Expressive Power and Decidability for Memory Logics

Carlos Areces[1], Diego Figueira[2], Santiago Figueira[3,4], and Sergio Mera[3]⋆

[1] INRIA Nancy Grand Est, France
[2] LSV, ENS Cachan, CNRS, INRIA, France
[3] Departamento de Computación, FCEyN, UBA,Argentina
[4] CONICET, Argentina

Abstract. Taking as inspiration the hybrid logic $\mathcal{HL}(\downarrow)$, we introduce a new family of logics that we call *memory logics*. In this article we present in detail two interesting members of this family defining their formal syntax and semantics. We then introduce a proper notion of bisimulation and investigate their expressive power (in comparison with modal and hybrid logics). We will prove that in terms of expressive power, the memory logics we discuss in this paper are more expressive than orthodox modal logic, but less expressive than $\mathcal{HL}(\downarrow)$. We also establish the undecidability of their satisfiability problems.

1 Memory Logics: Hybrid Logics with a Twist

Hybrid languages have been extensively investigated in the past years. $\mathcal{HL}$, the simplest hybrid language, is usually presented as the basic modal language $\mathcal{K}$ extended with special symbols (called *nominals*) to name individual states in a model. These new symbols are simply a new sort of atomic symbols $\{i, j, k, \ldots\}$ disjoint from the set of standard propositional variables. While they behave syntactically exactly as propositional variables do, their semantic interpretation differ: nominals denote elements in the model, instead of sets of elements. This simple addition already results in increased expressive power. For example the formula $i \wedge \langle r \rangle i$ is true in a state w, only if w is a reflexive point named by the nominal i. As the basic modal language is invariant under unraveling, there is no equivalent modal formula [1].

But as we said above, $\mathcal{HL}$ is just the simplest hybrid language. Once nominals have been added to the language, other natural extensions arise. Having names for states at our disposal we can introduce, for each nominal i, an operator $@_i$ that allows us to jump to the point named by i obtaining the language $\mathcal{HL}(@)$. The formula $@_i\varphi$ (read 'at i, φ') moves the point of evaluation to the state named by i and evaluates φ there. Intuitively, the $@_i$ operators internalize the satisfaction relation '$\models$' into the logical language: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M} \models @_i\varphi$, where i is a nominal naming w. For this reason, these operators are usually called *satisfaction operators*.

⋆ S. Mera is partially supported by a grant of Fundación YPF.

If nominals are names for individual states, why not introduce also binders. We would then be able to write formulas like $\forall i.\langle r\rangle i$, which will be true at a state w if it is related to all states in the domain. The $\forall$ quantifier is very expressive: the satisfiability problem of $\mathcal{HL}(\forall)$ ($\mathcal{HL}$ extended with the universal binder $\forall$) is undecidable [2]. Moreover, $\mathcal{HL}(@,\forall)$ is expressively equivalent to full first-order logic (over the appropriate signature).

From a modal perspective, other binders besides $\forall$ are possible. The $\downarrow$ binder binds nominals to the *current* point of evaluation. In essence, it enables us to create a name for the here-and-now, and refer to it later in the formula. For example, the formula $\downarrow i.\langle r\rangle i$ is true at a state w if and only if it is related to itself. The intuitive reading is quite straightforward: the formula says "call the current state i and check that i is reachable". The logic $\mathcal{HL}(\downarrow)$ is also very expressive but weaker than $\mathcal{HL}(\forall)$. Sadly, its satisfiability problem is also undecidable.

Different binders for hybrid logics have been investigated in detail (see [2]), but in this article we want to take a look at $\downarrow$ from a slightly different perspective: we will consider nominals and $\downarrow$ as ways for storing and retrieving information in the model.

Models as Information Storage. We should note that nominals and $\downarrow$ work nicely together. Whereas $\downarrow i$ stores the current point of evaluation in the nominal i, nominals act as checkpoints enabling us to retrieve stored information by verifying if the current point is named by a given nominal i. To make this point clear, let's define formally the semantics of $\mathcal{HL}(\downarrow)$.

Definition 1. *A hybrid signature $\mathcal{S}$ is a tuple $\langle \text{PROP}, \text{REL}, \text{NOM}\rangle$ where* PROP*,* REL*,* NOM *are mutually disjoint infinite enumerable sets (the sets of propositional symbols, relational symbols and nominals, respectively).*

Formulas of $\mathcal{HL}(\downarrow)$ are defined over a given $\mathcal{S}$ by the following rules

$$\text{FORMS} ::= p \mid i \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle r\rangle\varphi \mid \downarrow i.\varphi,$$

where $p \in \text{PROP}$, $i \in \text{NOM}$, $r \in \text{REL}$ and $\varphi, \varphi_1, \varphi_2 \in \text{FORMS}$. Formulas in which any nominal i appears in the scope of a binder $\downarrow i$ are called sentences*.*

A model for $\mathcal{HL}(\downarrow)$ over a signature $\mathcal{S}$ is a tuple $\langle W, (R_r)_{r\in\text{REL}}, V, g\rangle$ where $\langle W, (R_r)_{r\in\text{REL}}, V\rangle$ is a standard Kripke model (i.e., W is a non empty set, each R_r is a binary relation over W, and V is a valuation), and g is an assignment function from NOM *to W.*

Given a model $\mathcal{M} = \langle W, (R_r)_{r\in\text{REL}}, V, g\rangle$ the semantic conditions for the propositional and modal operators are defined as usual (see [1]), and in addition:

$$\langle W, (R_r)_{r\in\text{REL}}, V, g\rangle, w \models i \text{ iff } g(i) = w$$
$$\langle W, (R_r)_{r\in\text{REL}}, V, g\rangle, w \models \downarrow i.\varphi \text{ iff } \langle W, (R_r)_{r\in\text{REL}}, V, g^i_w\rangle, w \models \varphi$$
where g^i_w is the assignment identical to g except perhaps in that $g^i_w(i) = w$.

We can think that $\downarrow i$ is *modifying* the model (by storing the current point of evaluation into i), and that i is being evaluated in the modified model. We can see the assignment g as a particular type of 'information storage' in our model, and consider $\downarrow$ and i as our way to access this information storage for reading and writing.

