On the Expressiveness of TPTL and MTL[☆,☆☆]

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Abstract

TPTL and MTL are two classical timed extensions of LTL. In this paper, we prove the 20-year-old conjecture that TPTL is strictly more expressive than MTL. But we show that, surprisingly, the TPTL formula proposed in [AH90] for witnessing this conjecture *can* be expressed in MTL. More generally, we show that TPTL formulae using only modality **F** can be translated into MTL.

1. Introduction

Temporal logics.. Temporal logics are a widely used framework in the field of specification and verification of reactive systems [Pnu77]. In particular, Linear-time Temporal Logic (LTL) allows to express properties about each individual execution of a model, such as the fact that *any occurrence of a problem eventually triggers the alarm*. LTL has been extensively studied, both w.r.t its expressiveness [Kam68, GPSS80, Mar03] and for model-checking purposes [SC85, VW86, Var96].

Timed temporal logics.. At the beginning of the 90s, real-time constraints have naturally been added to temporal logics [Koy90, ACD90], in order to add quantitative constraints to temporal logic specifications. The resulting logics allow to express *e.g.* that any occurrence of a problem in a system will trigger the alarm *within at most 5 time units*.

When dealing with dense time, we may consider two different semantics for timed linear-time temporal logics, depending on whether the formulae are evaluated over *timed words* (*i.e.*, over a discrete sequence of observations of the system; this is the *pointwise semantics*) or over *timed state sequences* (*i.e.*, over the continuous observation of the system; this is the *interval-based semantics*). We refer to [AH92b, Hen98, Ras99] for surveys on linear-time timed temporal logics.

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Expressiveness of TPTL *and* MTL.. Two interesting timed extensions of LTL are MTL (Metric Temporal Logic) [Koy90, AH93] and TPTL (Timed Propositional Temporal Logic) [AH94].

MTL extends LTL by adding subscripts to temporal operators: for instance, the above property can be written in MTL as

$$\mathbf{G}\,(\texttt{problem} \Rightarrow \mathbf{F}_{\leq 5}\,\texttt{alarm}).$$

TPTL is "more temporal" [AH94] in the sense that it uses real clocks in order to assert temporal constraints. A TPTL formula can "reset" a formula clock at some point, and later compare the value of that clock to some integer. The property above would then be written as

$$\mathbf{G}\,(\texttt{problem} \Rightarrow x.\mathbf{F}\,(\texttt{alarm} \wedge x \leq 5))$$

where "$x.\varphi$" means that x is reset at the current position, before evaluating φ. This logic also allows to easily express that, for instance, within 5 time units after the occurrence of a problem, the system triggers the alarm and then enters a failsafe mode:

$$\mathbf{G}\,(\texttt{problem} \Rightarrow x.\mathbf{F}\,(\texttt{alarm} \wedge \mathbf{F}\,(\texttt{failsafe} \wedge x \leq 5))). \tag{1}$$

While it is clear that any MTL formula can be translated into an equivalent TPTL one, Alur and Henzinger state in [AH92b, AH93] that there is no intuitive MTL equivalent to formula (1). It has thus been conjectured that TPTL would be strictly more expressive than MTL [AH92b, AH93, Hen98], formula (1) being proposed as a possible witness not expressible in MTL.

Our contributions.. We consider that problem for the aforementioned semantics (*pointwise* and *interval-based*) over infinite sequences. We prove that

- the conjecture *does* hold for both semantics;
- for the pointwise semantics, formula (1) witnesses the expressiveness gap, *i.e.*, it cannot be expressed in MTL;
- for the interval-based semantics, formula (1) *can* be expressed in MTL, but we exhibit another TPTL formula that cannot be expressed in MTL, confirming the conjecture.

Our study also yields several interesting side-results:

- we prove that, for the interval-based semantics, MITL (a restriction of MTL where timing constraints are restricted to be non-singular [AFH96]) cannot express property (1). This result is counter-intuitive, since formula (1) does not involve any punctual constraint.
- MTL is strictly more expressive under the interval-based semantics than under the pointwise one, since it can express formula (1) only in the first case. This had recently and independently been remarked in [DP07] in the case of finite words;

- we also get that, for both semantics, MTL+Past and MITL+Past (where the past-time modality "since" is used [AFH96]) are strictly more expressive than their respective pure-future fragments.
- our main result also extends to the branching-time logic TCTL with explicit clock [HNSY94], which we prove is strictly more expressive than TCTL with subscripts [ACD93], as conjectured in [Alu91, Yov93].

Finally, we prove that, under the interval-based semantics, the fragment of TPTL where only the modality **F** is allowed (which we call the *existential fragment*[1] of TPTL, and write $\mathsf{TPTL}_{\mathbf{F}}$) can be translated into MTL (actually, into the corresponding *existential fragment* $\mathsf{MTL}_{\mathbf{F}}$ of MTL). This generalizes the fact that formula (1) can be expressed in MTL (in formula (1), the subformula under **G** is in $\mathsf{TPTL}_{\mathbf{F}}$, and can thus be expressed in MTL).

Those results are summarised on Figure 1 (where edges going upwards indicate gaps in expressiveness).

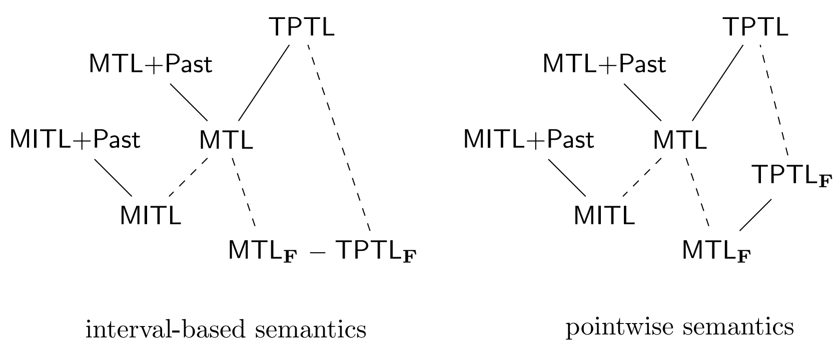


Figure 1: Summary of our expressiveness results (dashed edges indicate folk results)

Related work.. Over the last 15 years, many researches have focused on expressiveness questions for timed temporal logics (over both integer and real time). We refer to [AH92a, AH93, AH94, BL95, AFH96, RSH98, FR07] for original works, and to [Ost92, Hen98, Ras99] for surveys on that topic.

MTL and TPTL have also been studied for the purpose of verification. If the underlying time domain is discrete, then MTL and TPTL have decidable verification problems [AH93, AH94]. When considering dense time, verification problems (satisfiability, model-checking) become much harder: [AFH96] proves that the satisfiability problem for MTL is undecidable when considering the interval-based semantics. This result of course carries on to TPTL. It has recently been proved that model-checking and satisfiability are decidable (but non-primitive recursive) for MTL over finite words under the pointwise semantics [OW05], while they are still undecidable for TPTL [AH94].

[1] Not to be confused with the existential fragment of branching-time logics.

Recently, MTL and TPTL have been investigated in the scope of monitoring and path-checking. [TR05] proposes an (exponential) monitoring algorithm for MTL under the pointwise semantics. [MR06] shows that, in the interval-based semantics, MTL formulae can be verified on lasso-shaped timed state sequences in polynomial time, while TPTL formulae require at least polynomial space.

Plan of the paper.. The paper is organized as follows: in Section 2, we define the logics TPTL and MTL together with the two semantics we consider. In Section 3, we present our main result, namely that TPTL is strictly more expressive than MTL (for both semantics), whereas the last section (Section 4) focuses on the "existential" fragments of TPTL and MTL, where only the modality **F** is allowed.

2. Timed Linear-Time Temporal Logics

Basic definitions.. In the sequel, AP represents a non-empty, countable set of atomic propositions. We let $\mathbb{R}$ (resp. $\mathbb{R}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{Q}_{\geq 0}$, $\mathbb{N}$, $\mathbb{N}_{>0}$) denote the set of reals (resp. nonnegative reals, rationals, nonnegative rationals, nonnegative integers, positive integers). An *interval* is a convex subset of $\mathbb{R}$. An interval I' is *adjacent* to another interval I when $I \cap I' = \emptyset$, $I \cup I'$ is an interval and for all $x \in I$, for all $y \in I'$, $x < y$. Given an interval I and a real number t, we write $I - t$ for the interval $\{t' \in \mathbb{R} \mid t' + t \in I\}$. We denote by $\mathcal{I}_{\mathbb{R}}$ (resp. $\mathcal{I}_{\mathbb{R}_{\geq 0}}$, $\mathcal{I}_{\mathbb{Q}}$) the set of intervals whose bounds are in $\mathbb{R}$ (resp. $\mathbb{R}_{\geq 0}$, $\mathbb{Q}$).

Given a finite set X of variables called *clocks*, a *clock valuation* over X is a mapping $v\colon X \to \mathbb{R}_{\geq 0}$ which assigns to each clock a time value in $\mathbb{R}_{\geq 0}$. Given a valuation v and a nonnegative real t, we write $v[x \mapsto t]$ for the valuation v' such that $v'(x) = t$ and $v'(y) = v(y)$ for all $y \in X \setminus \{x\}$. We write $\mathbf{0}$ for the valuation such that $\mathbf{0}(x) = 0$ for all $x \in X$.

Timed state sequences and timed words.. An *interval sequence* over $\mathbb{R}_{\geq 0}$ is an infinite sequence $I = I_0 I_1 \ldots$ of non-empty intervals of $\mathcal{I}_{\mathbb{R}_{\geq 0}}$ satisfying the following properties:

- *(adjacency)* the intervals I_i and I_{i+1} are adjacent for all $i \geq 0$;
- *(progress)* every nonnegative real belongs to some interval I_i.

A *timed state sequence* over 2^{AP} is a pair $\kappa = (\sigma, I)$ where $\sigma = \sigma_0 \sigma_1 \ldots$ is an infinite sequence of elements of 2^{AP} and $I = I_0 I_1 \ldots$ is an interval sequence. A timed state sequence can equivalently be seen as an infinite sequence of elements in $2^{\mathsf{AP}} \times \mathcal{I}_{\mathbb{R}_{\geq 0}}$.

Let $\kappa = (\sigma, I)$ be a timed state sequence, and $t \in \mathbb{R}_{\geq 0}$. Let $i \in \mathbb{N}$ be the unique integer such that $t \in I_i$. We write $\kappa(t)$ for the set $\sigma_i \subseteq \mathsf{AP}$. We also define the *suffix* of κ at date t as being the timed state sequence $\kappa' = (\sigma', I')$ such that, for all $k \in \mathbb{N}$, $\sigma'_k = \sigma_{i+k}$ and $I'_k = (I_{i+k} - t) \cap \mathbb{R}_{\geq 0}$.

A *time sequence* over $\mathbb{R}_{\geq 0}$ is an infinite sequence $\tau = \tau_0\tau_1 \ldots$ of nonnegative reals satisfying the following properties:

- *(initialization)* $\tau_0 = 0$;
- *(monotonicity)* the sequence is nondecreasing: $\tau_{i+1} \geq \tau_i$ for any $i \in \mathbb{N}$;
- *(progress)* every time value is eventually reached: $\forall t \in \mathbb{R}_{\geq 0}.\ \exists i \in \mathbb{N}.\ \tau_i > t$.

A *timed word* over 2^{AP} is a pair $\rho = (\sigma, \tau)$, where $\sigma = \sigma_0\sigma_1 \ldots$ is an infinite sequence of elements of 2^{AP} and $\tau = \tau_0\tau_1 \ldots$ a time sequence over $\mathbb{R}_{\geq 0}$. It can equivalently be seen as an infinite sequence of elements $\langle \sigma_i, \tau_i \rangle$ of $2^{\mathsf{AP}} \times \mathbb{R}_{\geq 0}$.

Let $\rho = (\sigma, \tau)$ be a timed word, and $i \in \mathbb{N}$. We write $\rho(\tau_i)$ for $\sigma(i)$, and define the i-th suffix of ρ to be the timed word $\rho' = (\sigma', \tau')$ such that, for all $k \in \mathbb{N}$, $\sigma'_k = \sigma_{k+i}$ and $\tau'_k = \tau_{k+i} - \tau_i$.

We force timed words to satisfy $\tau_0 = 0$ in order to have a natural way of defining initial satisfiability of a temporal logic formula. This is no loss of generality since it can be obtained by adding a silent action to the alphabet.

*2.1. Timed Propositional Temporal Logic (*TPTL*)*

The logic TPTL [AH94, Ras99] is a timed extension of LTL [Pnu77] which uses extra variables (clocks) explicitly in the formulae. Formulae of TPTL are built from atomic propositions, boolean connectives, the modality "until", clock constraints and clock resets. Formally:

$$\mathsf{TPTL} \ni \varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi\, \mathbf{U}\, \varphi \mid x \sim c \mid x.\varphi$$

where p ranges over AP, x ranges over a finite set of *clock variables*, c ranges over $\mathbb{Q}$, and $\sim\, \in \{\leq, <, =, >, \geq\}$.

There are two main semantics for TPTL, the *interval-based* semantics which interprets TPTL formulae over timed state sequences, and the *pointwise* semantics, which interprets them over timed words. This last semantics is less general since, as we will see below, formulae can only be interpreted at points in time when actions occur.

In the literature, both semantics have been considered, and results highly depend on the underlying semantics. For example, a recent result by Ouaknine and Worrell [OW05] states that the satisfiability of a formula in MTL (a subset of TPTL, see below) is decidable under the pointwise semantics, whereas it is known to be undecidable under the interval-based semantics [AFH96].

