL model-checking problem on $\mathcal{M}$-weighted hybrid systems where the structure $\mathcal{M}$ is not Archimedian. Other problems can also be considered such as mean-payoff games or model-checking a weighted version of the game logic ATL.

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