But let us take a step back and consider the new picture. When we introduced the $\downarrow$ binder, our main aim was to define a binder which was weaker than the first-order quantifier. We thought of the semantics of $\downarrow$ first, and we suitably adjusted the way we updated the assignment later. But why do we need to restrict ourselves to binders and assignments?

Let us start with a standard Kripke models $\langle W, (R_r)_{r \in \text{REL}}, V \rangle$, and let us consider a very simple addition: just a set $S \subseteq W$. We can, for example, think of S as a set of states that are, for some reason, 'known' to us. Already in this very simple set up we can define the following operators

$$\langle W, (R_r)_{r \in \text{REL}}, V, S \rangle, w \models \textcircled{r}\varphi \text{ iff } \langle W, (R_r)_{r \in \text{REL}}, V, S \cup \{w\} \rangle, w \models \varphi$$
$$\langle W, (R_r)_{r \in \text{REL}}, V, S \rangle, w \models \textcircled{k} \text{ iff } w \in S.$$

As it is clear from the semantic definition, the 'remember' operator $\textcircled{r}$ (a unary modality) just marks the current state as being 'already visited', by storing it in our 'memory' S. On the other hand, the zero-ary operator $\textcircled{k}$ (for 'known') queries S to check if the current state has already been visited.

In this simple language we would have that $\langle W, (R_r)_{r \in \text{REL}}, V, \emptyset \rangle, w \models \textcircled{r}\langle r \rangle\textcircled{k}$ will be true only if w is reflexive. Is this new logic equivalent to $\mathcal{HL}(\downarrow)$? As we will prove in this article, the answer is negative: the new language is less expressive than $\mathcal{HL}(\downarrow)$ but more expressive than $\mathcal{K}$. Intuitively, in the new language we cannot discern between states stored in S, while an assignment g keeps a complete mapping between states and nominals.

Naturally, we can include structures which are richer than a simple set, in our models. Let us consider one example. Let S be now a stack of elements that we will represent as a list that 'grows to the right' (we will denote the act of pushing w in S as $S \cdot w$). Let us define the operators:

$$\langle W, (R_r)_{r \in \text{REL}}, V, S \rangle, w \models \texttt{(push)}\varphi \quad \text{iff} \quad \langle W, (R_r)_{r \in \text{REL}}, V, S \cdot w \rangle, w \models \varphi$$
$$\langle W, (R_r)_{r \in \text{REL}}, V, S \cdot w' \rangle, w \models \texttt{(pop)}\varphi \quad \text{iff} \quad \langle W, (R_r)_{r \in \text{REL}}, V, S \rangle, w \models \varphi$$
$$\langle W, (R_r)_{r \in \text{REL}}, V, [] \rangle, w \models \texttt{(pop)}\varphi \text{ never}$$
$$\langle W, (R_r)_{r \in \text{REL}}, V, S \cdot w' \rangle, w \models \texttt{top} \quad \text{iff} \quad w = w'.$$

We will call this new family of logics *memory logics* ($\mathcal{M}$) and in this article we will focus on $\mathcal{M}(\textcircled{r}, \textcircled{k})$, i.e., the logic $\mathcal{K}$ extended with the operators $\textcircled{r}$ and $\textcircled{k}$ introduced above, and investigate two possible variations.

More generally, our proposal is to take seriously the usual saying that 'modal languages are languages to talk about labeled graphs' but give us the freedom to choose what we want to 'remember' about a given graph and how we are going to store it.

To close this section, we formally define the syntax and semantics of the logics we will investigate in the rest of the article.

Syntax and semantics for $\mathcal{M}(ⓡ, ⓚ)$. Syntactically, we obtain $\mathcal{M}(ⓡ, ⓚ)$ by extending the basic modal language $\mathcal{K}$ with the ⓡ and ⓚ modalities.

Definition 2 (Syntax). *Let* $\text{PROP} = \{p_1, p_2, \ldots\}$ *(the propositional symbols) and* $\text{REL} = \{r_1, r_2, \ldots\}$ *(the relational symbols) be pairwise disjoint, countable infinite sets of symbols. The set* FORMS *of formulas of* $\mathcal{M}(ⓡ, ⓚ)$ *in the signature* $\langle \text{PROP}, \text{REL} \rangle$ *is defined as:*

$$\text{FORMS} ::= p \mid ⓚ \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle r \rangle \varphi \mid ⓡ\varphi,$$

where $p \in \text{PROP}$*,* $r \in \text{REL}$ *and* $\varphi, \varphi_1, \varphi_2 \in \text{FORMS}$*.*

While the syntax of the logics that we will discuss in this article is the same, they differ subtly in their semantics.

Definition 3 (Semantics). *Given a signature* $\mathcal{S} = \langle \text{PROP}, \text{REL} \rangle$*, a model for* $\mathcal{M}(ⓡ, ⓚ)$ *is a tuple* $\langle W, (R_r)_{r \in \text{REL}}, V, S \rangle$*, where* $\langle W, (R_r)_{r \in \text{REL}}, V \rangle$ *is a standard Kripke model and* $S \subseteq W$*. The semantics is defined as:*

$$\begin{aligned}
\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models p &\text{ iff } w \in V(p) \\
\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models \neg\varphi &\text{ iff } \langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \not\models \varphi \\
\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models \varphi \wedge \psi &\text{ iff } \langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models \varphi \\
&\quad\text{ and } \langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models \psi \\
\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models \langle r\rangle\varphi &\text{ iff there is } w' \text{ such that } R_r(w, w') \\
&\quad\text{ and } \langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w' \models \varphi \\
\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models ⓡ\varphi &\text{ iff } \langle W, (R_r)_{r\in\text{REL}}, V, S \cup \{w\}\rangle, w \models \varphi \\
\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle, w \models ⓚ &\text{ iff } w \in S
\end{aligned}$$

In this paper, we will be especially interested in the case where formulas are evaluated in models with no previously 'remembered' states, that is, the case where $S = \emptyset$. We will call $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ the logic that results from restricting the class of models to those with $S = \emptyset$.