***Interval-based semantics.*.** In the interval-based semantics, models are timed state sequences κ, and formulae are evaluated at a date $t \in \mathbb{R}_{\geq 0}$ with a valuation $v\colon X \to \mathbb{R}_{\geq 0}$ (where X is the set of formula clocks) representing the date at which each clock has been reset last. The satisfaction relation, denoted with

$\kappa, t, v \models_{ib} \varphi$ (we might omit the index ib, and simply write $\kappa, t, v \models \varphi$, when it is clear from the context), is defined inductively as follows:

$$
\begin{array}{rcl}
\kappa, t, v \models_{ib} p & \text{iff} & p \in \kappa(t), \\
\kappa, t, v \models_{ib} \varphi_1 \wedge \varphi_2 & \text{iff} & \kappa, t, v \models_{ib} \varphi_1 \text{ and } \kappa, t, v \models_{ib} \varphi_2, \\
\kappa, t, v \models_{ib} \neg\varphi & \text{iff} & \text{it is not the case that } \kappa, t, v \models_{ib} \varphi, \\
\kappa, t, v \models_{ib} \varphi_1 \,\mathbf{U}\, \varphi_2 & \text{iff} & \exists t' > t \text{ such that } \kappa, t', v \models_{ib} \varphi_2, \\
 & & \quad \text{and } \forall t < t'' < t',\ \kappa, t'', v \models_{ib} \varphi_1 \\
\kappa, t, v \models_{ib} x \sim c & \text{iff} & t - v(x) \sim c, \\
\kappa, t, v \models_{ib} x.\varphi & \text{iff} & \kappa, t, v[x \mapsto t] \models_{ib} \varphi.
\end{array}
$$

We write $\kappa \models_{ib} \varphi$ when $\kappa, 0, \mathbf{0} \models_{ib} \varphi$. We interpret "$x.\varphi$" as a reset operator. Note also that the semantics of $\mathbf{U}$ is strict in the sense that, in order to satisfy $\varphi_1 \,\mathbf{U}\, \varphi_2$, a timed state sequence is not required to satisfy φ_1; this semantics is more expressive than the non-strict semantics (see section 2.5).

In the following, we use classical shorthands: $\top$ stands for $p \vee \neg p$, $\varphi_1 \Rightarrow \varphi_2$ stands for $\neg\varphi_1 \vee \varphi_2$, $\mathbf{F}\,\varphi$ stands for $\top \,\mathbf{U}\, \varphi$ (and means that φ eventually holds at a strict future time), and $\mathbf{G}\,\varphi$ stands for $\neg(\mathbf{F}\,\neg\varphi)$ (and means that φ always holds in the strict future).

***Pointwise semantics.*.** In this semantics, models are timed words ρ, and satisfiability is no longer interpreted at a date $t \in \mathbb{R}_{\geq 0}$ but at a position $i \in \mathbb{N}$ along the timed word. For a timed word $\rho = (\sigma, \tau)$, with $\sigma = (\sigma_i)_{i \geq 0}$ and $\tau = (\tau_i)_{i \geq 0}$, a position $i \in \mathbb{N}$ and a valuation v, we define the satisfaction relation $\rho, i, v \models_{pw} \varphi$ inductively as follows:

$$
\begin{array}{rcl}
\rho, i, v \models_{pw} p & \text{iff} & p \in \sigma_i, \\
\rho, i, v \models_{pw} \varphi_1 \wedge \varphi_2 & \text{iff} & \rho, i, v \models_{pw} \varphi_1 \text{ and } \rho, i, v \models_{pw} \varphi_2, \\
\rho, i, v \models_{pw} \neg\varphi & \text{iff} & \text{it is not the case that } \rho, i, v \models_{pw} \varphi, \\
\rho, i, v \models_{pw} \varphi_1 \,\mathbf{U}\, \varphi_2 & \text{iff} & \exists j > i \text{ such that } \rho, j, v \models_{pw} \varphi_2, \\
 & & \quad \text{and } \forall i < k < j.\ \rho, k, v \models_{pw} \varphi_1, \\
\rho, i, v \models_{pw} x \sim c & \text{iff} & \tau_i - v(x) \sim c, \\
\rho, i, v \models_{pw} x.\varphi & \text{iff} & \rho, i, v[x \mapsto \tau_i] \models_{pw} \varphi.
\end{array}
$$

We write $\rho \models_{pw} \varphi$ whenever $\rho, 0, \mathbf{0} \models_{pw} \varphi$. We might omit the index pw when it is clear from the context.

Example 1. *Consider the timed word $\rho = \langle a, 0\rangle\langle a, 1.1\rangle\langle b, 2\rangle \ldots$, and the* TPTL *formula $\varphi = x.\mathbf{F}\,(x = 1 \wedge y.\mathbf{F}\,(y = 1 \wedge b))$. Then $\rho \not\models_{pw} \varphi$, because ρ contains no action at date* 1*.*

Now, a timed word can be seen as a special case of timed state sequence. For instance, ρ corresponds to the timed state sequence

$$\kappa = \langle\{a\}, [0, 0]\rangle\langle\emptyset, (0, 1.1)\rangle\langle\{a\}, [1.1, 1.1]\rangle\langle\emptyset, (1.1, 2)\rangle\langle\{b\}, [2, 2]\rangle \ldots$$

But in that case, $\kappa \models_{ib} \varphi$.

2.2. Metric Temporal Logic (MTL)

The logic MTL [Koy90, AH93] extends the logic LTL with time restrictions on "until" modalities. Formulae of MTL are built from atomic propositions, boolean connectives and time-constrained "until":

$$\mathsf{MTL} \ni \varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi\, \mathbf{U}_I\, \varphi$$

where p ranges over the set AP of atomic propositions, and I ranges over $\mathcal{I}_{\mathbb{Q}}$.

In the interval-based semantics, formulas of MTL are evaluated at a certain date $t \in \mathbb{R}_{\geq 0}$ along a timed state sequence κ:

$$\begin{aligned}
\kappa, t \models_{ib} p \quad &\text{iff} \quad p \in \kappa(t),\\
\kappa, t \models_{ib} \varphi_1 \wedge \varphi_2 \quad &\text{iff} \quad \kappa, t \models_{ib} \varphi_1 \text{ and } \kappa, t \models_{ib} \varphi_2,\\
\kappa, t \models_{ib} \neg\varphi \quad &\text{iff} \quad \text{it is not the case that } \kappa, t \models_{ib} \varphi,\\
\kappa, t \models_{ib} \varphi_1\, \mathbf{U}_I\, \varphi_2 \quad &\text{iff} \quad \exists t' > t \text{ such that } t' - t \in I \text{ and } \kappa, t' \models_{ib} \varphi_2,\\
&\qquad \text{and } \forall t < t'' < t',\ \kappa, t'' \models_{ib} \varphi_1.
\end{aligned}$$

Again, we use the shorthand $\kappa \models \varphi$ for $\kappa, 0 \models \varphi$ when $\varphi \in \mathsf{MTL}$.

The pointwise semantics of MTL is defined at a position $i \in \mathbb{N}$ along a timed word w as follows:

$$\begin{aligned}
\rho, i \models_{pw} p \quad &\text{iff} \quad p \in \sigma_i,\\
\rho, i \models_{pw} \varphi_1 \wedge \varphi_2 \quad &\text{iff} \quad \rho, i \models_{pw} \varphi_1 \text{ and } \rho, i \models_{pw} \varphi_2,\\
\rho, i \models_{pw} \neg\varphi \quad &\text{iff} \quad \text{it is not the case that } \rho, i \models_{pw} \varphi,\\
\rho, i \models_{pw} \varphi_1\, \mathbf{U}_I\, \varphi_2 \quad &\text{iff} \quad \exists j > i \text{ such that } \tau_j - \tau_i \in I \text{ and } \rho, j \models_{pw} \varphi_2,\\
&\qquad \text{and } \forall i < k < j.\ \rho, k \models_{pw} \varphi_1.
\end{aligned}$$

We omit the constraint on modality $\mathbf{U}$ when $(0, \infty)$ is assumed. We write $\mathbf{U}_{\sim c}$ for $\mathbf{U}_I$ when $I = \{t \mid t \sim c\}$. As previously, we use classical shorthands such as $\mathbf{F}_I$ or $\mathbf{G}_I$.

Note that we could have defined MTL as a fragment of TPTL: $\varphi_1\, \mathbf{U}_I\, \varphi_2$ is equivalent[2] to $x.(\varphi_1\, \mathbf{U}\, (x \in I \wedge \varphi_2))$. As a consequence, TPTL is at least as expressive as MTL.

Example 2. *In* MTL*, the formula φ of Example 1 can be expressed as $\mathbf{F}_{=1}\, \mathbf{F}_{=1}\, b$. In the interval-based semantics, this formula is equivalent to $\mathbf{F}_{=2}\, b$, but this is* not *the case in the pointwise semantics.*

2.3. Metric Interval Temporal Logic (MITL)

MITL [AFH96] is a restricted version of MTL where the interval decorating the "until" modality cannot be singular (*i.e.*, reduced to a single point). Relaxing punctuality has the great benefit of making model-checking and satisfiability decidable: under the interval-based semantics, both problems can be achieved in exponential space, while they are undecidable for MTL [AFH96].

[2] We leave it to the keen reader to formalize this statement.

2.4. Adding Past-Time Modalities

The logics defined above only allow formulas to deal with future time points. It is classical to also define a symmetric version of the "until" modality, named "since", which deals with events that occurred in the past [Kam68, LPZ85]. The semantics of that modality is defined symmetrically:

- For the interval-based semantics:

$$\kappa, t, v \models_{ib} \varphi_1 \mathbf{S}\, \varphi_2 \quad \text{iff} \quad \exists t' < t \text{ such that } \kappa, t', v \models_{ib} \varphi_2 \text{ and } \forall t' < t'' < t,\ \kappa, t'', v \models_{ib} \varphi_1.$$

- For the pointwise semantics:

$$\rho, i, v \models_{pw} \varphi_1 \mathbf{S}\, \varphi_2 \quad \text{iff} \quad \exists j < i \text{ such that } \rho, j, v \models_{pw} \varphi_2 \text{ and } \forall j < k < i\ \rho, k, v \models_{pw} \varphi_1.$$

The corresponding MTL modality $\mathbf{S}_I$ is defined in the obvious way. Then, for instance, the TPTL formula $x.(p\, \mathbf{S}\, (q \wedge x \leq -2))$ expresses that q held 2 time units ago or earlier, and that p has been holding since then. It would be written $p\, \mathbf{S}_{(-\infty,-2]}\, q$, or equivalently $p\, \mathbf{S}_{\leq -2}\, q$, in MTL.

We note MTL+Past (resp. MITL+Past, TPTL+Past) the logic MTL (resp. MITL, TPTL) extended with the "since" modality. Such extensions have been defined and studied in [AH92a, AH93].

2.5. Relative Expressiveness

Let $\mathcal{S}$ be a set of models, and $\mathcal{L}$ and $\mathcal{L}'$ two logical languages interpreted over models in $\mathcal{S}$. We say that a formula $\varphi \in \mathcal{L}$ is *equivalent* to $\varphi' \in \mathcal{L}'$ if for every $\pi \in \mathcal{S}$, π satisfies φ iff π satisfies φ'. The language $\mathcal{L}'$ is *at least as expressive* as $\mathcal{L}$ over $\mathcal{S}$ iff all formulae in $\mathcal{L}$ have an equivalent formula in $\mathcal{L}'$. It is *strictly more expressive* if, moreover, there exists a formula in $\mathcal{L}'$ which has no equivalent in $\mathcal{L}$. We say that $\mathcal{L}$ and $\mathcal{L}'$ are *equally expressive* whenever each language is at least as expressive as the other.

Let us mention some classical results about expressiveness of (untimed) linear-time temporal logics:

- first of all, it can be proved that (the logic made of boolean combinators and) the strict until is at least as expressive as (the logic with) the non-strict one. The converse inclusion does not hold in general: along a timed word, for instance, the strict until can distinguish between two consecutive occurrences of the same letter, while the non-strict one cannot [Rey03, FR07].

- adding past-time modalities to LTL does not increase its expressive power: any LTL+Past formula can be expressed in LTL [Kam68, GPSS80], even though there are cases where the resulting LTL formula is exponentially larger [LMS02, Mar03]. Those results don't carry on to timed temporal logics: [AH92a] shows that past-time modalities strictly increase the expressive power of MITL under the interval-based semantics.

Proving expressiveness results is sometimes involved. In order to prove that a given formula φ cannot be expressed in a logic $\mathcal{L}$, the naive technique is to build two models M and N that φ can *distinguish* (or *separate*) (*i.e.*, φ evaluates to true on one model and to false on the other one), and prove that no formula of $\mathcal{L}$ can distinguish those two models. That technique turns out to be too restrictive for proving that TPTL is strictly more expressive than MTL: consider any two models that TPTL can separate (*i.e.*, there is a TPTL formula that holds on only one of those models, and fails to hold on the other one). The models are therefore different: there exists an atomic proposition a and a date t such that the MTL formula $\mathbf{F}_{=t}\, a$ holds on one of the models and fails to hold on the other one[3]. This naive approach only compares the *distinguishing power* of the logics, which is coarser than the *expressive power*. The remark above indicates that TPTL and MTL have the same distinguishing power. Conversely, it can easily be seen that LTL has less distinguishing power than TPTL (*i.e.* there exists two models that TPTL can separate but that LTL cannot).

A more involved technique, that we will use in the sequel, consists in building two *families* of models (M_i) and (N_i) such that φ distinguishes between M_i and N_i for all i, and such that no formula in $\mathcal{L}$ with size less than i distinguishes between M_i and N_i. This technique has already been applied successfully *e.g.* in [EH86, Eme91, Lar95, BCL05].

