Probabilistic Automata on Finite Words: Decidable and Undecidable Problems

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Introduction

Example

1 is the initial state.

\{3\} is the set of accepting states.

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Accepted Language [Rabin, 63]

Let \( A \) a probabilistic automaton and a rational \( \lambda \):

\[ \mathcal{L}_A(\lambda) = \{ w \in A^* \mid \mathbb{P}_A(w) \geq \lambda \} \]
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Let \(A\) a probabilistic automaton and a rational \(\lambda\):

\[L_A(\lambda) = \{w \in A^* \mid P_A(w) \geq \lambda\}.

Formal definition

A probabilistic automaton is a tuple \(A = (Q, A, (M_a)_{a \in A}, q_0, F)\).
Outline

Emptiness Problem

Threshold isolation problem

♯-acyclic probabilistic automata

Conclusion
The Emptiness problem

Definition

Given a probabilistic automaton $A$, decide $\exists w \in A^* \text{ s.t. } \begin{cases} \mathbb{P}_A(w) \geq \frac{1}{2} \\ \mathbb{P}_A(w) > \frac{1}{2} \end{cases}$, in the strict version.

Theorem [Paz, 71]
The (strict) emptiness problem is undecidable.
Proof by reduction from a problem on context free grammars.
The Emptiness problem

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New undecidability proof

Lemma [Bertoni, 77]

Given a probabilistic automaton $A$, it is undecidable whether $\exists w \in A^*$ such that $P_A(w) = \frac{1}{2}$.

Remark [Gimbert, O.]

The emptiness problem is undecidable even for simple probabilistic automata. A probabilistic automaton is simple if $(M,a) \in A((s,t)) \in \{0,\frac{1}{2},1\}$, where $(s,t) \in Q$.  

Corollary [Gimbert, O.]

The following problem is undecidable:

$\text{Given a non deterministic automaton on finite words, is there a word such that at least half the computations are accepting?}$
New undecidability proof

Lemma [Bertoni, 77]
Given a probabilistic automaton $\mathcal{A}$, it is undecidable whether $\exists w \in A^*$ such that $\mathbb{P}_\mathcal{A}(w) = \frac{1}{2}$

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Automata with few probabilistic states

Proposition [Blondel, 03]
The emptiness problem is undecidable for any probabilistic automaton of size larger than 46.

Proposition [Gimbert, O.]
The emptiness problem for automata with 1 probabilistic transition is decidable in PSPACE.

Lemma [Gimbert, O.]
The following problem is undecidable: given a simple probabilistic automaton with 1 probabilistic transition and given a rational language $L \subseteq A^*$, decide $\exists w \in L \text{ s.t } P_A(w) \geq \frac{1}{2}$.

Corollary
The emptiness problem for automata with 2 probabilistic states is undecidable.
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Definition
Let $\mathcal{A}$ a probabilistic automaton. $\lambda$ is isolated with respect to $\mathcal{A}$ if:

$$\exists \varepsilon > 0, \forall w \in A^* : |P_{\mathcal{A}}(w) - \lambda| \geq \varepsilon.$$
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Theorem [Rabin, 63]
Let $A$ a probabilistic automaton and a rational $0 \leq \lambda \leq 1$. If $\lambda$ is isolated then the language $\mathcal{L}(\lambda)$ is rational.
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Let \( A \) a probabilistic automaton. \( \lambda \) is isolated with respect to \( A \) if :

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\]

Theorem [Rabin, 63]
Let \( A \) a probabilistic automaton and a rational \( 0 \leq \lambda \leq 1 \). If \( \lambda \) is isolated then the language \( L(\lambda) \) is rational.

Theorem [Bertoni, 77]
Given a probabilistic automaton \( A \) and a rational \( 0 < \lambda < 1 \), it is undecidable whether if \( \lambda \) isolated with respect to \( A \) or not.
Bertoni’s proof does not work in this case!
\( \lambda \in \{0, 1\} \)

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For the emptiness problem:
- When \( \lambda = 0 \) it is the emptiness problem for non-deterministic automata on finite words.
- When \( \lambda = 1 \) it is the emptiness problem for universal automata on finite words.
$\lambda \in \{0, 1\}$

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For the emptiness problem:

- When $\lambda = 0$ it is the emptiness problem for non deterministic automata on finite words.
- When $\lambda = 1$ it is the emptiness problem for universal automata on finite words.

- The case $\lambda = 1$ is called the value 1 problem.
The Value 1 Problem

Definition
Let $\mathcal{A}$ a probabilistic automaton. $\mathcal{A}$ has the value 1 if:

$$\forall \epsilon > 0, \exists w \in A^*, P_\mathcal{A}(w) \geq 1 - \epsilon$$
The Value 1 Problem

Definition
Let $A$ a probabilistic automaton. $A$ has the value 1 if:

$$\forall \varepsilon > 0, \exists w \in A^*, \mathbb{P}_A(w) \geq 1 - \varepsilon.$$ 

Theorem [Gimbert, O.]
The value 1 problem is undecidable.