2 Bisimulation

Here we will define a proper notion of bisimulation for $\mathcal{M}(ⓡ, ⓚ)$ and $\mathcal{M}_\emptyset(ⓡ, ⓚ)$, and use it to investigate their expressive power. We will use a presentation in terms of Ehrenfeucht games [3], but a relational presentation is also possible.

We start with some notation. Given $\mathcal{M} = \langle W, (R_r)_{r\in\text{REL}}, V, S\rangle$ and states $w_1, \ldots, w_n$, we define $\mathcal{M}[w_1, \ldots, w_n] = \langle W, (R_r)_{r\in\text{REL}}, V, S \cup \{w_1, \ldots, w_n\}\rangle$. The set of propositions that are true at a given state w is defined as $\mathsf{props}(w) = \{p \in \text{PROP} \mid w \in V(p)\}$. Given two models $\mathcal{M} = \langle W, (R_r)_{r\in\text{REL}}, V, S\rangle$ and $\mathcal{M}' = \langle W', (R'_r)_{r\in\text{REL}}, V', S'\rangle$, and states $w \in W$ and $w' \in W'$, we say that they *agree* if $\mathsf{props}(w) = \mathsf{props}(w')$ and $w \in S$ iff $w' \in S'$.

Bisimulation Games for $\mathcal{M}(ⓡ, ⓚ)$. Let $S = \langle \text{PROP}, \text{REL} \rangle$ be a standard modal signature. Let $\mathcal{M}_1 = \langle W_1, (R^1_r)_{r \in \text{REL}}, V_1, S_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, (R^2_r)_{r \in \text{REL}}, V_2, S_2 \rangle$ be models and let $w_1 \in W_1$ and $w_2 \in W_2$ be agreeing states. We define the *Ehrenfeucht game* $E(\mathcal{M}_1, \mathcal{M}_2, w_1, w_2)$ as follows. There are two players called *Spoiler* and *Duplicator*. In a play of the game, the players move alternatively. Spoiler always makes the first move. At every move, Spoiler starts by choosing in which model he will make a move. Let us set $s = 1$ and $d = 2$ in case he chooses $\mathcal{M}_1$; otherwise, let $s = 2$ and $d = 1$. He can then either:

1. Make a *memorizing step*. I.e., he extends S_s to $S_s \cup \{w_s\}$. The game then continues with $E(\mathcal{M}_1[w_1], \mathcal{M}_2[w_2], w_1, w_2)$.
2. Make a *move step*. I.e., he chooses $r \in \text{REL}$, and v_s, an R^s_r-successor of w_s. If w_s has no R^s_r-successors, then Duplicator wins. Duplicator has to chose v_d, an R^d_r-successor of w_d, such that v_s and v_d agree. If there is no such successor, Spoiler wins. Otherwise the game continues with $E(\mathcal{M}_1, \mathcal{M}_2, v_1, v_2)$.

In the case of an infinite game, Duplicator wins. Note that with this definition, exactly one of Spoiler or Duplicator wins each game.

Definition 4 (Bisimulation). *We say that two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar (and we write $\mathcal{M}_1 \underline{\leftrightarrow} \mathcal{M}_2$) when there exist $w_1 \in \mathcal{M}_1$ and $w_2 \in \mathcal{M}_2$ such that they agree and Duplicator has a winning strategy on $E(\mathcal{M}_1, \mathcal{M}_2, w_1, w_2)$. In this case we also say that w_1 and w_2 are bisimilar ($\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$).*

We are now ready to prove that the notion of bisimulation we just introduced is adequate. We will show that formulas of $\mathcal{M}(ⓡ, ⓚ)$ are preserved under bisimulation.

Definition 5 (Logic equivalence). *Given $\mathcal{M}_1, \mathcal{M}_2$ two models, $w_1 \in \mathcal{M}_1$, $w_2 \in \mathcal{M}_2$, we say that w_1 is* equivalent *(for some logic $\mathcal{L}$) to w_2 ($w_1 \leftrightsquigarrow w_2$) if for all φ (in $\mathcal{L}$) we have $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$.*

Theorem 1. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two models, $w_1 \in \mathcal{M}_1, w_2 \in \mathcal{M}_2$. If $w_1 \underline{\leftrightarrow} w_2$ then $w_1 \leftrightsquigarrow w_2$.*

Proof. We prove that if w_1 and w_2 agree and Duplicator has a winning strategy on $E(\mathcal{M}_1, \mathcal{M}_2, w_1, w_2)$ then $\forall \varphi \in \mathcal{M}(ⓡ, ⓚ)$, $\mathcal{M}_1, w_1 \models \varphi$ iff $\mathcal{M}_2, w_2 \models \varphi$. We proceed by induction on φ.

- The propositional and boolean cases are trivial.
- $\varphi = ⓚ$. This case follows from Definition 3 and because w_1 and w_2 agree.
- $\varphi = \langle r \rangle \psi$. This is the standard modal case. Preservation is ensured thanks to the *move steps* in the definition of the game.
- $\varphi = ⓡ\psi$. We prove that $\mathcal{M}_1, w_1 \models ⓡ\psi$ implies $\mathcal{M}_2, w_2 \models ⓡ\psi$. Suppose $\mathcal{M}_1, w_1 \models ⓡ\psi$ then $\mathcal{M}_1[w_1], w_1 \models \psi$. The following claim is clear.

  Claim. Let $\mathcal{M}_1, \mathcal{M}_2$ be two models, $w_1 \in \mathcal{M}_1, w_2 \in \mathcal{M}_2$. If Duplicator has a winning strategy on $E(\mathcal{M}_1, \mathcal{M}_2, w_1, w_2)$ then he has a winning strategy on $E(\mathcal{M}_1[w_1], \mathcal{M}_2[w_2], w_1, w_2)$.

By this claim, Duplicator has a winning strategy on $E(\mathcal{M}_1[w_1], \mathcal{M}_2[w_2], w_1, w_2)$. Applying inductive hypothesis and the fact that $\mathcal{M}_1[w_1], w_1 \models \psi$, we conclude $\mathcal{M}_2[w_2], w_2 \models \psi$ and then $\mathcal{M}_2, w_2 \models \textcircled{r}\psi$. The other direction is identical.