Other techniques involve translations of temporal logics to other formalisms, such as automata theory, language theory, algebraic structures or pebble games. Many examples can be found in the literature [Kam68, GPSS80, AH92a, TW96, Mar03].

3. TPTL is Strictly More Expressive Than MTL

3.1. Alur & Henzinger's Formula is not a Good Witness...

It has been conjectured in [AH92b, AH93, Hen98] that TPTL is strictly more expressive than MTL, and in particular that a TPTL formula such as

$$\mathbf{G}\,(a \Rightarrow x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge\ x \leq 2)))$$

cannot be expressed in MTL. The following proposition immediately entails that this formula is not a good witness formula for proving that TPTL is strictly more expressive than MTL.

Proposition 1. *The* TPTL *formula* $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge\ x \leq 2))$ *can be expressed in* MTL *for the interval-based semantics.*

PROOF. Let Φ be the TPTL formula $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge\ x \leq 2))$. This formula expresses that, along the timed state sequence, from the current point on, there

[3]It could be the case that $t \notin \mathbb{Q}_{\geq 0}$, and that the resulting formula is not in MTL. Still, since our models have finite variability, t could be replaced by some interval with rational bounds.

is a b followed by a c, and the delay before that occurrence of c is less than 2 time units. For proving the proposition, we build an MTL formula Φ' which is equivalent to Φ over timed state sequences. Formula Φ' is defined as the disjunction $\Phi' = \Phi'_1 \vee \Phi'_2 \vee \Phi'_3$ where:

$$\begin{cases} \Phi'_1 &= (\mathbf{F}_{\leq 1}\, b) \wedge (\mathbf{F}_{[1,2]}\, c) \\ \Phi'_2 &= \mathbf{F}_{\leq 1}\, (b \wedge \mathbf{F}_{\leq 1}\, c) \\ \Phi'_3 &= \mathbf{F}_{\leq 1}\, [(\mathbf{F}_{\leq 1}\, b) \wedge (\mathbf{F}_{=1}\, c)] \end{cases}$$

Let κ be a timed state sequence. If $\kappa \models \Phi'$, it is clear enough that $\kappa \models \Phi$. Suppose now that $\kappa \models \Phi$; then there exists $0 < t_1 < t_2 \leq 2$ such that $\kappa, t_1, \mathbf{0} \models b$ and $\kappa, t_2, \mathbf{0} \models c$. If $t_1 \leq 1$ then κ satisfies Φ'_1 or Φ'_2 (or both) depending on t_2 being smaller or greater than 1. If $t_1 \in]1, 2]$ then there exists a date t' in $(0, 1]$ such that $\kappa, t' \models (\mathbf{F}_{\leq 1}\, b) \wedge (\mathbf{F}_{=1}\, c)$ which implies that $\kappa \models \Phi'_3$. We illustrate the three possible cases on Figure 2. □

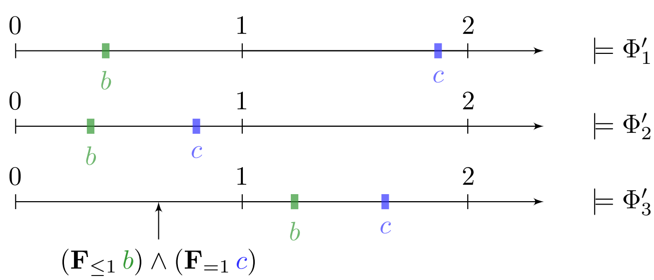


Figure 2: Translation of TPTL formula Φ in MTL

From the proposition above we get that the TPTL formula $\mathbf{G}\,(a \Rightarrow \Phi)$ is equivalent over timed state sequences to the MTL formula $\mathbf{G}\,(a \Rightarrow \Phi')$.

3.2. The Detriment of Relaxing Punctuality

The MTL formula proposed in the previous section involves a *punctual* constraint $\mathbf{F}_{=1}$. It is natural to wonder if it is really needed since, at first sight, the original property does not involve punctuality. Surprisingly:

Proposition 2. *The formula* $\Phi = x.(\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2)))$ *cannot be expressed in* MITL *for the interval-based semantics.*

We need several extra definitions before tackling the proof. Given a formula φ, we define its *granularity* p by $p = \prod_{\frac{a}{b} \text{ appears in } \varphi} b$. Clearly enough, any constant that appears in a formula φ is a multiple of $\frac{1}{p}$, where p is the granularity of φ. We write MITL_p (resp. MTL_p) for the set of MITL- (resp. MTL-) formulae with granularity p.

Proof. We construct two families of (timed words seen as) timed state sequences $(\mathcal{A}_n)_{n\in\mathbb{N}_{>0}}$ and $(\mathcal{B}_n)_{n\in\mathbb{N}_{>0}}$ such that:

(a) $\mathcal{A}_n \models \Phi$ whereas $\mathcal{B}_n \not\models \Phi$ for every $n \in \mathbb{N}_{>0}$,

(b) for any $p \in \mathbb{N}_{>0}$ and any $\varphi \in \mathsf{MITL}_p$, $\mathcal{A}_p \models \varphi \iff \mathcal{B}_p \models \varphi$.

Proposition 2 immediately follows: if Φ were to have an MITL equivalent Ψ, then Ψ would satisfy both (a) and (b), which is contradictory.

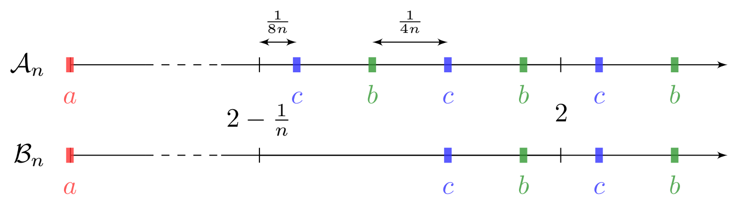


Figure 3: Models $\mathcal{A}_n$ and $\mathcal{B}_n$

The two families of models are depicted in Figure 3. Note that, along $\mathcal{A}_n$ and $\mathcal{B}_n$, there is an a at date 0 and no action between dates 0 and $2-\frac{1}{n}$. After date $2-\frac{1}{n}$, the word $\mathcal{A}_n$ is periodic with a period $\frac{1}{2n}$: atomic proposition c holds at dates $2-\frac{7}{8n}+\frac{i}{2n}$ with $i \geq 0$ whereas b holds at dates $2-\frac{5}{8n}+\frac{i}{2n}$. The word $\mathcal{B}_n$ is obtained from $\mathcal{A}_n$ by dropping the second and third events.

We first show that, for any $p \in \mathbb{N}_{>0}$, any MITL_p formula is uniformly true or false on certain intervals of $\mathcal{A}_p$ and $\mathcal{B}_p$. For any integers p and i with $1 \leq i \leq 2p$, we write $J_{i,p}$ for the interval $(2-\frac{i}{p}-\frac{1}{8p}, 2-\frac{i}{p}+\frac{5}{8p}) \cap \mathbb{R}_{\geq 0}$.

Lemma 3. *For any integers p and i with $1 \leq i \leq 2p$, any $\varphi \in \mathsf{MITL}_p$, and any $x, y \in J_{i,p}$,*

$$\mathcal{B}_p, x \models \varphi \iff \mathcal{B}_p, y \models \varphi.$$

Proof. We prove this lemma by induction on i. We first prove the induction step: assume the result holds up to $i-1$. We show the result for i by a second induction on the structure of φ. This induction is obvious if φ is an atomic proposition, or if it is the conjunction or negation of smaller subformulae.

The last case is when $\varphi = \varphi_1 \mathbf{U}_I \varphi_2$. In the sequel, q stands for $\frac{1}{p}$. Since $\varphi \in \mathsf{MITL}_p$, then I is one of $I = (k_1q, k_2q)$, $I = [k_1q, k_2q)$, $I = (k_1q, k_2q]$ or $I = [k_1q, k_2q]$, with $k_1 < k_2$. We show the induction hypothesis for all four cases by proving the stronger fact that, if there exists $x \in J_{i,p}$ such that $\mathcal{B}_p, x \models \varphi_1 \mathbf{U}_{[k_1q,k_2q]} \varphi_2$, then for all $y \in J_{i,p}$, $\mathcal{B}_p, y \models \varphi_1 \mathbf{U}_{(k_1q,k_2q)} \varphi_2$.

To prove this implication, we assume the existence of a position x of $J_{i,p}$ such that $\varphi_1 \mathbf{U}_{[k_1q,k_2q]} \varphi_2$ holds in that position along $\mathcal{B}_p$. We pick a position $y \in J_{i,p}$, and prove that $\mathcal{B}_p, y \models \varphi_1 \mathbf{U}_{(k_1q,k_2q)} \varphi_2$.

By construction of x, $\mathcal{B}_p, x+t \models \varphi_2$ for some $k_1q \leq t \leq k_2q$, and for any $t' \in (0,t)$, $\mathcal{B}_p, x+t' \models \varphi_1$. By induction hypothesis for φ_1, we know that for any $z \in J_{i,p}$, $\mathcal{B}_p, z \models \varphi_1$. This holds in particular between y and x if $y < x$.

We now have to distinguish between several cases depending on the values of x, y and t:

- Case $k_1 q < x + t - y < k_2 q$: this is the case where the witness for x is also correct for y. Taking $t' = x + t - y$, we know that φ_1 holds between y and x, so that $\mathcal{B}_p, y \models \varphi_1 \mathbf{U}_{(k_1 q, k_2 q)} \varphi_2$.
- Case $x + t - y \leq k_1 q$ (in particular $x \leq y$):
  - If $k_1 = 0$, then we have $x + t \leq y < 2 - iq + \frac{5q}{8}$. So by i.h., we have that φ_1 and φ_2 are satisfied everywhere in the interval $J_{i,p}$, so $\mathcal{B}_p, y \models \varphi_1 \mathbf{U}_{(k_1 q, k_2 q)} \varphi_2$.
  - If $k_1 \geq i$: w.l.o.g., we assume that $|x - y| < \frac{q}{2}$. The general case can be recovered by considering $z = \frac{(x+y)}{2}$ and applying the lemma twice. We have that $x + t > 2 - iq - \frac{q}{8} + k_1 q \geq 2 - \frac{q}{8}$. As the suffixes of $\mathcal{B}_p$ starting at $x + t$ and $x + t + \frac{q}{2}$ are the same, we have that $\mathcal{B}_p, x + t + \frac{q}{2} \models \varphi_2$. The point $x + t + \frac{q}{2}$ will be the witness for $\varphi_1 \mathbf{U}_{(k_1 q, k_2 q)} \varphi_2$ being true in y. We have to ensure
    1. that φ_1 is satisfied between the old witness $(x + t)$ and the new one $(x + t + \frac{q}{2})$: this holds because for any $0 \leq z < \frac{q}{2}$, the suffixes of $\mathcal{B}_p$ starting at $x + t + z$ and $x + t + z - \frac{q}{2}$ are identical, and $\mathcal{B}_p, x + t + z - \frac{q}{2}$ satisfies φ_1.
    2. that $t' = x + t - y + \frac{q}{2}$ is in the interval $(k_1 q, k_2 q)$: $t' \leq k_1 q + \frac{q}{2} < k_2 q$ (because $x + t - y \leq k_1 q$) and $t' > t \geq k_1 q$ (because $|x - y| < \frac{q}{2}$).
  - If $0 < k_1 < i$ (which entails that $i > 1$), we prove the existence of a witness in the interval $J_{i-k_1,p}$.
    We have that $x + t > 2 - iq - \frac{q}{8} + k_1 q = 2 - (i - k_1)q - \frac{q}{8}$, and $x + t = x + t - y + y < k_1 q + 2 - iq + \frac{5q}{8}$, so that $x + t$ is in $J_{i-k_1,p}$. We apply the i.h. at level $i - k_1$, and get that φ_1 and φ_2 are satisfied everywhere in $J_{i-k_1,p}$. Taking $t' = k_1 q + \frac{2 - iq + 5q/8 - y}{2}$, it is easily verified that $k_1 q < t' < k_2 q$ and $y + t' \in J_{i-k_1,p}$.
- Case $x + t - y \geq k_2 q$ (in particular $x > y$):
  - If $k_2 \geq i$: we again assume that $|x - y| < \frac{q}{2}$.
    Since $x + t = x + t - y + y > k_2 q + 2 - iq - \frac{q}{8} \geq 2 - \frac{q}{8}$ and $\mathcal{B}_p, x + t \models \varphi_2$, we get that $\mathcal{B}_p, x + t - \frac{q}{2} \models \varphi_2$. There remains to show that the new witness $t' = x + t - y - \frac{q}{2}$ is in the correct interval: we have $t' < t \leq k_2 q$ since $|x - y| < \frac{q}{2}$, and $t' \geq k_2 q - \frac{q}{2} > k_1 q$. Also, as shown earlier, φ_1 is satisfied between y and x.
  - If $k_2 < i$ (thus $i > 1$), we build another witness in the interval $J_{i-k_2,p}$. Again, $x + t = x + t - y + y > 2 + k_2 q - iq - \frac{q}{8}$ and $x + t < 2 - iq + 5\frac{q}{8} + k_2 q$ so $x + t$ is in $J_{i-k_2,p}$. We apply the i.h. at level $i - k_2$, and get that both φ_1 and φ_2 are satisfied everywhere in $J_{i-k_2,p}$. Taking $t' = k_2 q - \frac{y - (2 - iq - q/8)}{2}$, we easily conclude that it is a witness for $\varphi_1 \mathbf{U}_{(k_1 q, k_2 q)} \varphi_2$ being true in y.

