On the Expressiveness and Complexity of ATL

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Recherches en vérification automatique

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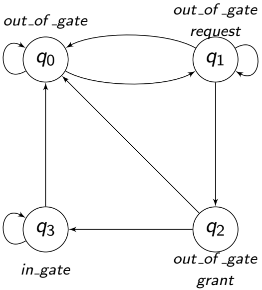
Overview of CTL

CTL

- A Kripke structure
- Quantification over paths ($\mathbf{E}$/ $\mathbf{A}$)



Example

There exists a run such that the train is always out of gate.

$$\mathbf{EG}\ \mathit{out\_of\_gate}$$

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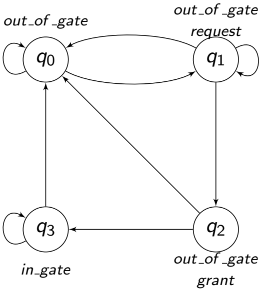

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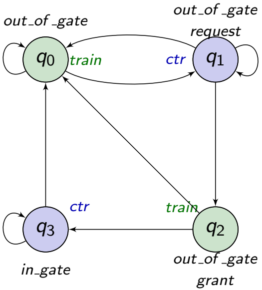

Example

Whenever the train is out of gate, the controller cannot force it to enter the gate.

$$\mathbf{AG}\,(out\_of\_gate \implies \neg \langle\langle ctr \rangle\rangle\, \mathbf{F}\, in\_gate)$$

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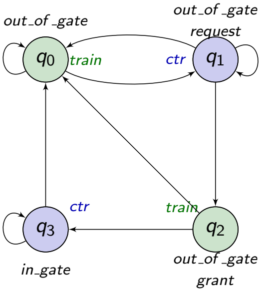

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Outline of the talk

1. Introduction
   - Overview of CTL and ATL
2. Definitions
   - Multi-agent models
   - Strategy and outcomes
   - ATL (Alternating-time Temporal Logic)
3. Expressiveness
   - Weak Until
4. Complexity
   - Model checking ATL on CGSs
   - Model checking ATL on ATSs
5. Conclusion

Outline of the talk

CGS definition

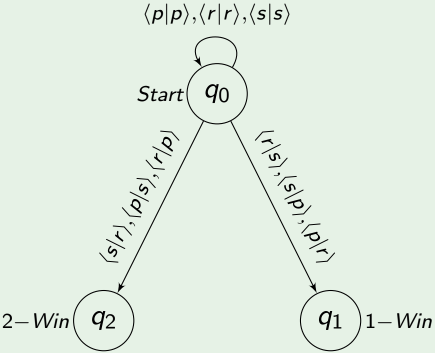
Definition

A *CGS* $\mathcal{C}$ is a 5-tuple (Loc, Lab, Agt, Chc, Edg) s.t:

- Loc: a finite set of *locations*;
- Lab: Loc $\rightarrow 2^{\mathsf{AP}}$: a labeling function;
- Agt $= \{A_1, ..., A_k\}$: a set of *agents* (or *players*);
- Chc: Loc $\times$ Agt $\rightarrow \mathbb{N}_{\geq 1}$ the choice function.
  Chc(ℓ, A_i) = number of possible moves for A_i from ℓ.
- Edg: Loc $\times\ \mathbb{N}^k \rightarrow$ Loc: the transition table.

Example of a CGS

Example



		Player 2		
	q_0	p	r	s
Player 1	p	q_0	q_1	q_2
	r	q_2	q_0	q_1
	s	q_1	q_2	q_0

Figure: Paper, rock and scissors

Semantics of CGSs

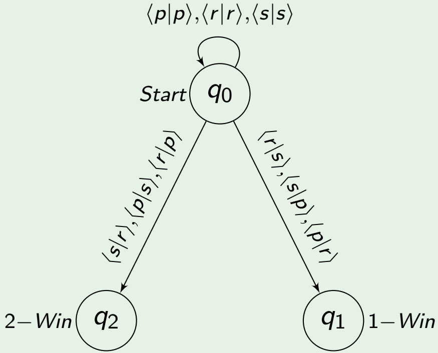
- From a location ℓ, each agent A_i chooses some m_{A_i} with
$$m_{A_i} < \mathrm{Chc}(\ell, A_i).$$
- $\mathrm{Edg}(\ell, m_{A_1}, \cdots, m_{A_k})$ gives the new location.