This concludes the proof.

The converse of Theorem 1 holds for image-finite models (i.e., models in which the set of successors of any state in the domain is finite). The proof is exactly the same as for $\mathcal{K}$, as $\textcircled{r}$ and $\textcircled{k}$ do not interact with the accessibility relation [1].

Theorem 2 (Hennessy-Milner Theorem). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two image finite models. Then for every $w_1 \in \mathcal{M}_1$ and $w_2 \in \mathcal{M}_2$, $w_1 \leftrightsquigarrow w_2$ then $w_1 \underline{\leftrightarrow} w_2$.*

Clearly, as Theorems 1 and 2 hold for arbitrary models, the results hold also for $\mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$.

3 Expressivity

In this section we compare the expressive power of memory logics with respect to both the modal and hybrid logics. But comparing the expressive power of these logics poses a complication because, strictly speaking, each of them uses a different class of models. We would like to be able to define a natural mapping between models of each logic, similar to the natural mapping that exists between Kripke models and first-order models [1].

Such a mapping is easy to define in the case of $\mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$: each Kripke model $\langle W, (R_r)_{r \in \mathrm{REL}}, V\rangle$ can be identified with the $\mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$ model $\langle W, (R_r)_{r \in \mathrm{REL}}, V, \emptyset\rangle$. Similarly, for formulas which are sentences, the $\mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$ model $\langle W, (R_r)_{r \in \mathrm{REL}}, V, \emptyset\rangle$ can be identified with the hybrid model $\langle W, (R_r)_{r \in \mathrm{REL}}, V, g\rangle$ (for g arbitrary). As we will discuss below, it is harder to find such a natural way to transform models for the case of $\mathcal{M}(\textcircled{r}, \textcircled{k})$: the most natural way seems to involve a shift in the signature of the language.

Definition 6 ($\mathcal{L} \leq \mathcal{L}'$). *We say that $\mathcal{L}$ is* not more expressive than *$\mathcal{L}'$ (notation $\mathcal{L} \leq \mathcal{L}'$) if it is possible to define a function Tr between formulas of $\mathcal{L}$ and $\mathcal{L}'$ such that for every model $\mathcal{M}$ and every formula φ of $\mathcal{L}$ we have that*

$$\mathcal{M} \models_{\mathcal{L}} \varphi \text{ iff } \mathcal{M} \models_{\mathcal{L}'} \mathsf{Tr}(\varphi).$$

We say that $\mathcal{L}$ is strictly less expressive than $\mathcal{L}'$ (notation $\mathcal{L} < \mathcal{L}'$) if $\mathcal{L} \leq \mathcal{L}'$ but not $\mathcal{L}' \leq \mathcal{L}$.

$\mathcal{K}$ is strictly less expressive than $\mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$. It is easy to see intuitively that $\textcircled{r}$ and $\textcircled{k}$ do bring additional expressive power into the language: with their help we can detect cycles in a given model, while formulas of $\mathcal{K}$ are invariant under unraveling.

Showing that $\mathcal{K} \leq \mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$ is straightforward as $\mathcal{K}$ is a sublanguage of $\mathcal{M}_\emptyset(\textcircled{r}, \textcircled{k})$. Hence, we can take Tr to be the identity function.

Theorem 3. $\mathcal{K} \leq \mathcal{M}_\emptyset(ⓡ, ⓚ)$.

Proving that $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ is strictly more expressive is only slightly harder.

Theorem 4. $\mathcal{K} \neq \mathcal{M}_\emptyset(ⓡ, ⓚ)$

Proof. Let $\mathcal{M}_1 = \langle \{w\}, \{(w, w)\}, \emptyset \rangle$ and $\mathcal{M}_2 = \langle \{u, v\}, \{(u, v), (v, u)\}, \emptyset \rangle$ be two Kripke models. It is known that they are $\mathcal{K}$ bisimilar (see [1]). On the other hand, the equivalent $\mathcal{M}(ⓡ, ⓚ)$ models are distinguishable by $\varphi = ⓡ\langle r \rangle ⓚ$.

$\mathcal{M}_\emptyset(ⓡ, ⓚ)$ *is strictly less expressive than* $\mathcal{HL}(\downarrow)$. We will define a translation that maps formulas of $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ into sentences of $\mathcal{HL}(\downarrow)$. Intuitively, it is clear that we can use $\downarrow$ to simulate ⓡ, but ⓚ does not distinguishes between different memorized states (while nominals binded by $\downarrow$ do distinguish them). We can solve this using disjunction to gather together all previously remembered states.

Theorem 5. $\mathcal{M}_\emptyset(ⓡ, ⓚ) \leq \mathcal{HL}(\downarrow)$.

Proof. See the technical appendix.

Finally we arrive to the most interesting question in this section: as we already mentioned, $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ seems to be weaker than $\mathcal{HL}(\downarrow)$ because it allows us to remember that we have already visited a given state, but we cannot distinguish among different visited states. Indeed, we can prove that $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ is strictly less expressive than $\mathcal{HL}(\downarrow)$, but the proof is slightly involved.

Theorem 6. $\mathcal{M}_\emptyset(ⓡ, ⓚ) \neq \mathcal{HL}(\downarrow)$.

Proof. Let $\mathcal{M}_1 = \langle \omega, R_1, \emptyset, \emptyset \rangle$ and $\mathcal{M}_2 = \langle \omega, R_2, \emptyset, \emptyset \rangle$, where $R_1 = \{(n, m) \mid n \neq m\} \cup \{(0, 0)\}$ and $R_2 = \{(n, m) \mid n \neq m\} \cup \{(0, 0), (1, 1)\}$ (the models are shown in Figure 1, the accessibility relation is the non-reflexive transitive closure of the arrows shown in the picture).