The case $i = 1$ follows from the proof above, since the induction hypothesis is only needed in cases where $i > 1$. $\square$

We now easily deduce the following result:

Lemma 4. *For all $p \in \mathbb{N}_{>0}$, for all $\varphi \in \mathsf{MITL}_p$, $\mathcal{A}_p, 0 \models \varphi \iff \mathcal{B}_p, 0 \models \varphi$.*

Proof. Let φ be in MITL_p. Then

$$\begin{aligned}\mathcal{A}_p, 0 \models \varphi &\iff \mathcal{B}_p, \frac{p}{2} \models \varphi \text{ since the suffixes } \mathcal{A}_p, 0 \text{ and } \mathcal{B}_p, \frac{p}{2} \text{ are equal,}\\ &\iff \forall x \in \left[0, \frac{5p}{8}\right) \mathcal{B}_p, x \models \varphi \text{ by Lemma 3,}\\ &\iff \mathcal{B}_p, 0 \models \varphi\end{aligned}$$

$\square$

As a side result, Propositions 1 and 2 entail the following theorem:

Theorem 5. MTL *is strictly more expressive than* MITL *in the continuous semantics.*

This result was already known: MITL formulas can be translated into timed automata [AFH96], and thus can only express time-regular properties, while MTL can express non-timed-regular languages.

3.3. TPTL *vs* MTL *in the Pointwise Semantics*

We now show the following result:

Proposition 6. *The* TPTL *formula $\Phi = x.(\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2)))$ has no equivalent* MTL *formula for the pointwise semantics.*

Proof. We keep the notations of Section 3.2, and in particular we consider again the families of models (now seen as timed words) $(\mathcal{A}_n)_{n \in \mathbb{N}_{>0}}$ and $(\mathcal{B}_n)_{n \in \mathbb{N}_{>0}}$ depicted on Figure 3. As previously, MTL_p denotes the fragment of MTL with formulae of granularity p. As in the previous section, we will prove that:

(a) $\mathcal{A}_p \models \Phi$ whereas $\mathcal{B}_p \not\models \Phi$ for every $p \in \mathbb{N}_{>0}$,

(b) for all $p \in \mathbb{N}_{>0}$ and all $\varphi \in \mathsf{MTL}_p$, $\mathcal{A}_p \models \varphi \iff \mathcal{B}_p \models \varphi$.

Equation (a) is obvious. We prove Equation (b) with the following two lemmas:

Lemma 7. *For any $p \in \mathbb{N}_{>0}$, for any $k, k' \geq 1$ such that $k = k' \mod 2$, and for any formula $\varphi \in \mathsf{MTL}_p$,*

$$\mathcal{A}_p, k \models \varphi \iff \mathcal{B}_p, k \models \varphi \iff \mathcal{A}_p, k' \models \varphi \iff \mathcal{B}_p, k' \models \varphi.$$

This result is straightforward, since the suffixes $\mathcal{A}_p, k$, $\mathcal{B}_p, k$, $\mathcal{A}_p, k'$, and $\mathcal{B}_p, k'$ are the same.

Lemma 8. *For all $p \in \mathbb{N}_{>0}$ and all $\varphi \in \mathsf{MTL}_p$, $\mathcal{A}_p, 0 \models \varphi \iff \mathcal{B}_p, 0 \models \varphi$.*

We proceed by induction on the structure of formula φ. The case of atomic propositions is easy, as well as the induction steps for conjunction and negation. Again, we write q for $\frac{1}{p}$.

Assume $\varphi = \varphi_1 \, \mathbf{U}_I \, \varphi_2$. Note that for all $k \in \mathbb{N}$ there is no action at time kq in $\mathcal{A}_p$ or $\mathcal{B}_p$. It follows that for all $k_1, k_2 \in \mathbb{N}$,

$$\begin{aligned}\mathcal{A}_p \models \varphi_1 \, \mathbf{U}_{[k_1 q, k_2 q]} \, \varphi_2 \iff \mathcal{A}_p \models \varphi_1 \, \mathbf{U}_{(k_1 q, k_2 q]} \, \varphi_2 \\ \iff \mathcal{A}_p \models \varphi_1 \, \mathbf{U}_{[k_1 q, k_2 q)} \, \varphi_2 \iff \mathcal{A}_p \models \varphi_1 \, \mathbf{U}_{(k_1 q, k_2 q)} \, \varphi_2\end{aligned}$$

and the same holds for $\mathcal{B}_p$. W.l.o.g., we can assume that $I = (k_1 q, k_2 q)$.

In what follows, we write τ_i for the date associated to position i in $\mathcal{A}_p$, and τ'_j for the time associated to position j in $\mathcal{B}_p$.

- We first suppose that $\mathcal{A}_p, 0 \models \varphi$, and show that $\mathcal{B}_p, 0 \models \varphi$. We know that there exists $i > 0$ with $\tau_i \in I$, $\mathcal{A}_p, i \models \varphi_2$, and such that for any $0 < k < i$, $\mathcal{A}_p, k \models \varphi_1$. We distinguish between two subcases:
  - If $i \geq 3$: we take $j = i - 2$; then $\tau'_j = \tau_i$, and $\tau'_j \in I$. By Lemma 7, we get that $\mathcal{B}_p, j \models \varphi_2$. Since $\mathcal{A}_p, 1$ and $\mathcal{A}_p, 2$ satisfy φ_1, this lemma also entails that for any $k > 0$, $\mathcal{B}_p, k \models \varphi_1$. Thus $\mathcal{B}_p, 0 \models \varphi_1 \, \mathbf{U}_I \, \varphi_2$
  - If $1 \leq i \leq 2$: then $\tau_i \in \{2 - \frac{7q}{8}, 2 - \frac{5q}{8}\}$, which entails $k_2 q \geq 2$. Taking $j = i$ and applying Lemma 7, we obtain that $\mathcal{B}_p, j \models \varphi_2$ and, for all $0 < k < j$, $\mathcal{B}_p, k \models \varphi_1$. Since $j = i$, we have $\tau'_j = \tau_i + \frac{q}{2}$, so that $\tau'_j \geq \tau_i > k_1 q$ and $\tau'_j \leq 2 - \frac{q}{8} \leq 2 \leq k_2 q$. Thus, $\mathcal{B}_p, 0 \models \varphi_1 \, \mathbf{U}_{(k_1 q, k_2 q)} \, \varphi_2$, and *a fortiori* $\mathcal{B}_p, 0 \models \varphi_1 \, \mathbf{U}_I \, \varphi_2$
- Conversely, assume that $\mathcal{B}_p, 0 \models \varphi$. Then there exists $j > 0$ such that $\tau'_j \in I$, $\mathcal{B}_p, j \models \varphi_2$, and for any $0 < k < j$, $\mathcal{B}_p, k \models \varphi_1$. Two subcases may arise:
  - If $j \geq 3$: we then take $i = j + 2$. In that case, $\tau_i = \tau'_j$, and $\tau_i \in I$. From Lemma 7, we deduce that $\mathcal{A}_p, i \models \varphi_2$. Again, since $\mathcal{B}_p, 1$ and $\mathcal{B}_p, 2$ satisfy φ_1, Lemma 7 entails that for any $k > 0$, $\mathcal{A}_p, k \models \varphi_1$. Thus $\mathcal{A}_p, 0 \models \varphi_1 \, \mathbf{U}_I \, \varphi_2$
  - If $1 \leq j \leq 2$: then $\tau'_j \in \{2 - \frac{3q}{8}, 2 - \frac{q}{8}\}$, which entails that $k_1 q \leq 2 - q$. We take $i = j$: Lemma 7 ensures that $\mathcal{A}_p, i \models \varphi_2$ and that, for any $0 < k < i$, $\mathcal{A}_p, k \models \varphi_1$. We also have $\tau_i = \tau'_j - \frac{q}{2}$, so that $\tau_i < \tau'_j \leq k_2 p$ and $\tau_i \geq 2 - \frac{7q}{8} > 2 - q \geq k_1 q$. So $\mathcal{A}_p, 0 \models \varphi_1 \, \mathbf{U}_{(k_1 q, k_2 q)} \, \varphi_2$, and $\mathcal{A}_p, 0 \models \varphi_1 \, \mathbf{U}_I \, \varphi_2$. □

As a direct corollary of Proposition 6, we have:

Theorem 9. TPTL *is strictly more expressive than* MTL *for the pointwise semantics.*

Since the $\mathsf{MITL{+}Past}$ formula $\mathbf{F}_{\leq 2} \, (c \wedge \top \, \mathbf{S} \, b)$ also distinguishes between the families $(\mathcal{A}_p)_{p \in \mathbb{N}_{>0}}$ and $(\mathcal{B}_p)_{p \in \mathbb{N}_{>0}}$, we get the following corollary:

Corollary 10. *Under the pointwise semantics,* MTL+Past *and* MITL+Past *are strictly more expressive than* MTL *and* MITL*, resp.*

The result for MITL was already proved differently in [AH92b]. To our knowledge, the result concerning MTL was not known before (though it was expected since, on finite timed words, MTL is decidable while MTL+Past is not). Note that Corollary 10 constitutes a main difference between the timed and the untimed framework, where it is well-known that adding past-time modalities does not increase the expressive power of LTL over discrete time [Kam68, GPSS80].

3.4. TPTL *vs* MTL *in the Interval-Based Semantics*

According to Proposition 1, the formula which has been used for the pointwise semantics can not be used for the interval-based semantics. We will instead prove the following proposition:

Proposition 11. *The* TPTL *formula* $\Phi = x.\mathbf{F}\,(a \wedge x \leq 1 \wedge \mathbf{G}\,(x \leq 1 \Rightarrow \neg b))$ *has no equivalent in* MTL *for the interval-based semantics.*

Proof. Let $p \in \mathbb{N}_{>0}$, and $q = \frac{1}{p}$. Assume that Φ is equivalent to an MTL formula Ψ. Even if it means increasing the temporal height (*i.e.*, the maximal number of nested modalities), we may assume that Ψ only involves constraints of the form $\sim q$, with $\sim \in \{<, =, >\}$. We write $\mathsf{MTL}^-_{p,n}$ for the fragment of MTL using only $\sim q$ constraints and with temporal height at most n, and assume that $\Psi \in \mathsf{MTL}^-_{p,n_0}$ for some n_0.

The proof consists in building two families of timed state sequences $(\mathcal{A}_{p,n})_{n \geq 3}$ and $(\mathcal{B}_{p,n})_{n \geq 3}$ such that, for any $n \geq 3$,

(a) Φ holds initially in $\mathcal{A}_{p,n}$ but not in $\mathcal{B}_{p,n}$.

(b) $\mathcal{A}_{p,n}$ and $\mathcal{B}_{p,n}$ cannot be distinguished by any formula in $\mathsf{MTL}^-_{p,n-3}$.

We first define $\mathcal{A}_{p,n}$. Along that timed state sequence, atomic proposition a holds exactly at time points $\frac{q}{4n} + \alpha\frac{q}{2n}$, where α may be any nonnegative integer. Atomic proposition b will hold exactly at times $(\alpha + 1) \cdot \frac{q}{2} - \frac{4q}{6n}$, with $\alpha \in \mathbb{N}$.

As for $\mathcal{B}_{p,n}$, it has exactly the same a's, while b holds exactly at time points $(\alpha + 1) \cdot \frac{q}{2} - \frac{q}{6n}$, with $\alpha \in \mathbb{N}$.

The portions between 0 and $\frac{q}{2}$ of both timed state sequences is represented on Figure 4. Both timed state sequences are in fact periodic, with period $\frac{q}{2}$. Note that the situation around time point 1 is similar to the situation around $\frac{q}{2}$. Hence Φ holds in $\mathcal{A}_{p,n}$ and fails to hold in $\mathcal{B}_{p,n}$.

The following lemma is straightforward since, for each equivalence, the suffixes of the paths are the same.

Lemma 12. *For any positive p and n, for any nonnegative real x, and for any*

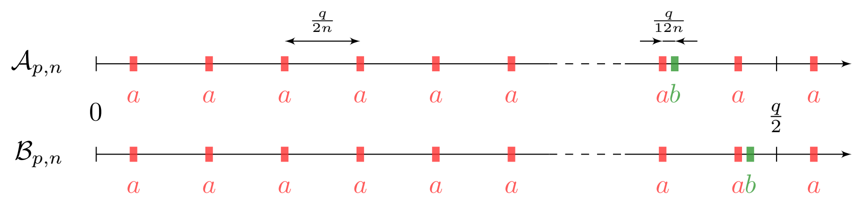


Figure 4: Two timed paths $\mathcal{A}_{p,n}$ and $\mathcal{B}_{p,n}$

MTL-*formula* φ, *letting* $q = \frac{1}{p}$, *we have the following properties:*

$$\mathcal{A}_{p,n}, x \models \varphi \iff \mathcal{B}_{p,n}, x + \frac{q}{2n} \models \varphi \tag{2}$$

$$\mathcal{A}_{p,n}, x \models \varphi \iff \mathcal{A}_{p,n}, x + \frac{q}{2} \models \varphi \tag{3}$$

$$\mathcal{B}_{p,n}, x \models \varphi \iff \mathcal{B}_{p,n}, x + \frac{q}{2} \models \varphi \tag{4}$$

We can now prove the following lemma:

Lemma 13. *Let* $p \in \mathbb{N}_{>0}$, *and* $q = \frac{1}{p}$. *For any* $k \leq n$, *for any* $\varphi \in \mathsf{MTL}^{-}_{p,k}$, *for any* $x \in \left[0, \frac{q}{2} - \frac{(k+2)q}{2(n+3)}\right)$, *for any* $\alpha \in \mathbb{N}$, *we have*

$$\mathcal{A}_{p,n+3}, \alpha\frac{q}{2} + x \models \varphi \iff \mathcal{B}_{p,n+3}, \alpha\frac{q}{2} + x \models \varphi$$

Proof. The proof is by induction on both k and the structure of the formula φ. In order to (try to) improve readability, we write $\mathcal{A}$ and $\mathcal{B}$ for $\mathcal{A}_{p,n+3}$ and $\mathcal{B}_{p,n+3}$, resp., and we let $\delta = \frac{q}{2(n+3)}$.