Proof.
- Reduction from the strict emptiness problem.
- Inspired by Baier, Bertrand and Größer for the emptiness of probabilistic Büchi automata.
Sketch of The Proof

Let $A_x$ the following automaton:
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\[
(val_A = 1) \iff (x > \frac{1}{2}).
\]
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Classical subset construction

\[(a, \frac{1}{2}), b\]

\[\{1\} \cdot a = \{1, 2\} \]
\[\{1, 2\} \cdot a = \{1, 2\} \]
$\#$-acyclic probabilistic automata

Classical subset construction

\[
\begin{align*}
(a, \frac{1}{2}), b & \quad \text{Transition: } a, \frac{1}{2} \\
1 & \quad \text{State: } 1 \\
2 & \quad \text{State: } 2
\end{align*}
\]

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Operator $\#$

\[
\begin{align*}
\frac{1}{2} & \quad \text{Transition: } \frac{1}{2} \\
1 & \quad \text{State: } 1 \\
2 & \quad \text{State: } 2
\end{align*}
\]

\[
\{1, 2\} \cdot a^\# = \{2\}.
\]

The state 2 is recurrent.
The state 1 is transient.
Support graph
Support graph

\[
\begin{align*}
\text{1} & \rightarrow \text{2} & \text{2} & \rightarrow \text{1} & \text{3} & \rightarrow \text{4} & \text{4} & \rightarrow \text{3} \\
& \quad & \quad & \quad & \quad & \quad & \quad & \\
\end{align*}
\]

\[
\begin{align*}
\{1, 3\} & \rightarrow \{1, 2, 3, 4\} & \{1, 2, 3, 4\} & \rightarrow \{2, 4\} \\
& \quad & \quad & \quad & \quad & \quad & \quad & \\
\end{align*}
\]
Support graph
Decidability of $\#$-acyclic probabilistic automata

Theorem [Gimbert, O.]

The value 1 problem for $\#$-acyclic probabilistic automata is decidable.
Decidability of ♯-acyclic probabilistic automata

Theorem [Gimbert, O.]
The value 1 problem for ♯-acyclic probabilistic automata is decidable.

Definition

- Reachability in the support graph is called ♯-reachability.
- A set $T$ is said to be limit-reachable from another set $S$ if there exists a sequence $w_0, w_1, \cdots \in A^*$ such that:

$$\lim_{n\to\infty} P_A(S \xrightarrow{w_n} T) = 1.$$
Decidability of \(\#\)-acyclic probabilistic automata

Theorem [Gimbert, O.]
The value 1 problem for \(\#\)-acyclic probabilistic automata is decidable.

Definition
- Reachability in the support graph is called \(\#\)-reachability.
- A set \(T\) is said to be limit-reachable from another set \(S\) if there exists a sequence \(w_0, w_1, \cdots \in A^*\) such that:
  \[
P_A(S \xrightarrow{w_n} T) \xrightarrow{n \to \infty} 1.
  \]

First remark
- The \(\#\)-reachability implies limit-reachability. But the converse is not true in general.
Example

\begin{center}
\begin{tikzpicture}[node distance=2cm,>=latex]
  \node (1) [circle, draw] {1};
  \node (2) [circle, draw, right of=1] {2};
  \node (3) [circle, draw, right of=2] {3};
  \path [->]
  (1) edge [loop below] node {$b$} (1)
  (1) edge [bend left] node {$a$} (2)
  (2) edge [bend left] node {$a$} (1)
  (2) edge [loop above] node {$a$} (2)
  (2) edge [bend left] node {$b$} (3)
  (3) edge [bend left] node {$b$} (2)
  (3) edge [loop below] node {$b$} (3)
  (3) edge [loop above] node {$a$} (3);
\end{tikzpicture}
\end{center}
Example

The diagram shows a transition system with states labeled 1, 2, and 3, and transitions labeled with symbols $a$ and $b$. The states are connected by directed edges indicating possible transitions. The states are also associated with sets, such as $\{2\}$, $\{3\}$, $\{1, 2, 3\}$, $\{1, 2\}$, and $\{1\}$, which might represent language elements or states in a formal language or automaton. The transitions include loops labeled with $b\#$, indicating self-loops on states 2 and 3. The transitions between states are labeled with $a$ and $b$, and the direction of the transitions is indicated by arrows. The state transitions and set associations are key components of this diagram, potentially illustrating a formal language or automaton.
Example
Sketch of the proof

For ♯-acyclic probabilistic automata, limit-reachability $\iff$ ♯-reachability.

\[ \forall a \in A, Q. a \# = Q. \]

If $Q$ is limit-reachable from $S$ then $Q$ is ♯-reachable from $S$.

\[ \forall a \in A, Q. a \# = Q. \]

$Q$ is the unique set limit-reachable support from $Q$.

\[ \forall a \in A, S. a \# = S. \]

Every limit-reachable set from $Q$ contains $S$, $S$ in unique.
Sketch of the proof

For $\#$-acyclic probabilistic automata, limit-reachability $\implies$ $\#$-reachability.

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If $Q$ is limit-reachable from $S$ then $Q$ is $\#$-reachable from $S$. 

Sketch of the proof

For \( \# \)-acyclic probabilistic automata, limit-reachability \( \implies \) \( \# \)-reachability.

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\forall a \in A, \ Q.a\# = Q .
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If \( Q \) is limit-reachable from \( S \) then \( Q \) is \( \# \)-reachable from \( S \).

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$$\forall a \in A, \ Q.a = Q .$$

$$\forall a \in A, \ S.a\# = S .$$

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Contribution

- Short and simple proof for the emptiness problem.
- Strengthening Paz result.
- Undecidability of the value one problem.
- Defining a non trivial family of automata for which the value 1 problem is decidable.
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- Strengthening Paz result.
- Undecidability of the value one problem.
- Defining a non trivial family of automata for which the value 1 problem is decidable.

agenda

- Extend the result for \#-acyclic probabilistic automata to a larger class.
- Non trivial class of probabilistic automata for which the emptiness problem is decidable.