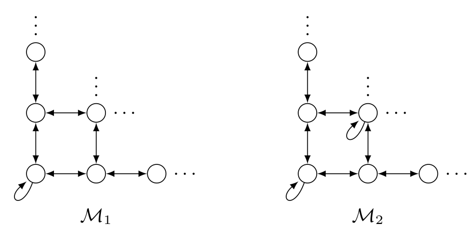

Fig. 1. Two $\mathcal{M}_\emptyset(ⓡ, ⓚ)$-bisimilar models

We prove that $\mathcal{M}_1, 0 \underline{\leftrightarrow} \mathcal{M}_2, 0$ showing the winning strategy for duplicator. Intuitively, the strategy for Duplicator consists in the following idea: whenever one player is in $(\mathcal{M}_1, 0)$ the other will be in $(\mathcal{M}_2, 0)$ or $(\mathcal{M}_2, 1)$, and conversely whenever a player is in $(\mathcal{M}_1, n)$, $n > 0$ the other will be in $(\mathcal{M}_2, m)$, $m > 1$. This is maintained until Spoiler (if ever) decides to remember a state. Once this is done, then any strategy will be a winning one for Duplicator.

Being a bit more formal, the winning strategy will have two stages. While Spoiler does not remember any reflexive state, Duplicator plays with the following strategy: if Spoiler chooses 0 in any model, Duplicator chooses 0 in the other one; if Spoiler chooses $n > 0$ in $\mathcal{M}_1$, Duplicator plays $n + 1$ in $\mathcal{M}_2$; if Spoiler chooses $n > 0$ in $\mathcal{M}_2$, Duplicator plays $n - 1$ in $\mathcal{M}_1$.

Notice that with this strategy Spoiler chooses a reflexive state if and only if Duplicator answers with a reflexive one. This is clearly a winning strategy. If ever Spoiler decides to remember a reflexive state, Duplicator starts using the following strategy: if Spoiler selects a state n, Duplicator answers with an agreeing state m of the opposite model. Notice that this is always possible since both n and m see infinitely many non remembered states and at least one remembered state. Therefore $\mathcal{M}_1, w \underline{\leftrightarrow} \mathcal{M}_2, w$.

On the other hand, let φ be the formula $\downarrow i.\langle r\rangle(i \wedge \langle r\rangle(\neg i \wedge \downarrow i.\langle r\rangle i))$. It is easy to see that $\mathcal{M}_1, w \not\models \varphi$ but $\mathcal{M}_2, w \models \varphi$.

The basic idea behind the previous proof is that if the relations R_1 and R_2 extend the set $\{(n,m) \mid n \neq m\}$, then $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ can distinguish between irreflexive and non irreflexive frames, but it cannot distinguish frames with a different number of reflexive nodes.

There is a number of interesting remarks to be made above the previous proof. First, notice that it is essential for the winning strategy of Duplicator that each state in a model is related to infinitely many others. The question of whether $\mathcal{M}_\emptyset(ⓡ, ⓚ) < \mathcal{HL}(\downarrow)$ on image-finite models is still open. Second, notice that the $\mathcal{HL}(\downarrow)$ sentence that we used in the proof uses only one nominal. Hence, we have actually proved that $\mathcal{HL}_1(\downarrow) \not\leq \mathcal{M}_\emptyset(ⓡ, ⓚ)$, where $\mathcal{HL}_1(\downarrow)$ is $\mathcal{HL}(\downarrow)$ restricted to only one nominal. But actually, it is also the case that $\mathcal{M}_\emptyset(ⓡ, ⓚ) \not\leq \mathcal{HL}_1(\downarrow)$.

Proposition 1. *The logics $\mathcal{HL}_1(\downarrow)$ and $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ are incomparable in terms of expressive power.*

Proof. See technical appendix.

Actually, this incomparability result can be extended to $\mathcal{HL}(\downarrow)$ restricted to any fixed number of nominals, by taking cliques of the appropriate size.

Theorem 7. *For any fixed k, the logics $\mathcal{HL}_k(\downarrow)$ and $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ are incomparable in terms of expressive power.*

We will now briefly discuss the case of $\mathcal{M}(ⓡ, ⓚ)$. As we already mentioned at the beginning of this section, the first required step to compare expressivity is to be able to define a natural mapping between models of the different logics involved. Consider a model $\langle W, (R_r)_{r\in\text{REL}}, V, S\rangle$ for $\mathcal{M}(ⓡ, ⓚ)$; if we want to associate a Kripke model we have to decide how to deal with the set S. The only natural choice seems to be to extend the signature with a special propositional variable *known*, and let V' be identical to V excepts that $V'(\mathit{known}) = S$. And the same can be done to obtain a hybrid model from a $\mathcal{M}(ⓡ, ⓚ)$ model.

Theorem 8. *The following results concerning expressive power can be established*

1. $\mathcal{K}$ *over the signature* $\langle \text{PROP} \cup \{known\}, \text{REL} \rangle$ *is strictly less expressive than* $\mathcal{M}(ⓡ, ⓚ)$ *over the signature* $\langle \text{PROP}, \text{REL} \rangle$.
2. $\mathcal{M}(ⓡ, ⓚ)$ *over the signature* $\langle \text{PROP}, \text{REL} \rangle$ *is strictly less expressive than* $\mathcal{HL}(\downarrow)$ *over the signature* $\langle \text{PROP} \cup \{known\}, \text{REL}, \text{NOM} \rangle$.
3. $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ *over the signature* $\langle \text{PROP} \cup \{known\}, \text{REL} \rangle$ *is equivalent to* $\mathcal{M}(ⓡ, ⓚ)$ *over the signature* $\langle \text{PROP}, \text{REL} \rangle$

Proof. See technical appendix for details.

To close this section, we mention that the satisfaction preserving translations defined in the proof can actually be used to transfer known results, for example, from $\mathcal{HL}(\downarrow)$ to $\mathcal{M}(ⓡ, ⓚ)$ and $\mathcal{M}_\emptyset(ⓡ, ⓚ)$. For instance, both logics are compact and their formulas are preserved by generated submodels (see [4]).

4 Infinite Models and Undecidability

The last issue that we will discuss in this paper is the undecidability of the satisfiability problem for both $\mathcal{M}(ⓡ, ⓚ)$ and $\mathcal{M}_\emptyset(ⓡ, ⓚ)$. The proof is an adaptation of the proof of undecidability of $\mathcal{HL}(\downarrow)$ presented in [2].