Notations:

- $\mathrm{Next}(\ell) = \{\mathrm{Edg}(\ell, \cdots m_{A_i} \cdots) \mid \forall m_{A_i} \cdot 1 \leq i \leq k\}$
- $\mathrm{Next}(\ell, A_j, m) = \left\{\mathrm{Edg}(\ell, \cdots, m_{A_{j-1}}, m, m_{A_{j+1}} \cdots)\right\}$

CGS example

Example



Player 2

Player 1	q_0	p	r	s
	p	q_0	q_1	q_2
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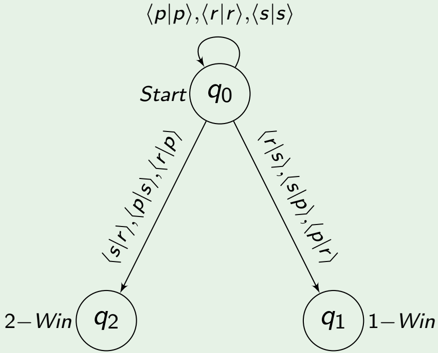

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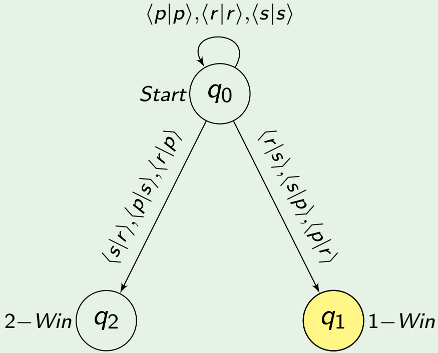

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ATS definition

Definition

An *ATS* $\mathcal{A}$ is a 4-tuple $(\mathsf{Loc}, \mathsf{Lab}, \mathsf{Agt}, \mathsf{Chc})$ where:

- Loc, Lab and Agt are the same as in CGSs;
- a move is a set of locations:

  $$\mathsf{Chc}\colon \mathsf{Loc} \times \mathsf{Agt} \rightarrow \mathcal{P}(\mathcal{P}(\mathsf{Loc}))$$

  with the following requirement: for any location ℓ and for moves $Q_i \in \mathsf{Chc}(\ell, A_i)$,

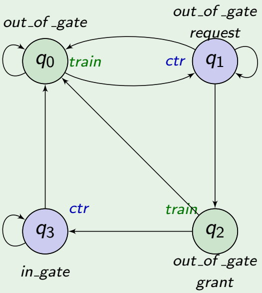
  $$\bigcap_{i \leq k} Q_i \text{ must be a singleton.}$$

The next location is precisely the location that belongs to all the choices of the agents.

$\mathsf{Next}(\ell)$ and $\mathsf{Next}(\ell, A_i, m)$ are defined in the obvious way.

ATS example

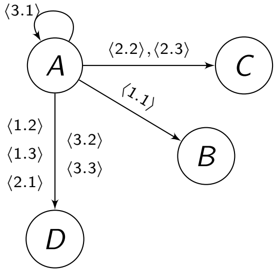
Example



- $\delta(q_0, train) = \{\{q_0\}, \{q_1\}\}$.
- $\delta(q_1, ctr) = \{\{q_0\}, \{q_1\}, \{q_2\}\}$.
- $\delta(q_2, train) = \{\{q_0\}, \{q_3\}\}$.
- $\delta(q_3, ctr) = \{\{q_0\}, \{q_3\}\}$.
- $\delta(q_0, ctr) = \delta(q_1, train) = \delta(q_2, ctr) = \delta(q_3, train) = \{\mathrm{Loc}\}$.

Figure: Train controller

Translation $CGS \leftrightarrow ATS$



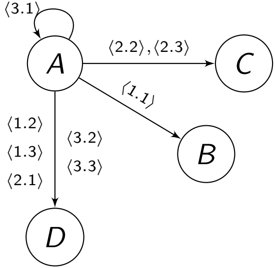
Naive approach

Move	*Player* 1	*Player* 2
1	$\{B, D\}$	$\{A, B, D\}$
2	$\{C, D\}$	$\{C, D\}$
3	$\{A, D\}$	$\{C, D\}$

Figure: Converting an CGS into an ATS

Cost of the translation:

Translation $CGS \leftrightarrow ATS$



Correct approach

Move	*Player* 1	*Player* 2
1	$\{B_{1.1}, D_{1.2}, D_{1.3}\}$	$\{A_{3.1}, B_{1.1}, D_{2.1}\}$
2	$\{C_{2.2}, C_{2.3}, D_{2.1}\}$	$\{C_{2.2}, D_{1.2}, D_{3.2}\}$
3	$\{A_{3.1}, D_{3.2}, D_{3.3}\}$	$\{C_{2.3}, D_{1.3}, D_{3.3}\}$