- The case where $k = 0$ is easy, since φ may only be an atomic proposition, and all positions in the interval we consider are labeled with the same propositions.
- Assume the result holds for some $k < n$. We prove it for $k + 1$.
  - the case of atomic propositions and boolean combinations is still straightforward.
  - Assume $\varphi = \varphi_1 \mathbf{U}_{=q} \varphi_2$: pick some value $x \in \left[0, \frac{q}{2} - ((k+1)+2)\delta\right)$ and $\alpha \in \mathbb{N}$, and assume $\mathcal{A}, \alpha\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{=q} \varphi_2$. Then φ_2 holds at position $(\alpha+2)\frac{q}{2} + x$, and φ_1 holds at all intermediate positions. Applying the induction hypothesis, we get that $\mathcal{B}, (\alpha+2)\frac{q}{2} + x \models \varphi_2$. We also obtain that φ_1 holds along $\mathcal{B}$ at positions between $\alpha\frac{q}{2} + x$ and $\alpha\frac{q}{2} + x + \delta$. It also holds at positions between $\alpha\frac{q}{2} + x + \delta$ and $(\alpha+2)\frac{q}{2} + x$ thanks to equation (2). This entails that $\mathcal{B}, \alpha\frac{q}{2} + x \models \varphi$. Conversely, assume that $\mathcal{B}, \alpha\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{=q} \varphi_2$. With the induction hypothesis, we get that $\mathcal{A}, (\alpha+2)\frac{q}{2} + x \models \varphi_2$. From equation (2), we

know that φ_1 holds between $\alpha\frac{q}{2} + x$ and $(\alpha+2)\frac{q}{2} + x - \delta$ along $\mathcal{B}$. Last, equation (3) ensures that it also holds between $(\alpha+2)\frac{q}{2} + x - \delta$ and $(\alpha+2)\frac{q}{2} + x$, which completes the proof.

– Assume $\varphi = \varphi_1 \mathbf{U}_{<q} \varphi_2$: pick some value $x \in \big[0, \frac{q}{2} - ((k+1)+2)\delta\big)$ and $\alpha \in \mathbb{N}$, and assume $\mathcal{A}, \alpha\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{<q} \varphi_2$.

  ∗ If the witness for φ_2 lies between $\alpha\frac{q}{2} + x$ and $(\alpha+1)\frac{q}{2} + x$, then by applying equation (2), we get that $\mathcal{B}, \alpha\frac{q}{2} + x + \delta \models \varphi_1 \mathbf{U}_{<\frac{q}{2}} \varphi_2$. The induction hypothesis ensures that φ_1 holds on timed state sequence $\mathcal{B}$ between $\alpha\frac{q}{2} + x$ and $\alpha\frac{q}{2} + x + \delta$, and we deduce that $\mathcal{B}, \alpha\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{<q} \varphi_2$.

  ∗ Now, if the witness lies between $(\alpha+1)\frac{q}{2} + x$ and $(\alpha+2)\frac{q}{2} + x$, with equation (3), there is also a possible witness between $\alpha\frac{q}{2} + x$ and $(\alpha+1)\frac{q}{2} + x$, and we apply the previous proof.

  Conversely, assume $\mathcal{B}, \alpha\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{<q} \varphi_2$. We still consider two cases:

  ∗ If the witness for φ_2 lies between $\alpha\frac{q}{2} + x$ and $\alpha\frac{q}{2} + x + \delta$, we can apply the induction hypothesis to φ_1 and φ_2, and we get the result.

  ∗ Otherwise, it suffices to apply equation (2).

– Last, assume that $\varphi = \varphi_1 \mathbf{U}_{>q} \varphi_2$: Pick some value x in the interval $\big[0, \frac{q}{2} - ((k+1)+2)\delta\big)$ and $\alpha \in \mathbb{N}$, and assume $\mathcal{A}, \alpha\frac{q}{2}+x \models \varphi_1 \mathbf{U}_{>q} \varphi_2$. By applying equation (2), and the induction hypothesis for φ_1, we get that $\mathcal{B}, \alpha\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{>q} \varphi_2$.

  Conversely, if $\mathcal{B}, (\alpha+2)\frac{q}{2} + x \models \varphi_1 \mathbf{U}_{>q} \varphi_2$, if the witnessing position for φ_2 lies after $\alpha\frac{q}{2}+x+\delta$, it suffices to apply equation (2). Otherwise, equation (3) ensures that we can find another witness for φ_2 satisfying this condition. This completes the proof. □

As a corollary of this lemma, when $k = n$ and $\alpha = x = 0$, we get that any formula in $\mathsf{MTL}^-_{p,n}$ cannot distinguish between models $\mathcal{A}_{p,n+3}$ and $\mathcal{B}_{p,n+3}$. In particular, formula Ψ should satisfy both (a) and (b) for $n = n_0$, which is contradictory. This concludes the proof of Proposition 11.

The following theorem immediately follows:

Theorem 14. TPTL *is strictly more expressive than* MTL *for the interval-based semantics.*

Note that the formula $x.\mathbf{F}\,(a \wedge x \leq 1 \wedge \mathbf{G}\,(x \leq 1 \Rightarrow \neg b))$ does not use modality $\mathbf{U}$, so the fragment of TPTL using only modalities $\mathbf{F}$ and $\mathbf{G}$ is also strictly more expressive than the corresponding fragment of MTL. This is not the case for the fragment of TPTL using only the $\mathbf{F}$ modality (see Section 4).

Now, clearly enough, the $\mathsf{MTL}+\mathsf{Past}$ formula $\mathbf{F}_{=1}(\neg b\,\mathbf{S}\,a)$ distinguishes between the two families of models[4] $(\mathcal{A}_{p,n})_{p\in\mathbb{N}_{>0},n\in\mathbb{N}_{>0}}$ and $(\mathcal{B}_{p,n})_{p\in\mathbb{N}_{>0},n\in\mathbb{N}_{>0}}$. So does the more involved $\mathsf{MITL}+\mathsf{Past}$ formula

$$\mathbf{F}_{\geq 1}(\neg a \wedge \mathbf{F}^{-1}_{\geq -1}(\mathbf{G}^{-1}\neg a) \wedge \neg b\,\mathbf{S}\,a). \tag{5}$$

Indeed, the subformula $\mathbf{F}^{-1}_{\geq -1}(\mathbf{G}^{-1}\neg a)$ requires that there is a point not too far away in the past (at most 1 time unit ago) such that a has never been true in the past. That point is necessarily between dates 0 and $\frac{q}{4n}$, and $\mathbf{F}^{-1}_{\geq -1}(\mathbf{G}^{-1}\neg a)$ is true precisely between dates 0 and $1+\frac{q}{4n}$. Thus, formula (5) states that there is a point between dates 1 and $1+\frac{q}{4n}$ at which $\neg b\,\mathbf{S}\,a$ holds. This formula is satisfied in $\mathcal{A}_{p,n}$, for any n and p, and it is not satisfied in any $\mathcal{B}_{p,n}$. We then get the following corollary:

Corollary 15. $\mathsf{MTL}+\mathsf{Past}$ *(resp.* $\mathsf{MITL}+\mathsf{Past}$*) is strictly more expressive than* MTL *(resp.* MITL*) for the interval-based semantics.*

To our knowledge, these are the first expressiveness results for timed linear-time temporal logics using past-time modalities under the interval-based semantics.

4. On the Existential Fragments of MTL and TPTL

$\mathsf{TPTL}_{\mathbf{F}}$ is the fragment of TPTL which only uses modality $\mathbf{F}$ (and not the general modality $\mathbf{U}$) and restricts negation to atomic propositions. Formally, $\mathsf{TPTL}_{\mathbf{F}}$ is defined by the following grammar:

$$\mathsf{TPTL}_{\mathbf{F}} \ni \varphi ::= p \mid \neg p \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{F}\,\varphi \mid x \sim c \mid x.\varphi.$$

An example of a $\mathsf{TPTL}_{\mathbf{F}}$ formula is $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2))$ (see Subsection 3.1). Similarly we define the fragment $\mathsf{MTL}_{\mathbf{F}}$ of MTL where only $\mathbf{F}$-modalities are allowed:

$$\mathsf{MTL}_{\mathbf{F}} \ni \varphi ::= p \mid \neg p \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{F}_I\,\varphi.$$

From Subsection 3.3, we know that, under the pointwise semantics, $\mathsf{TPTL}_{\mathbf{F}}$ is strictly more expressive than $\mathsf{MTL}_{\mathbf{F}}$, since formula $x.\mathbf{F}\,(b \wedge \mathbf{F}\,(c \wedge x \leq 2))$ has no equivalent in MTL (thus in $\mathsf{MTL}_{\mathbf{F}}$). On the other hand, when considering the interval-based semantics, we proved that the formula above can be expressed in $\mathsf{MTL}_{\mathbf{F}}$ (see Subsection 3.1). In this section, we generalize the construction of Subsection 3.1, and prove that $\mathsf{TPTL}_{\mathbf{F}}$ and $\mathsf{MTL}_{\mathbf{F}}$ have the same expressive power in the interval-based semantics.

Theorem 16. $\mathsf{TPTL}_{\mathbf{F}}$ *and* $\mathsf{MTL}_{\mathbf{F}}$ *are equally expressive for the interval-based semantics.*

[4] Note that this formula is *not* equivalent to the formula used in Proposition 11, but that it is sufficient for our purpose that it distinguishes between the two families of models.

PROOF. We assume w.l.o.g. that all constants appearing in formulae of $\mathsf{TPTL}_{\mathbf{F}}$ are integers. For every $\mathsf{TPTL}_{\mathbf{F}}$ formula, we build an equivalent $\mathsf{MTL}_{\mathbf{F}}$ formula for the interval-based semantics. The construction proceeds in six steps. Example 4, displayed on page 26, illustrates the whole transformation.

1. Normal form of $\mathsf{TPTL}_{\mathbf{F}}$ *formulae..* Even if it means adding extra clocks, we assume that all occurrences of the $\mathbf{F}$-modality are directly embedded into some reset operator "$x.$", and that any clock x appearing in the formula is reset only once. Thus, we only consider formulae of the logic defined by

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid x \sim c \mid x.\mathbf{F}\,\varphi \tag{6}$$

and such that each clock appears at most once in a reset quantifier "$x.\mathbf{F}$ ".

We now recursively build a normal form for $\mathsf{TPTL}_{\mathbf{F}}$ formulae, which is some kind of disjunctive normal form. We call *atom* an atomic proposition or its negation.

Definition 17. *A* $\mathsf{TPTL}_{\mathbf{F}}$ *formula is* simple *if it is generated by the grammar*

$$\psi ::= a \mid x \sim c \mid x.\mathbf{F}\,\psi \mid \psi \wedge \psi$$

where a is an atom and $x \sim c$ is a clock constraint.

The following lemma is straightforward, using the property that $x.\mathbf{F}\,(\varphi_1 \vee \varphi_2)$ is equivalent to $(x.\mathbf{F}\,\varphi_1) \vee (x.\mathbf{F}\,\varphi_2)$.

Lemma 18. *Every* $\mathsf{TPTL}_{\mathbf{F}}$ *formula is equivalent to some positive Boolean combination of simple* $\mathsf{TPTL}_{\mathbf{F}}$ *formulae.*

The initial problem thus reduces to constructing equivalent $\mathsf{MTL}_{\mathbf{F}}$ formulae for simple $\mathsf{TPTL}_{\mathbf{F}}$ formulae.

2. From simple $\mathsf{TPTL}_{\mathbf{F}}$ *formulae to systems of difference inequations..* In this part, we recursively transform a $\mathsf{TPTL}_{\mathbf{F}}$ formula into a system of inequations, where we will associate with every eventuality $\varphi = x.\mathbf{F}\,\psi$ a date y_ψ at which ψ will hold, and a date y_φ at which φ will hold. This yields conditions between variables and the other dates and clocks which already appear in the transformation.

We first define what we call *systems of difference inequations*, which will be associated to $\mathsf{TPTL}_{\mathbf{F}}$ formulae.

Definition 19. *Let X be a finite set of clocks, and Y be a finite set of variables, disjoint from X. A* system $\mathcal{S}$ *over X and Y is a pair $(V, \mathcal{J})$ where $V : Y \to \mathsf{MTL}_{\mathbf{F}}$ associates with every variable $y \in Y$ an $\mathsf{MTL}_{\mathbf{F}}$ formula $V(y)$, and $\mathcal{J}$ is a Boolean combination of (difference) inequations of the form $x - x' \sim c$ or $x \sim c$ where x, x' are elements of $X \cup Y$, $\sim \in \{<, \leq, =, \geq, >\}$, and $c \in \mathbb{Z}$ is an integer.*

Intuitively, such a system $\mathcal{S}$ represents a property over timed state sequences where $\mathsf{MTL}_{\mathbf{F}}$ formulae given by V have to be satisfied at dates satisfying the constraints given by $\mathcal{J}$.