We first prove that both languages lack the finite model property [1].

Theorem 9. *There is a formula* $Inf \in \mathcal{M}_\emptyset(ⓡ, ⓚ)$ *such that* $\mathcal{M}, w \models Inf$ *implies that the domain of* $\mathcal{M}$ *is an infinite set.*

Proof. The formula *Inf* states that there is a nonempty subset of W that is an unbounded strict partial order. See the technical appendix for details.

To prove failure of the finite model property for the case $\mathcal{M}(ⓡ, ⓚ)$ we first notice that the following lemma is easy to establish (we only state it for the monomodal case; a similar result is true in the multimodal case). Failure of the finite model property is then a direct consequence.

Lemma 1. *Let* φ *be a formula of modal depth* d. *If* $\langle W, R_r, V, S \rangle, w \models \left(\bigwedge_{i=0}^{d} [r]^i \neg ⓚ \right) \wedge \varphi$ *then* $\langle W, R_r, V, \emptyset \rangle, w \models \varphi$.

Corollary 1. $\mathcal{M}(ⓡ, ⓚ)$ *lacks the finite model property.*

Proof. Using Lemma 1, one can easily see that the formula $Inf \wedge \left(\bigwedge_{i=0}^{4} [r]^i \neg ⓚ \right)$, where *Inf* is the one in the proof of Theorem 9, forces an infinite model.

We now turn to undecidability. We show that $\mathcal{M}(ⓡ, ⓚ)$ and $\mathcal{M}_\emptyset(ⓡ, ⓚ)$ are undecidable by encoding the $\omega \times \omega$ tiling problem (see [5]). Following the idea in [2], we construct a spy point over the relation S which has access to every tile. The relations U and R represent moving up and to the right, respectively, from one tile to the other. We code each type of tile with a fixed propositional symbol t_i. With this encoding we define for each tiling problem T, a formula φ^T such that the set of tiles T tiles $\omega \times \omega$ iff φ^T has a model.

Theorem 10. *The satisfiability problem for* $\mathcal{M}_\emptyset(\circled{r}, \circled{k})$ *is undecidable.*

Proof. See the technical appendix for details.

Corollary 2. *The satisfiability problem for* $\mathcal{M}(\circled{r}, \circled{k})$ *is undecidable.*

Proof. Using Lemma 1 and the formula φ^T in Theorem 10, we obtain a formula ψ such that if $\mathcal{M}, w \models \psi$ then $\mathcal{M}$ is a tiling of $\omega \times \omega$. For the converse, we can build exactly the same model as in the above proof.

5 Conclusions and Further Work

In this paper we investigate two members of a family of logics that we called *memory logics*. These logics were inspired by the hybrid logic $\mathcal{HL}(\downarrow)$: the $\downarrow$ operator can be thought of as a storage command, and our aim is to carry this idea further investigating different ways in which information can be stored. We have proved that, in terms of expressive power, the memory logics $\mathcal{M}(\circled{r}, \circled{k})$ and $\mathcal{M}_\emptyset(\circled{r}, \circled{k})$ lay between the basic modal logic $\mathcal{K}$ and the hybrid logic $\mathcal{HL}(\downarrow)$. Unluckily, the reduced expressive power is not sufficient to ensure good computational behavior: both $\mathcal{M}(\circled{r}, \circled{k})$ and $\mathcal{M}_\emptyset(\circled{r}, \circled{k})$ fail to have the finite model property and moreover their satisfiability problems are undecidable.

Despite the negative result concerning decidability, we believe that the new perspective we pursue in this paper is appealing. Clearly, it opens up the way to many new interesting modal languages (we discuss some examples in Sect. 1). As in the case of modal and hybrid languages, all of them seem to share some common behavior, and the challenge is now to discover and understand it.

Much work rest to be done. We are currently working on complete axiomatizations of $\mathcal{M}(\circled{r}, \circled{k})$ and $\mathcal{M}_\emptyset(\circled{r}, \circled{k})$, and on model theoretic characterizations. Extending the language with nominals is a natural step, and then adapting the internalized hybrid tableau method [6] to the new languages is straightforward. More interesting is to explore new languages of the family (like `(push)`, `(pop)`, or `(forget)`), and interaction between the memory operators and the modalities.

For example, if we restrict the class of models to those in which we are forced to memorize the current state each time we take a step via the accessibility relation, then the logic turns decidable (even though it is still strictly more expressive than $\mathcal{K}$). More precisely, changing the semantic definition of $\langle r \rangle$ to be

$$\begin{aligned}\langle W, (R_r)_{r\in \mathrm{REL}}, V, S\rangle, w \models \langle r\rangle\varphi \text{ iff } & \exists w' \in W, R_r(w, w') \text{ and} \\ & \langle W, (R_r)_{r\in \mathrm{REL}}, V, S \cup \{w\}\rangle, w' \models \varphi\end{aligned}$$

and calling the resulting logic $\mathcal{M}^-(\circled{r}, \circled{k})$, then $\mathcal{K} < \mathcal{M}^-(\circled{r}, \circled{k}) < \mathcal{M}(\circled{r}, \circled{k})$. Moreover, $\mathcal{M}^-(\circled{r}, \circled{k})$ has the bounded tree model property: every satisfiable formula φ of $\mathcal{M}^-(\circled{r}, \circled{k})$ is satisfied in a tree of size bounded by a computable funcion over the size of φ. Hence, the satisfiability problem of $\mathcal{M}^-(\circled{r}, \circled{k})$ is decidable.

The work presented in this paper is somehow related in spirit with the work on Dynamic Epistemic Logic and other update logics [**?**,**?**], but as we discuss in the introduction, our inspiration was rooted in a new interpretation of the $\downarrow$ binder.