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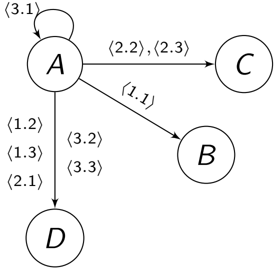

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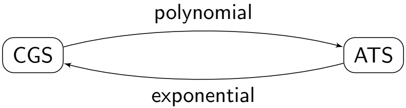

Strategies and outcomes

Definition

- A computation is an infinite sequence $\rho = \ell_0\ell_1\cdots$ such that $\forall i, \ell_{i+1} \in \text{Next}(\ell_i)$.
- A strategy is a function f_{A_i} s.t. $f_{A_i}(\ell_0, \cdots, \ell_m) =$ a possible move for A_i from ℓ_m.
- The outcomes $\text{Out}(\ell, f_{A_i})$ are the set of computations from ℓ that agree with the strategy f_{A_i} of A_i.
- Notations: given $A \subseteq \text{Agt}$, we note:
  - $F_A = \{f_{A_i} | A_i \in A\}$
  - $\text{Out}(\ell, F_A)$

Syntax of ATL

Definition ([AHK97])

The syntax of ATL is defined by the following grammar:

$$\begin{aligned} \mathsf{ATL} \ni \varphi_s, \psi_s \quad &::= \quad p \mid \neg\varphi_s \mid \varphi_s \vee \psi_s \mid \langle\!\langle A \rangle\!\rangle\, \varphi_p \\ \varphi_p \quad &::= \quad \mathbf{X}\, \varphi_s \mid \mathbf{G}\, \varphi_s \mid \varphi_s\, \mathbf{U}\, \psi_s. \end{aligned}$$

where p ranges over the set AP and A over the subsets of Agt.

ATL subsumes CTL, since we have:

$$\mathbf{E}\varphi_p \equiv \langle\!\langle \mathsf{Agt} \rangle\!\rangle\, \varphi_p,$$

$$\mathbf{A}\varphi_p \equiv \langle\!\langle \varnothing \rangle\!\rangle\, \varphi_p.$$

Semantics

Definition

- Semantics

$$\ell \models \langle\!\langle A \rangle\!\rangle\, \varphi_p \quad \text{iff} \quad \exists F_A \in \mathrm{Strat}(A).\ \forall \rho \in \mathrm{Out}(\ell, F_A).\ \rho \models \varphi_p$$

$$\rho \models \varphi_s\, \mathbf{U}\, \psi_s \quad \text{iff} \quad \exists i.\rho[i] \models \psi_s \text{ and } \forall 0 \leq j < i.\rho[j] \models \varphi_s$$

- We have $\langle\!\langle A \rangle\!\rangle\, \varphi \Rightarrow \neg\, \langle\!\langle \mathrm{Agt} \setminus A \rangle\!\rangle\, \neg\varphi$,
  but
  $$\neg\, \langle\!\langle A \rangle\!\rangle\, \varphi \not\Rightarrow \quad \langle\!\langle \mathrm{Agt} \setminus A \rangle\!\rangle\, \neg\varphi.$$

- We denote $[\![A]\!]\, \varphi$ for $\neg\, \langle\!\langle A \rangle\!\rangle\, \neg\varphi$

Outline of the talk

Can ATL express weak until?

Definition

$\varphi \mathbf{W} \psi \equiv \varphi \mathbf{U} \psi \vee \mathbf{G} \varphi$

CTL

- $\mathbf{E}\varphi \mathbf{W} \psi \equiv \mathbf{EG} \varphi \vee \mathbf{E}\varphi \mathbf{U} \psi$
- $\mathbf{A}\varphi \mathbf{W} \psi \equiv \neg \mathbf{E}(\neg\psi) \mathbf{U} (\neg\varphi \wedge \neg\psi)$

Question

Can we express $\langle\langle A\rangle\rangle\, a \mathbf{W} b$ in ATL?

- $\langle\langle A\rangle\rangle (\mathbf{G} \varphi \vee \varphi \mathbf{U} \psi)$ is not an ATL formula,
- $\langle\langle A\rangle\rangle \varphi \mathbf{W} \psi \not\equiv \langle\langle A\rangle\rangle \mathbf{G} \varphi \vee \langle\langle A\rangle\rangle \varphi \mathbf{U} \psi$.

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Idea:

- We present two families of models that cannot be distinguished by ATL formulae of any given size.
- One model satisfies $\langle\langle A\rangle\rangle\, a\, \mathbf{W}\, b$ while the other does not.