Let $\mathcal{S} = (V, \mathcal{J})$ be a system over X and Y, κ be a timed state sequence, $v\colon Y \to \mathbb{R}_{\geq 0}$ be a function assigning a time-point to every variable $y \in Y$, and $v'\colon X \to \mathbb{R}_{\geq 0}$ a valuation for clocks in X. We say that $\kappa, v, v' \vdash \mathcal{S}$ when, writing $v \sqcup v'$ for the function naturally extending v and v', the following properties are satisfied:

$$v \sqcup v' \models \mathcal{J} \quad \text{and} \quad \forall y \in Y,\ \kappa, v(y) \models_{ib} V(y).$$

The satisfaction relation for systems is then defined by[5]:

$$\kappa, t, v' \models \mathcal{S} \quad \text{iff} \quad \exists v\colon Y \to \mathbb{R}_{\geq 0} \text{ s.t. } \kappa, v, v' \vdash \mathcal{S}, \\ \forall\, y \in Y,\ v(y) \geq t \text{ and } \exists y_0 \in Y,\ v(y_0) = t.$$

Let φ be a simple $\mathsf{TPTL}_{\mathbf{F}}$ formulas with set of formula clocks X_φ. We explain how to inductively build a system $\mathcal{S}_\varphi = (V_\varphi, \mathcal{J}_\varphi)$ over X_φ and some set of variables Y_φ such that:

$$\kappa, 0, v' \models_{ib} \varphi \quad \text{iff} \quad \kappa, 0, v' \models \mathcal{S}_\varphi. \tag{7}$$

- If φ is an atom, the set of variables Y_φ contains a single variable y_φ, the system has no constraint, and $V_\varphi(y_\varphi) = \varphi$.

- If φ is a clock constraint $x \sim c$, the set Y_φ contains a single variable y_φ, the system $\mathcal{J}_\varphi$ is $(y_\varphi - x \sim c)$, and $V_\varphi(y_\varphi) = \top$. Intuitively, y_φ will represent the date at which $x \sim c$ needs to hold, whereas x will represent the date at which clock x is reset.

- We assume that φ is of the form $x.\mathbf{F}\,\psi$. We assume we have already computed a system $\mathcal{S}_\psi = (V_\psi, \mathcal{J}_\psi)$ over X_ψ and Y_ψ which corresponds to ψ in the sense of equivalence (7). The construction of the system $\mathcal{S}_\varphi = (V_\varphi, \mathcal{J}_\varphi)$ is then done as follows. The set of variables Y_φ is $Y_\psi \cup \{y_\varphi\}$ where y_φ is a fresh variable representing the date at which formula φ will hold. For every variable $y \in Y_\varphi$, $V_\varphi(y) = V_\psi(y)$ if $y \in Y_\psi$, and $V_\varphi(y_\varphi) = \top$. The system $\mathcal{J}_\varphi$ is defined as $\bigwedge_{y \in Y_\psi} (y_\varphi < y) \wedge \mathcal{J}_\psi[x \leftarrow y_\varphi]$, where $\mathcal{J}_\psi[x \leftarrow y_\varphi]$ is the system $\mathcal{J}_\psi$ in which variable x has been replaced by y_φ (roughly, the current date, represented by variable y_φ, corresponds to the date at which clock x is reset).

- We assume that φ is of the form $\bigwedge_{k=1}^{h} \varphi_k$, where φ_k is a simple $\mathsf{TPTL}_{\mathbf{F}}$ formula. We assume we have already computed, for each $1 \leq k \leq h$, a system $\mathcal{S}_{\varphi_k} = (V_{\varphi_k}, \mathcal{J}_{\varphi_k})$ over X_{φ_k} and Y_{φ_k} which corresponds to φ_k in

[5] As the system is evaluated at time t, at least one of the variables of the valuation will be mapped to t.

the sense of equivalence (7). The construction of the system $\mathcal{S}_\varphi = (V_\varphi, \mathcal{J}_\varphi)$ is then achieved as follows. The set of variables Y_φ is $\bigcup_{k=1}^{h}(Y_{\varphi_k} \setminus \{y_{\varphi_k}\}) \cup \{y_\varphi\}$, where y_φ is a fresh variable representing the date at which the subformula φ will hold. The system $\mathcal{S}_\varphi = (V_\varphi, \mathcal{J}_\varphi)$ is then defined as follows: $V_\varphi(y_\varphi) = \bigwedge_{1\leq k\leq h} V_{\varphi_k}(y_{\varphi_k})$, and $V_\varphi(y) = V_{\varphi_k}(y)$ if $y \in Y_{\varphi_k} \setminus \{y_{\varphi_k}\}$. The system $\mathcal{J}_\varphi$ is defined as

$$\left(\bigwedge_{k=1}^{h} \mathcal{J}_{\varphi_k}[y_{\varphi_k} \leftarrow y_\varphi]\right) \wedge \left(\bigwedge_{y\in Y_\varphi} y_\varphi \leq y\right)$$

Remark 1.
- *Note that, by construction, for every formula φ, there is a variable $y_\varphi \in Y_\varphi$ such that $\mathcal{J}_\varphi$ implies $y_\varphi \leq y$ for every $y \in Y_\varphi$.*
- *Note that writing $\mathcal{S}_\varphi = (V_\varphi, \mathcal{J}_\varphi)$, if φ is closed (i.e., if every clock $x \in X_\varphi$ is under the scope of the resetting operator 'x.'), then there are no constraints on variables of X_φ in the inequation system $\mathcal{J}_\varphi$.*

Example 3. *For the formula $x_1.\mathbf{F}\,(a \wedge x_2.\mathbf{F}\,(b \wedge x_1 \leq 2))$, the system obtained from the above inductive transformation is:*

$$\mathcal{S} = \begin{cases} V\colon \quad y_1 \mapsto a \\ \qquad\; y_2 \mapsto b \\ \mathcal{J} = (y_2 - y_0 \leq 2) \wedge (y_1 < y_2) \wedge (y_0 < y_1) \wedge (y_0 < y_2) \end{cases}$$

It is just a technical matter to prove the following lemma, establishing the correctness of the construction:

Lemma 20. $\kappa, t, v' \models_{ib} \varphi \iff \kappa, t, v' \models \mathcal{S}_\varphi$.

Proof. The proof is by induction on the structure of φ. The case of atoms and clock constraints is obvious. We next assume that φ is of the form $x.\mathbf{F}\,\psi$.

$$\kappa, t, v' \models_{ib} x.\mathbf{F}\,\psi$$

$$\iff \exists t' > t \text{ s.t. } \kappa, t', v'[x \mapsto t] \models_{ib} \psi$$

$$\iff \exists t' > t \text{ s.t. } \kappa, t', v'[x \mapsto t] \models \mathcal{S}_\psi \qquad \text{(by induction hypothesis)}$$

$$\iff \exists t' > t.\ \exists v\colon Y_\psi \to \mathbb{R}_{\geq 0} \text{ s.t. } \begin{cases} \kappa, v, v'[x \mapsto t] \vdash \mathcal{S}_\psi \\ \forall y \in Y_\psi.\ v(y) \geq t' \\ v(y_\psi) = t' \end{cases}$$

(by definition, and because y_ψ is the smallest variable in $\mathcal{J}_\psi$)

$$\iff \exists t' > t.\ \exists v\colon Y_\varphi \to \mathbb{R}_{\geq 0} \text{ s.t. } \begin{cases} v_{|Y_\psi} \sqcup v'[x \mapsto t] \models \mathcal{J}_\psi \\ \forall y \in Y_\psi.\ \kappa, v(y) \models_{ib} V_\psi(y) \\ \forall y \in Y_\psi.\ v(y) \geq t' \\ v(y_\psi) = t' \\ v(y_\varphi) = t \\ \kappa, v(y_\varphi) \models_{ib} V_\varphi(y_\varphi) = \top \end{cases}$$

$$\iff \exists v\colon Y_\varphi \to \mathbb{R}_{\geq 0} \text{ s.t. } \begin{cases} v \sqcup v'[x \mapsto t] \models \mathcal{J}_\varphi \\ \forall y \in Y_\varphi.\ \kappa, v(y) \models_{ib} V_\varphi(y) \\ \forall y \in Y_\varphi.\ v(y) \geq t \\ v(y_\varphi) = t \end{cases}$$

(by definition of $\mathcal{S}_\varphi$)

$$\iff \kappa, t, v' \models \mathcal{S}_\varphi \qquad \text{(because } \mathcal{J}_\varphi \text{ does not constrain variable } x\text{)}$$

We finally assume that φ is of the form $\bigwedge_{k=1}^{h} \varphi_k$.

$$\kappa, t, v' \models_{ib} \bigwedge_{k=1}^{h} \varphi_k \iff \forall 1 \leq k \leq h.\ \kappa, t, v' \models_{ib} \varphi_k$$

$$\iff \forall 1 \leq k \leq h.\ \kappa, t, v' \models \mathcal{S}_{\varphi_k}$$

(by induction hypothesis)

$$\iff \begin{cases} \forall 1 \leq k \leq h.\ \exists v_k\colon Y_{\varphi_k} \to \mathbb{R}_{\geq 0} \text{ s.t.} \\ \qquad \begin{cases} \forall y \in Y_{\varphi_k}.\ v_k(y) \geq t, \\ v_k(y_{\varphi_k}) = t, \\ v_k \sqcup v' \models \mathcal{J}_{\varphi_k}, \\ \forall y \in Y_{\varphi_k},\ \kappa, v_k(y) \models_{ib} V_{\varphi_k}(y) \end{cases} \end{cases}$$

$$\iff \exists v\colon Y_\varphi \to [t, +\infty) \text{ s.t. } \begin{cases} v(y_\varphi) = t, \\ v \sqcup v' \models \mathcal{J}_\varphi, \\ \forall y \in Y_\varphi,\ \kappa, v(y) \models V_\varphi(y) \end{cases}$$

$$\iff \kappa, t, v' \models \mathcal{S}_\varphi$$

This concludes the proof of Lemma 20. □

3. Some properties of systems of difference inequations.. Let $\mathcal{S}$ be a system over Y and ψ be an $\mathsf{MTL}_{\mathbf{F}}$ formula. We say that $\mathcal{S}$ and ψ are *equivalent* if, for every timed state sequence κ,

$$\kappa, 0, \mathbf{0} \models \mathcal{S} \quad \text{iff} \quad \kappa, 0 \models_{ib} \psi.$$

Our goal is thus to build an $\mathsf{MTL}_{\mathbf{F}}$ formula ψ equivalent to $\mathcal{S}_\varphi$, where φ is a simple $\mathsf{TPTL}_{\mathbf{F}}$ formula.

We say that two systems $\mathcal{S} = (V, \mathcal{J})$ and $\mathcal{S}' = (V', \mathcal{J}')$ are *equivalent* whenever $V = V'$, and $\mathcal{J}$ and $\mathcal{J}'$ have the same solutions. Note that two equivalent systems represent $\mathsf{TPTL}_{\mathbf{F}}$ formulae that are equivalent over timed state sequences.

The following lemma holds rather straightforwardly.

Lemma 21. *Let $\mathcal{S}_1 = (V, \mathcal{J}_1)$ and $\mathcal{S}_2 = (V, \mathcal{J}_2)$ be two systems over X and Y. Let $\mathcal{S} = (V, \mathcal{J})$ be a system over X and Y such that the set of solutions of $\mathcal{J}$ is the union of the sets of solutions of $\mathcal{J}_1$ and $\mathcal{J}_2$. Then*

$$\kappa, t, v' \models \mathcal{S} \iff \kappa, t, v' \models \mathcal{S}_1 \text{ or } \kappa, t, v' \models \mathcal{S}_2.$$

PROOF. Assume that $\kappa, t, v' \models \mathcal{S}$. There exists $v\colon Y \to \mathbb{R}_{\geq 0}$ such that $\kappa, v, v' \vdash \mathcal{S}$, for all $y \in Y$, $v(y) \geq t$, and there exists $y_0 \in Y$ such that $v(y_0) = t$. By definition of the $\vdash$ satisfaction relation, we have that $v \sqcup v' \models \mathcal{J}$, and for every $y \in Y$, $\kappa, v(y), v' \models_{ib} V(y)$. As the set of solutions of $\mathcal{J}$ is the union of the sets of solutions of $\mathcal{J}_1$ and $\mathcal{J}_2$, there exists $i \in \{1,2\}$ such that $v \sqcup v' \models \mathcal{J}_i$. Thus we get that $\kappa, t, v' \models \mathcal{S}_i$.

Conversely, assume that $\kappa, t, v' \models \mathcal{S}_i$ for some $i \in \{1,2\}$. There exists $v\colon Y \to \mathbb{R}_{\geq 0}$ such that $\kappa, v, v' \vdash \mathcal{S}_i$, for all $y \in Y$, $v(y) \geq t$, and there exists $y_0 \in Y$ such that $v(y_0) = t$. By definition of the $\vdash$ satisfaction relation, we have that $v \sqcup v' \models \mathcal{J}_i$, and for every $y \in Y$, $\kappa, v(y), v' \models_{ib} V(y)$. As the set of solutions of $\mathcal{J}$ is the union of the sets of solutions of $\mathcal{J}_1$ and $\mathcal{J}_2$, we get that $v \sqcup v' \models \mathcal{J}$, and thus that $\kappa, t, v' \models \mathcal{S}$. □

Thanks to this lemma, we have the following property: if φ_i is an $\mathsf{MTL}_{\mathbf{F}}$ formula equivalent to a system $\mathcal{S}_i$ (for $i \in \{1,2\}$), then $\varphi_1 \vee \varphi_2$ is an $\mathsf{MTL}_{\mathbf{F}}$ formula equivalent to $\mathcal{S}$.