References

1. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press (2001)
2. Blackburn, P., Seligman, J.: Hybrid languages. Journal of Logic, Language and Information **4** (1995) 251–272
3. Ebbinghaus, H., Flum, J., Thomas, W.: Mathematical Logic. Springer-Verlag (1984)
4. Areces, C., Blackburn, P., Marx, M.: Hybrid logics: characterization, interpolation and complexity. The Journal of Symbolic Logic **66**(3) (2001) 977–1010
5. Börger, E., Grädel, E., Gurevich, Y.: The classical decision problem. Springer Verlag (1997)
6. Blackburn, P.: Internalizing labelled deduction. Journal of Logic and Computation **10**(1) (2000) 137–168

Technical Appendix

Proof (Theorem 5). The translation Tr, taking $\mathcal{M}(ⓡ, ⓚ)$ formulas over the signature $\langle \textsc{prop}, \textsc{rel}\rangle$ to $\mathcal{HL}(\downarrow)$ sentences over the signature $\langle \textsc{prop}, \textsc{rel}, \textsc{nom}\rangle$ is defined for any finite set $N \subseteq \textsc{nom}$ as follows:

$$\begin{aligned} \mathsf{Tr}_N(p) &= p \quad p \in \textsc{prop} \\ \mathsf{Tr}_N(ⓚ) &= \textstyle\bigvee_{i \in N} i \\ \mathsf{Tr}_N(\neg\varphi) &= \neg \mathsf{Tr}_N(\varphi) \\ \mathsf{Tr}_N(\varphi_1 \wedge \varphi_2) &= \mathsf{Tr}_N(\varphi_1) \wedge \mathsf{Tr}_N(\varphi_2) \\ \mathsf{Tr}_N(\langle r\rangle\varphi) &= \langle r\rangle \mathsf{Tr}_N(\varphi) \\ \mathsf{Tr}_N(ⓡ\varphi) &= \downarrow i.\mathsf{Tr}_{N\cup\{i\}}(\varphi) \quad \text{where } i \notin N. \end{aligned}$$

A simple induction shows that $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, g, w \models \mathsf{Tr}_\emptyset(\varphi)$, for any g.

Proof (Proposition 1). As we said, $\mathcal{HL}_1(\downarrow) \not\leq \mathcal{M}_\emptyset(ⓡ, ⓚ)$ is a direct consequence of the proof of Theorem 6. To prove $\mathcal{M}_\emptyset(ⓡ, ⓚ) \not\leq \mathcal{HL}_1(\downarrow)$, let $\mathcal{M}_1 = \langle\{1, 2, 3\}, \{(i, j) \mid 1 \leq i, j \leq 3\}, \emptyset, \emptyset\rangle$ (a clique of size 3) and $\mathcal{M}_2 = \langle\{1, 2\}, \{(i, j) \mid 1 \leq i, j \leq 2\}, \emptyset, \emptyset\rangle$ (a clique of size 2). It is easy to check that $\mathcal{M}_1, 1 \underline{\leftrightarrow}_{\mathcal{HL}_1(\downarrow)} \mathcal{M}_2, 1$. However, the formula $\varphi = ⓡ\langle r\rangle(\negⓚ \wedge ⓡ\langle r\rangle(\negⓚ \wedge ⓡ\langle r\rangle\negⓚ))$ distinguishes the models: $\mathcal{M}_1, 1 \models \varphi$ but $\mathcal{M}_2, 1 \not\models \varphi$.

Proof (Theorem 8). All proofs are similar to (and sometimes easier than) the ones presented above. We only discuss 2. To prove $\mathcal{M}(ⓡ, ⓚ) \leq \mathcal{HL}(\downarrow)$ (over the appropriate signatures) we adapt the translation Tr with the following clause for ⓚ

$$\mathsf{Tr}_N(ⓚ) = \left(\textstyle\bigvee_{i \in N} i\right) \vee \mathit{known}.$$

$\mathcal{HL}(\downarrow) \not\leq \mathcal{M}(ⓡ, ⓚ)$ can be shown using the following models. Let $\mathcal{M}_1 = \langle\{w\}, \{(w, w)\}, \emptyset, \{w\}\rangle$ and $\mathcal{M}_2 = \langle\{u, v\}, \{(u, v), (v, u)\}, \emptyset, \{u, v\}\rangle$. Duplicator always wins on $E(\mathcal{M}_1, \mathcal{M}_2, w, u)$ and thus $\mathcal{M}_1, w \underline{\leftrightarrow}_{\mathcal{M}(ⓡ,ⓚ)} \mathcal{M}_2, u$. On the other hand, $\mathcal{M}'_1, w \models_{\mathcal{HL}(\downarrow)} \downarrow i.\langle r\rangle i$ but $\mathcal{M}'_2, u \not\models_{\mathcal{HL}(\downarrow)} \downarrow i.\langle r\rangle i$, for $\mathcal{M}'_1, \mathcal{M}'_2$ the models corresponding to $\mathcal{M}_1$ and $\mathcal{M}_2$.

Proof (Theorem 9). Consider the following formulas:

$$\begin{array}{rl}
(\mathit{Back}) & p \wedge [r]\neg p \wedge \langle r\rangle\top \wedge \textcircled{r}([r]\langle r\rangle\textcircled{k})\\
(\mathit{Spy}) & \textcircled{r}([r][r](\neg p \to \textcircled{r}(\langle r\rangle(p \wedge \textcircled{k} \wedge \langle r\rangle(\neg p \wedge \textcircled{k})))))\\
(\mathit{Irr}) & [r]\textcircled{r}\neg\langle r\rangle\textcircled{k}\\
(\mathit{Succ}) & [r]\langle r\rangle\neg p\\
(\mathit{3cyc}) & \neg(\langle r\rangle\textcircled{r}\langle r\rangle(\neg p \wedge \langle r\rangle(\neg p \wedge \neg\textcircled{k} \wedge \langle r\rangle\textcircled{k})))\\
(\mathit{Tran}) & [r]\textcircled{r}[r](\neg p \to ([r](\neg p \to (\textcircled{r}\langle r\rangle(p \wedge \langle r\rangle(\textcircled{k} \wedge \langle r\rangle\textcircled{k}))))))
\end{array}$$

Let *Inf* be $\mathit{Back} \wedge \mathit{Spy} \wedge \mathit{Irr} \wedge \mathit{Succ} \wedge \mathit{3cyc} \wedge \mathit{Tran}$. Let $\mathcal{M} = \langle W, R, V, \emptyset\rangle$. We show that if $\mathcal{M}, w \models \mathit{Inf}$, then W is infinite.