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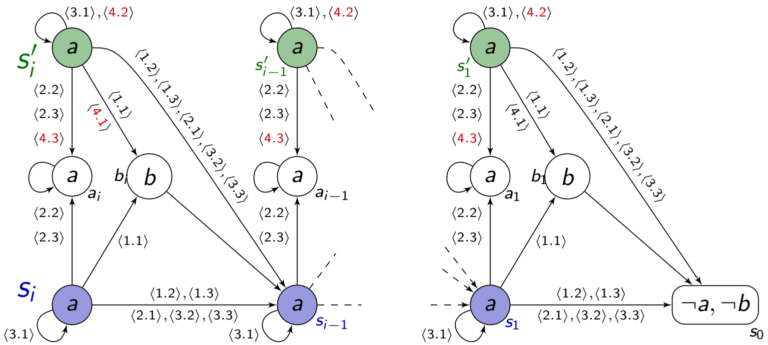

Lemma

$\forall i > 0, \forall \psi \in ATL$ *with* $|\psi| \leq i$ *we have:* $s_i \models \psi$ *iff* $s_i' \models \psi$.

Outline of the talk

ATL model checking over CGSs

Theorem ([AHK02])

Model checking ATL over CGSs is PTIME-complete.

- *Pre* $(A, L) = \{\ell \mid \exists m_A \cdot \mathrm{Next}(\ell, A, m_A) \subseteq L\}$

$\varphi = \langle\!\langle A \rangle\!\rangle\, \theta_1 \,\mathbf{U}\, \theta_2$

$L := [\mathit{false}]$; $T := [\theta_2]$;
while $T \not\subseteq L$ do
    $L := L \cup T$;
    $T := \mathit{Pre}(A, L) \cap [\theta_1]$
od;
$[\varphi] := T$

Overall complexity: $O(|\mathrm{Edg}| \cdot |\varphi|)$, thus PTIME.

Implicit CGS

Definition

An implicit CGS is a CGS where:

- The transition function: in each ℓ it's given $((\varphi_0, \ell_0), \cdots, (\varphi_n, \ell_n))$ where $\ell_i \in \text{Loc}$, φ_i is a boolean combination of propositions $A_j = c$.
- $\text{Edg}(\ell, m_{A_1}, \cdots, m_{A_k}) = \ell_j$ s.t.

$$j = min(i \mid \varphi_i(\ell, m_{A_1}, \cdots, m_{A_k}) = \top).$$

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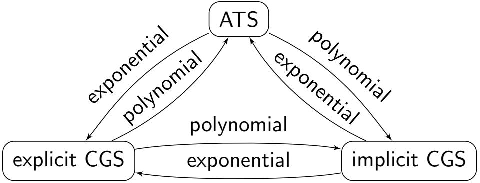

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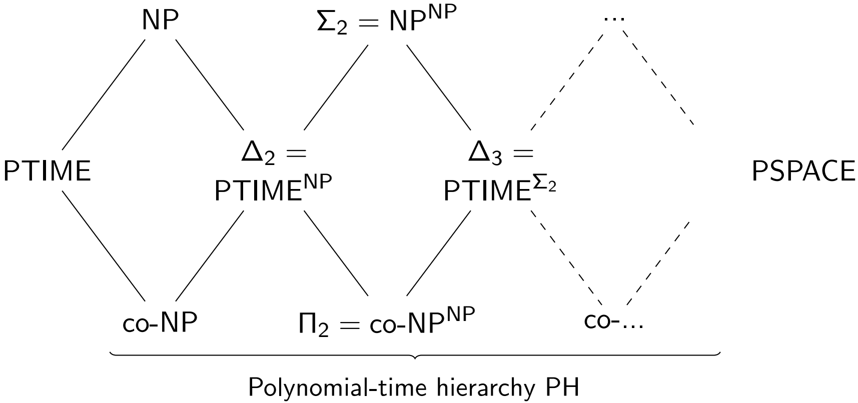

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Model checking ATL over implicit CGSs in Δ_3-complete.

Membership in Δ_3.

- Σ_2 algorithm proposed in [JD05]: correctly handles positive formulas (*i.e.* of the form $\langle\!\langle A \rangle\!\rangle\, \varphi$).
- That algorithm is used as an oracle, called a polynomial number of times.

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Hardness in Σ_2. [JD05]

EQSAT_2:

Input: a boolean formula φ over variables in $X \cup Y$.