4. Reduction to bounded systems of difference inequations.. We fix a system $\mathcal{S} = (V, \mathcal{J})$, assuming $\mathcal{J} = \{x_i - x_j \prec_{i,j} m_{i,j} \mid i,j = 0 \dots n\}$ is a set of constraints in normal form (*i.e.*, all constraints are tightened) with $x_0 = 0$. We assume in addition (even if it means duplicating the system, adding constraints of the form $x_i \leq x_j$, renaming variables, and applying Lemma 21) that constraints in $\mathcal{J}$ imply that $x_{i-1} \leq x_i$ for every $0 < i \leq n$, and we let M be the maximal constant appearing in $\mathcal{J}$. For every $\mathsf{b}\colon \{1, \dots, n\} \to \{\leq, >\}$, we define a new set of constraints $\mathcal{J}^{\mathsf{b}}$ where constraints $\{x_i - x_{i-1}\ \mathsf{b}(i)\ M \mid 1 \leq i \leq n\}$ are added to $\mathcal{J}$. We claim the following two lemmas:

Lemma 22. *$(a_i)_{0 \leq i \leq n}$ is a solution of $\mathcal{J}$ iff it is a solution of $\mathcal{J}^{\mathsf{b}}$ for some $\mathsf{b}\colon \{1, \dots, n\} \to \{\leq, >\}$.*

Lemma 23. *We pick some $\mathsf{b}\colon \{1, \dots, n\} \to \{\leq, >\}$ such that $\mathcal{J}^{\mathsf{b}}$ is consistent (i.e., $\mathcal{J}^{\mathsf{b}}$ has a solution), and write $\equiv_{\mathsf{b}}$ for the following equivalence on indices:*

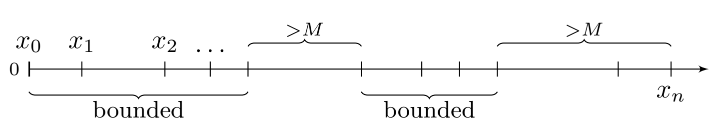
$$i \equiv_{\mathsf{b}} j \quad \text{iff} \quad \text{for all } i \leq k < j,\ \mathsf{b}(k) = \leq.$$

Then $\mathcal{J}^{\mathsf{b}}$ is equivalent to

$$\{x_i - x_j \prec_{i,j} m_{i,j} \mid i \equiv_{\mathsf{b}} j\} \cup \{x_i - x_{i-1}\ \mathsf{b}(i)\ M \mid 1 \leq i \leq n\}.$$

This is a straightforward consequence of the fact that M is the maximal constant appearing in $\mathcal{J}$, and of the fact that $x_{i-1} \leq x_i$ for every $0 < i \leq n$.

Lemma 23 can be depicted as follows:



On this picture, each point on the line represents a variable, and a part denoted "bounded" gathers variables whose differences are bounded by the system of inequations $\mathcal{J}^{\mathsf{b}}$. Two "bounded" parts are separated by more than M time units.

From Lemmas 22 and 21, if ψ^{b} is an $\mathsf{MTL}_{\mathbf{F}}$ formula equivalent to $\mathcal{S}^{\mathsf{b}}$, then the disjunction of all ψ^{b}'s, when b ranges over the whole set of functions $\{1,\ldots,n\} \to \{\leq, >\}$, is equivalent to $\mathcal{S}$. It remains to explain how we construct a formula equivalent to a system $\mathcal{S}^{\mathsf{b}}$.

We fix a $\mathsf{b}\colon \{1,\ldots,n\} \to \{\leq, >\}$, and denote by $(I_i)_{0\leq i\leq p}$ the equivalence classes for $\equiv_{\mathsf{b}}$ (in increasing order). For each $0 \leq i \leq p$, we denote by n_i the largest index in I_i. We assume we have a procedure for computing $\mathsf{MTL}_{\mathbf{F}}$ formulae equivalent to systems $\mathcal{S} = (V, \mathcal{J})$ where $\mathcal{J}$ implies that all variables are bounded. We will describe such a procedure at step 6 below. The resulting $\mathsf{MTL}_{\mathbf{F}}$ formula is denoted by $\Psi(\mathcal{S})$. By a decreasing induction on i, we define systems $(\mathcal{S}_i)_{0\leq i\leq p}$ as follows: $\mathcal{S}_i = (V_i, \mathcal{J}_i)$ is a system over $\{x_j \mid j \in I_i\}$ and

$$\left\{\begin{array}{l} \left\{\begin{array}{ll} V_i(x_j) = V^{\mathsf{b}}(x_j) & \text{if } i = p \text{ and } j \in I_i, \text{ or if } j \in I_i \setminus \{n_i\} \\ V_i(x_{n_i}) = V^{\mathsf{b}}(x_{n_i}) \wedge \mathbf{F}_{>M}\, \Psi(\mathcal{S}_{i+1}) & \text{if } i \neq p \end{array}\right. \\ \mathcal{J}_i = \mathcal{J}^{\mathsf{b}}_{|I_i} \text{ is the restriction of } \mathcal{J}^{\mathsf{b}} \text{ to variables } \{x_j \mid j \in I_i\} \end{array}\right.$$

From Lemmas 20 and 23, formula ψ^{b} is equivalent to formula $\Psi(\mathcal{S}_0)$ defined above. That way, we have reduced our initial problem to that of finding $\mathsf{MTL}_{\mathbf{F}}$ formulae equivalent to systems $\mathcal{S} = (V, \mathcal{J})$ where constraints in $\mathcal{J}$ imply that all variables are bounded.

5. Decomposition of bounded systems of difference inequations.. We fix $\mathcal{S} = (V, \mathcal{J})$. We assume that the variables involved in $\mathcal{J}$ are $\{x_i \mid 0 \leq i \leq n\}$, and that they are bounded by M. Following region decompositions of timed automata [AD94], we split $\mathcal{J}$ into systems where constraints are regions. Roughly, a region specifies in which elementary intervals (interval of the form $(c; c+1)$ or singleton $\{c\}$ for $c \leq M$) lie the differences $x_i - x_j$. It is then sufficient to find $\mathsf{MTL}_{\mathbf{F}}$ formulae for systems $\mathcal{S}_R = (V_R, \mathcal{J}_R)$ where $\mathcal{J}_R$ represents a bounded region: indeed, if ψ_R is an $\mathsf{MTL}_{\mathbf{F}}$ formula equivalent to the system $\mathcal{S}_R = (V, \mathcal{J}_R)$ where $\mathcal{J}_R$ contains all the constraints of $\mathcal{J}$ and all constraints defining the region R (which equivalently means that $\mathcal{J}_R$ corresponds to R because R is either included in $\mathcal{J}$ or disjoint from $\mathcal{J}$), then the formula $\bigvee_{R\subseteq \mathcal{J}} \psi_R$ is equivalent to $\mathcal{S}$ (applying Lemma 21).

A region R can be equivalently characterized by an integral value for every variable x_i ($0 \leq i \leq n$) and by variables $(X_i)_{0\leq i\leq p}$ (that form a partition of $\{x_i \mid 0 \leq i \leq n\}$) such that[6]

[6]In the sequel, $\langle x\rangle$ represents the fractional part of x, and $\lfloor x\rfloor$ represents the lower bound of the interval in which variable x lies in R (if x is in $\{c\}$ or $(c; c+1)$, then $\lfloor x\rfloor$ is c).

- $x \in X_0$ if, and only if, $\langle x \rangle = 0$ (where $\langle x \rangle$ denotes the fractional part of clock x),
- $x, y \in X_i$ if, and only if, $\langle x \rangle = \langle y \rangle$,
- $x \in X_i$ and $y \in X_j$ with $i < j$ implies $\langle x \rangle < \langle y \rangle$,

Let $\mathcal{S}' = (V', \mathcal{J}')$ be the system over $\{X_i \mid 1 \le i \le p\}$ (X_i's are viewed as variables here) such that for every $1 \le i \le p$, $V'(X_i) = \bigwedge_{x \in X_i} \mathbf{F}_{=\lfloor x \rfloor} V(x)$, and $\mathcal{J}'$ is the system $0 < X_1 < \ldots < X_p < 1$. If ψ' is an $\mathsf{MTL}_{\mathbf{F}}$ formula equivalent to $\mathcal{S}'$, then the formula $\left(\bigwedge_{x \in X_0} \mathbf{F}_{=\lfloor x \rfloor} V(x)\right) \wedge \psi'$ is equivalent to the whole system $\mathcal{S}$.

6. $\mathsf{MTL}_{\mathbf{F}}$ *formulae for simple systems..* It remains to find $\mathsf{MTL}_{\mathbf{F}}$ formulae $\Psi_{[1\ldots p],r}$ equivalent to systems $\mathcal{S}_{p,r} = (V, \mathcal{J}_{p,r})$ over $\{X_i \mid 1 \le i \le p\}$, where r is any rational and $\mathcal{J}_{p,r}$ is the set of constraints $0 < X_1 < \cdots < X_p < r$. Note that, even if this is an abuse of notation, we assume we have a unique function V which is used for all systems $\mathcal{S}_{p,r}$. We inductively build formulae $\Psi_{[h\ldots h+k],r}$, which handle the case of variables X_h to X_{h+k} on the interval $(0, r)$. When $k < 0$, the formula is `true`. When $k = 0$, we have $\Psi_{[h],r} = \mathbf{F}_{<r} V(X_h)$. For $k+1$ variables X_h to X_{h+k}, $\Psi_{[h\ldots h+k],r}$ is the disjunction of the following four formulae Φ_1 to Φ_4, distinguishing between the possible positions of the variables:

- if there is no variable in the interval $\left(0, \frac{r}{k+1}\right]$ and all variables $(X_{h+i})_{0 \le i \le k}$ are in the interval $\left(\frac{r}{k+1}, r\right)$:

  $$\Phi_1 = \Theta_1 \vee \mathbf{F}_{<\frac{r}{k+1}} \left(\Theta_1\right).$$

  where

  $$\Theta_1 = \mathbf{F}_{<\frac{r}{k+1}} \left[\bigvee_{i=1}^{k} \left(\left(\mathbf{F}_{=r-\frac{ir}{k+1}} V(X_{h+k}) \right) \wedge \Psi_{[h\ldots h+k-1], r-\frac{ir}{k+1}} \right) \right]$$

  The formula Φ_1 distinguishes between the possible positions for the last variable X_{h+k}: it is in one of the punctual intervals $\left[r - \frac{ir}{k+1}, r - \frac{ir}{k+1}\right]$ or in one of the open intervals $\left(r - \frac{ir}{k+1}, r - \frac{(i-1)r}{k+1}\right)$ for some $1 \le i \le h+k$. Note that Φ_1 does not exactly express the above property: it may contain some more cases, but it always implies that $0 < X_1 < \cdots < X_p < r$. The same remark also applies for the other three formulae.

- if there are $1 \le l \le k$ variables in the interval $\left(0, \frac{r}{k+1}\right)$ and $k - l + 1$ variables in the interval $\left(\frac{r}{k+1}, r\right)$:

  $$\Phi_2 = \bigvee_{l=1}^{k} \left(\Psi_{[h\ldots h+l-1], \frac{r}{k+1}} \wedge \mathbf{F}_{=\frac{r}{k+1}} \left(\Psi_{[h+l\ldots h+k], r-\frac{r}{k+1}} \right) \right).$$

- if there are $0 \leq l \leq k$ variables in the interval $\left(0, \frac{r}{k+1}\right)$, one variable at date $\frac{r}{k+1}$, and $k - l$ variables in the interval $\left(\frac{r}{k+1}, r\right)$:

$$\Phi_3 = \bigvee_{l=0}^{k} \left(\Psi_{[h \ldots h+l-1], \frac{r}{k+1}} \wedge \mathbf{F}_{= \frac{r}{k+1}} \left(V(X_{h+l}) \wedge \Psi_{[h+l+1 \ldots h+k], r - \frac{r}{k+1}} \right) \right).$$

- finally, if all variables are in the interval $\left(0, \frac{r}{k+1}\right)$:

$$\Phi_4 = \mathbf{F}_{< \frac{r}{k+1}} \left(V(X_h) \wedge \mathbf{F}_{< \frac{r}{k+1}} \left(V(X_{h+1}) \wedge (\; \cdots \;) \right) \right)$$

It can easily be proved, by induction, that the resulting formula is equivalent to $\mathcal{S}_{p,r}$. $\square$

Example 4. *We illustrate all the steps of the above construction on the formula:*

$$\varphi \;=\; x.\mathbf{F} \left(a \wedge x \geq 1 \wedge \mathbf{F}\,(b \wedge x \leq 3) \wedge y.\mathbf{F}\,(\neg a \wedge x \leq 3 \wedge y > 1) \right)$$

Step 1. The normal form of φ is

$$x.\mathbf{F} \left(a \wedge x \geq 1 \wedge z.\mathbf{F}\,(b \wedge x \leq 3) \wedge y.\mathbf{F}\,(\neg a \wedge x \leq 3 \wedge y > 1) \right)$$

Step 2. Then, the system associated with this simple formula is

$$\mathcal{J} = \left\{ \begin{array}{ll} z_0 = 0 & z_2 > z_1 \\ z_1 - z_0 \geq 1 & z_3 > z_1 \\ z_2 - z_0 \leq 3 & z_1 > z_0 \\ z_3 - z_0 \leq 3 & z_2 > z_0 \\ z_3 - z_1 > 1 & z_3 > z_0 \end{array} \right\} \qquad V : \left\{ \begin{array}{l} z_0 \mapsto \top \\ z_1 \mapsto a \\ z_2 \mapsto b \\ z_3 \mapsto \neg a \end{array} \right.$$

Schematically, these constraints can be understood as follows:

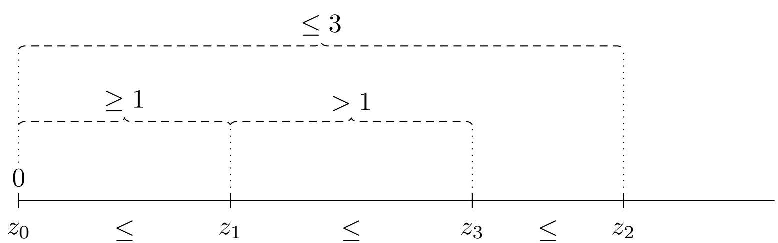
$$\underset{\substack{\uparrow \\ z_0}}{x}.\underset{\substack{\uparrow \\ z_1}}{\mathbf{F}} \left(a \wedge x \geq 1 \wedge z.\underset{\substack{\uparrow \\ z_2}}{\mathbf{F}}\,(b \wedge x \leq 3) \wedge y.\underset{\substack{\uparrow \\ z_3}}{\mathbf{F}}\,(\neg a \wedge x \leq 3 \wedge y > 1) \right)$$

where:

- *z_0 represents the initial time, i.e. the date at which formula φ has to hold (typically $z_0 = 0$);*
- *z_1, z_2 and z_3 are three witness dates for the three eventualities (i.e., the three parenthesised subformulas).*

Step 4. *The above system of inequations does not constrain the order of z_2 and z_3; there are solutions of the system in which $z_2 < z_3$, and other solutions in which $z_3 \le z_2$. We thus split the system of inequations into two systems, that will be dealt separately; the formula for the global system will be the disjunction of the two formulas obtained from each new system. One of two systems will correspond to the previous constraints plus $z_2 < z_3$—we write $\mathcal{S}_{z_2<z_3}$ for the resulting system—, and the other system will correspond to the previous constraints plus $z_3 \le z_2$—we write $\mathcal{S}_{z_3\le z_2}$ for the resulting system. Below we will first focus on the system $\mathcal{S}_{z_3\le z_2}$, and then explain how we can deal with the difficult part of the system $\mathcal{S}_{z_2<z_3}$.*

The system $\mathcal{J}_{z_3\le z_2}$, illustrated on the next picture, is bounded (two consecutive clocks are never separated by more than 3 time units, the maximal constant); there is no need to further split the system.



Step 5. *This system of inequations we are focusing on can be decomposed into regions. For example, it contains the region defined by the constraints:*

$$\left\{\begin{array}{cc} z_0 = 0 & 1 < z_1 < 2 \\ 2 < z_2 < 3 & 2 < z_3 < 3 \\ z_2 - z_1 = 1 & 0 < z_3 - z_1 < 1 \\ 0 < z_2 - z_3 < 1 & \end{array}\right\}$$

We want to have only constants 0 and 1, we thus shift the above system and get the following one:

$$\mathcal{J}' = \left\{\begin{array}{cc} z'_0 = 0 & 0 < z'_1 < 1 \\ 0 < z'_2 < 1 & 0 < z'_3 < 1 \\ z'_2 - z'_1 = 0 & 0 < z'_1 - z'_3 < 1 \\ 0 < z'_2 - z'_3 < 1 & \end{array}\right\} \quad V'\colon \left\{\begin{array}{l} z'_0 \mapsto \top \\ z'_1 \mapsto \mathbf{F}_{=1}\, a \\ z'_2 \mapsto \mathbf{F}_{=2}\, b \\ z'_3 \mapsto \mathbf{F}_{=2}\, \neg a \end{array}\right.$$

Setting $X_0 = \{z'_0\}$, $X_1 = \{z'_1, z'_2\}$ and $X_2 = \{z'_3\}$, we get the new system

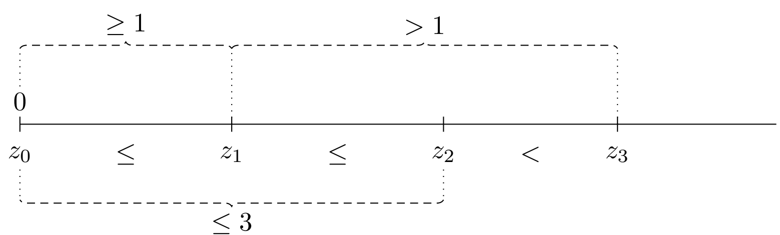
$$\mathcal{J}'' = \{0 = X_0 < X_1 < X_2 < 1\} \quad V''\colon \left\{\begin{array}{l} X_0 \mapsto \top \\ X_1 \mapsto (\mathbf{F}_{=1}\, a) \wedge (\mathbf{F}_{=2}\, b) \\ X_2 \mapsto \mathbf{F}_{=2}\, \neg a \end{array}\right.$$

Step 6. *We now build the formula corresponding to the region we have selected in $\mathcal{S}_{z_3 \le z_2}$: $\Phi = \Phi_1 \vee \Phi_2 \vee \Phi_3 \vee \Phi_4$ with*

$$\begin{aligned}
\Phi_1 &= \Theta_1 \vee \mathbf{F}_{<0.5}\,\Theta_1 \qquad \textit{with } \Theta_1 = \mathbf{F}_{<0.5}\,(\mathbf{F}_{=1}\,a \wedge \mathbf{F}_{=2}\,b) \wedge \mathbf{F}_{=2.5}\,\neg a) \\
\Phi_2 &= \mathbf{F}_{<0.5}\,(\mathbf{F}_{=1}\,a \wedge \mathbf{F}_{=2}\,b) \wedge \mathbf{F}_{(2.5,3)}\,\neg a \\
\Phi_3 &= \mathbf{F}_{=0.5}\,(\mathbf{F}_{=1}\,a \wedge \mathbf{F}_{=2}\,b \wedge \mathbf{F}_{(2,2.5)}\,\neg a) \vee \\
&\qquad\qquad (\mathbf{F}_{=0.5}\,(\mathbf{F}_{=1}\,a \wedge \mathbf{F}_{=2}\,b) \wedge \mathbf{F}_{=2.5}\,\neg a) \\
\Phi_4 &= \mathbf{F}_{<0.5}\,(\mathbf{F}_{=1}\,a \wedge \mathbf{F}_{=2}\,b \wedge \mathbf{F}_{(2,2.5)}\,\neg a).
\end{aligned}$$

Note that this formula is only one part *of the* $\mathsf{MTL}_{\mathbf{F}}$ *formula equivalent to our original formula φ. There are other formulas which come from the decompositions we have made in the 4-th and 5-th steps. To illustrate all aspects of the construction, we now consider one subcase of the system $\mathcal{S}_{z_2 < z_3}$.*

Step 4bis. *At the end of Step 4, we had selected the system $\mathcal{S}_{z_3 \le z_2}$ because it was bounded, meaning that two consecutive variables were not separated by more than 3 time units. We now consider the system $\mathcal{S}_{z_2 < z_3}$, which is illustrated below.*



In this system, nothing prevents $z_3 - z_2$ from being larger than 3. We thus split the system into two systems: the first one with the constraint $z_3 - z_2 \le 3$, and the second one with the constraint $z_3 - z_2 > 3$. The first case is bounded, its treatment being similar to what we have previously done. We thus only focus on the second system, which reduces to:

$$\widetilde{\mathcal{J}} = \left\{ \begin{array}{cc} \widetilde{z}_0 = 0, & \widetilde{z}_1 - \widetilde{z}_0 \ge 1 \\ \widetilde{z}_2 - \widetilde{z}_1 > 0 & \widetilde{z}_2 - \widetilde{z}_0 \le 3 \end{array} \right\} \qquad \widetilde{V} : \left\{ \begin{array}{l} \widetilde{z}_0 \mapsto \textsf{true} \\ \widetilde{z}_1 \mapsto a \\ \widetilde{z}_2 \mapsto b \wedge \mathbf{F}_{>3}\,\neg a \end{array} \right.$$

because the only constraint on z_3 is $z_3 - z_2 > 3$, hence replacing z_2 by variable $\widetilde{z}_2$, we write that b must hold at $\widetilde{z}_2$, and later, strictly after 3 time units, $\neg a$ has to hold (former position z_3).

Step 5bis. *As previously, we select one region included in the previous zone, for instance:*

$$\widetilde{\mathcal{J}} = \left\{ \begin{array}{cc} \widetilde{z}_0 = 0, & 1 < \widetilde{z}_1 - \widetilde{z}_0 < 2 \\ 0 < \widetilde{z}_2 - \widetilde{z}_1 < 1 & 2 < \widetilde{z}_2 - \widetilde{z}_0 < 3 \end{array} \right\} \quad \widetilde{V} : \left\{ \begin{array}{l} \widetilde{z}_0 \mapsto \textsf{true} \\ \widetilde{z}_1 \mapsto a \\ \widetilde{z}_2 \mapsto b \wedge \mathbf{F}_{>3}\,\neg a \end{array} \right.$$

We then shift the constraints to only obtain constants 0 *and* 1*, and we get the system:*

$$\widetilde{\mathcal{J}} = \{0 = \widetilde{z}'_0 < \widetilde{z}'_2 < \widetilde{z}'_1 < 1\} \qquad \widetilde{V}\colon \begin{cases} \widetilde{z}'_0 \mapsto \textsf{true} \\ \widetilde{z}'_1 \mapsto \mathbf{F}_{=1}\, a \\ \widetilde{z}'_2 \mapsto \mathbf{F}_{=2}\,(b \wedge \mathbf{F}_{>3}\, \neg a) \end{cases}$$

Step 6bis. We get the following formula for the selected subsystem of $\mathcal{S}_{z_2<z_3}$*:*

$$\begin{array}{ll} & \mathbf{F}_{[0,0.5)}\,(\mathbf{F}_{<2.5}\,(b \wedge \mathbf{F}_{>3}\,\neg a) \wedge \mathbf{F}_{=1.5}\, a) \\ \vee & \mathbf{F}_{<2.5}\,(b \wedge \mathbf{F}_{>3}\,\neg a) \wedge \mathbf{F}_{=1.5}\, a \\ \vee & \mathbf{F}_{=0.5}\,(\mathbf{F}_{=2}\,(b \wedge \mathbf{F}_{>3}\,\neg a) \wedge \mathbf{F}_{<1.5}\, a) \\ \vee & \mathbf{F}_{<0.5}\,(\mathbf{F}_{=2}\,(b \wedge \mathbf{F}_{>3}\,\neg a) \wedge \mathbf{F}_{1.5}\, a). \end{array}$$

Our construction from $\mathsf{TPTL}_{\mathbf{F}}$ to $\mathsf{MTL}_{\mathbf{F}}$ is exponential. We first compute the normal form of the $\mathsf{TPTL}_{\mathbf{F}}$ formula φ by choosing for every disjunction one of the disjuncts: the normal form is then the disjunction of all the formulae obtained by such choices. This gives an exponential number of formulae whose disjunction corresponds to φ, the size of each formula being linear in the size of φ. The reduction to bounded systems produces for each formula an exponential number of systems (whose size is polynomial in the size of φ). Then for each system we compute the corresponding MTL formula which has an exponential size in the size of the system. The MTL formula for φ is finally a combination of this exponential number of exponential formulae, its size is thus simply exponential.

Our construction above also yields a procedure for the satisfiability of a $\mathsf{TPTL}_{\mathbf{F}}$ formula. It is known [AFH96] that the satisfiability problem for TPTL and MTL is undecidable for the interval-based semantics, whereas it has been proved recently that the satisfiability problem for MTL over finite paths is decidable but non primitive recursive for the pointwise semantics [OW05]. With the construction above, we get:

Corollary 24. *The satisfiability problem for* $\mathsf{TPTL}_{\mathbf{F}}$ *(and* $\mathsf{MTL}_{\mathbf{F}}$*) is* NP*-complete for the interval-based semantics.*

Proof. If ψ is a $\mathsf{TPTL}_{\mathbf{F}}$ formula, first guess for each disjunction of ψ one of the disjuncts, and build the system $\mathcal{S} = (V, \mathcal{J})$ for the new formula which is directly in normal form (this is achieved in polynomial time); then guess an order on the variables which is consistent with the constraints in $\mathcal{J}$; finally solve a simple linear programming problem. For each guess, the problem can be solved in polynomial time and all guesses are independent, we thus get that the problem is in NP. Hardness in NP directly follows from that of 3SAT (an instance of 3SAT can be viewed as a special instance of $\mathsf{MTL}_{\mathbf{F}}$ or $\mathsf{TPTL}_{\mathbf{F}}$ satisfiability). □

5. Conclusion

We have proved the conjecture (first proposed in [AH90]) that the logic TPTL is strictly more expressive than MTL. In the meantime, many interesting and surprising expressiveness properties have appeared as side results: expressiveness of past-time operators, expressiveness of MITL, ...

We also derived a surprisingly efficient algorithm for the satisfiability of $\mathsf{TPTL}_{\mathbf{F}}$ under the interval-based semantics: it is not harder than boolean satisfiability, while satisfiability of MTL or TPTL is undecidable.

Linear models we have used for proving our expressiveness results can be viewed as special cases of branching-time models. Our main result thus applies to the branching-time logic TCTL (by replacing the modality **U** with the modality **AU**, and translates as: TCTL with explicit clocks [HNSY94] is strictly more expressive than TCTL with subscripts [ACD93], as conjectured in [Alu91, Yov93].

Studying the expressiveness of various timed temporal logics is still a very active topic [HR06, DP06, DMMP06, HR07, FR07, DP07, DHV07]. In particular, our work has opened the way for several works on the expressiveness of timed temporal logics, *e.g.* [DP07, DHV07], which discuss the relative expressiveness of MTL+Past (resp. TPTL+Past) in the pointwise and continuous semantics, or [DP06, DMMP06], which discuss the expressiveness of different fragments of MTL+Past.

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