Suppose $\mathcal{M}, w \models \mathit{Inf}$. Notice that if $\textcircled{k}$ holds in a state, is because it was previously remembered by the evaluating formula. Let $B = \{b \in W \mid wRb\}$. Because *Back* is satisfied, $w \notin B$, $B \neq \emptyset$ and for all $b \in B$, bRw. Because *Spy* is satisfied, if $a \neq w$ and a is a successor of an element of B then a is also an element of B. As *Irr* is satisfied at w, every state in B is irreflexive. As *Succ* is satisfied at w, every point in B has a successor distinct from w. As *3cyc* is satisfied, there cannot be 3 different elements in B forming a cycle, and this sentence together with *Tran* force R to transitively order B.

It follows that B is an unbounded strict partial order, hence infinite, and so is W.

Proof (Theorem 10). Let $T = \{T_1, \dots, T_n\}$ be a set of tile types. Given a tile type T_i, $u(T_i)$, $r(T_i)$, $d(T_i)$, $l(T_i)$ will represent the colors of the up, right, down and left edges of T_i r espectively. Define

$$\begin{array}{rl}
(\mathit{Back}) & p \wedge [S]\neg p \wedge \langle S\rangle\top \wedge \textcircled{r}([S]\langle S\rangle\textcircled{k}) \wedge \textcircled{r}([S][S]\textcircled{k})\\
(\mathit{Spy}) & \textcircled{r}[S][\dagger]\textcircled{r}\langle S\rangle(\textcircled{k} \wedge p \wedge \langle S\rangle(\textcircled{k} \wedge \neg p)), \qquad \text{where } \dagger \in \{U, R\}\\
(\mathit{Grid}) & [S][U]\neg p \wedge [S][R]\neg p \wedge [S]\langle U\rangle\top \wedge [S]\langle r\rangle\top\\
(\mathit{Func}) & \textcircled{r}[S]\textcircled{r}\langle\dagger\rangle\textcircled{r}\langle S\rangle\langle S\rangle(\textcircled{k} \wedge \langle\dagger\rangle\textcircled{k} \wedge [\dagger]\textcircled{k}), \qquad \text{where } \dagger \in \{U, R\}\\
(\mathit{Irr}) & [S]\textcircled{r}[\dagger]\neg\textcircled{k}, \qquad \text{where } \dagger \in \{U, R\}\\
(\mathit{2cyc}) & [S]\textcircled{r}[\dagger][\dagger]\neg\textcircled{k}, \qquad \text{where } \dagger \in \{U, R\}\\
(\mathit{Confluent}) & [S]\textcircled{r}\langle U\rangle\langle r\rangle\textcircled{r}\langle S\rangle\langle S\rangle(\textcircled{k} \wedge \langle U\rangle\langle r\rangle\textcircled{k} \wedge \langle r\rangle\langle U\rangle\textcircled{k})\\
(\mathit{UR\text{-}Irr}) & [S]\textcircled{r}[U][R]\neg\textcircled{k}\\
(\mathit{UR\text{-}2cyc}) & [S]\textcircled{r}[U][R][U][R]\neg\textcircled{k}\\
(\mathit{Unique}) & [S]\left(\bigvee_{1\le i\le n} t_i \wedge \bigwedge_{1\le i<j\le n}(t_i \to \neg t_j)\right)\\
(\mathit{Vert}) & [S]\bigwedge_{1\le i\le n}\left(t_i \to \langle U\rangle\bigvee_{1\le j\le n, u(T_i)=d(T_j)} t_j\right)\\
(\mathit{Horiz}) & [S]\bigwedge_{1\le i\le n}\left(t_i \to \langle r\rangle\bigvee_{1\le j\le n, r(T_i)=l(T_j)} t_j\right)
\end{array}$$

Let the formula φ^T be the conjunction of all the above formulas. We show that T tiles $\omega \times \omega$ iff φ^T is satisfiable.

Suppose $\mathcal{M}, w \models \varphi^T$. Observe that (*Back*) and (*Spy*) impose w to be a spy point over all its S-accessible states of $\mathcal{M}$. These S-accessible states will be the tiles. From this it follows that $[S]\psi$ holds at w iff ψ is true at every tile. Additionally, $\langle S\rangle\langle S\rangle\psi$ holds at tile v iff ψ is true at some tile (maybe the same one).

Taking the above points into account, one can establish the following. (*Grid*) states that from every tile there is another tile moving up (that is, following the U-relation). The same holds for the right direction (following the R-relation). (*Func*) forces that U and R are both functionals, given that (*Irr*) and (*2cyc*) guarantee irreflexivity and asymmetry of U and R respectively. (*Confluent*) imposes that the tiles are arranged in a grid pattern. To make its job, (*Confluent*) needs the composed relation $U \circ R$ to be irreflexive and asymmetric, and this is done by (*UR-Irr*) and (*UR-2cyc*) respectively.

All the formulas we discuss up to now configure the grid. The last three ensure that every tile has a unique type t_i, and that the colors of the tiles match properly. From this, it easily follows that $\mathcal{M}$ is a tiling of $\omega \times \omega$.

For the converse, suppose $f : \omega \times \omega \to T$ is a tiling of $\omega \times \omega$. We define the model $\mathcal{M} = \langle W, \{S, U, R\}, V, \emptyset \rangle$ as follows:

- $W = \omega \times \omega \cup \{w\}$
- $S = \{(w, v), (v, w) \mid v \in \omega \times \omega\}$ (hence w will act as the spy point)
- $U = \{((x, y), (x, y + 1)) \mid x, y \in \omega\}$
- $R = \{((x, y), (x + 1, y)) \mid x, y \in \omega\}$
- $V(p) = \{w\}$; $V(t_i) = \{x \mid x \in \omega \times \omega, f(x) = T_i\}$

The reader may verify that, by construction, $\mathcal{M}, w \models \varphi^T$.