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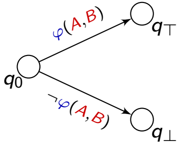
Lemma

The instance of EQSAT$_2$ is positive iff

$$q_0 \models \langle\langle A_1, \ldots, A_n \rangle\rangle \mathbf{X}\, q_\top.$$



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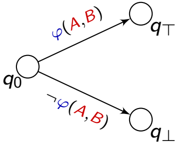

ATL model checking over implicit CGSs

Theorem

Model checking ATL over implicit CGSs in Δ_3-complete.

Hardness in Δ_3 (sketch).

SNSAT_2:

Input: formulas φ_i over variables in $X_i \cup Y_i \cup \{z_1, \ldots, z_{i-1}\}$.

Output: the value of z_m defined by:

$$\begin{cases} z_1 \stackrel{\text{def}}{=} \exists X_1.\ \forall Y_1.\ \varphi_1(X_1, Y_1) \\ z_2 \stackrel{\text{def}}{=} \exists X_2.\ \forall Y_2.\ \varphi_2(z_1, X_2, Y_2) \\ z_3 \stackrel{\text{def}}{=} \exists X_3.\ \forall Y_3.\ \varphi_3(z_1, z_2, X_3, Y_3) \\ \quad \ldots \\ z_m \stackrel{\text{def}}{=} \exists X_m.\ \forall Y_m.\ \varphi_m(z_1, \ldots, z_{m-1}, X_m, Y_m) \end{cases}$$

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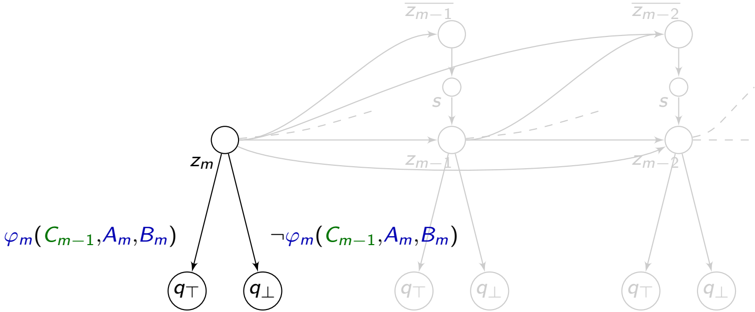

$$\psi_m = \langle\langle AC \rangle\rangle (\neg s)\, \mathbf{U}\, (q_\top \wedge \mathbf{EX}\, (s \wedge \mathbf{EX}\, \neg\psi_{m-1})).$$

ATL model checking over ATSs

Theorem ([AHK97])

Model checking ATL over ATSs is PTIME-complete.

Proof. Similar to the case of CGSs. □

But... Transitions of an ATS are not given explicitely.

The algorithm is polynomial *in the size of the underlying CGS* (which might be exponential).

Theorem ([JD05])

Model checking ATL over ATSs is PTIME-complete if the number of agents is fixed.

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NP-*hardness:* Reduction from 3-SAT.

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NP-*hardness:* Reduction from 3-SAT.

$$C = p \vee \neg q \vee r \qquad \rightsquigarrow \qquad \begin{cases} c_0 = \neg p \vee \neg q \vee \neg r \\ c_1 = \neg p \vee \neg q \vee r \\ c_2 = \neg p \vee q \vee \neg r \\ c_3 = \neg p \vee q \vee r \\ c_4 = p \vee \neg q \vee \neg r \\ c_5 = p \vee \neg q \vee r \\ c_6 = p \vee q \vee \neg r \\ c_7 = p \vee q \vee r \end{cases}$$

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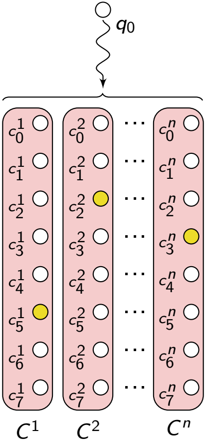

ATL model checking over ATSs

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1 player (P_1 to P_k) per atomic proposition:

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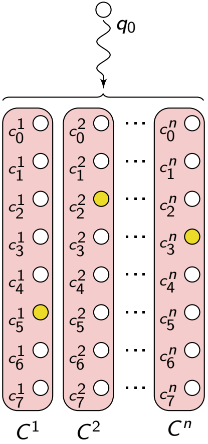

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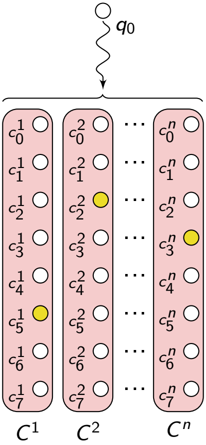
Once those players have chosen their moves, exactly one clause c^i_j per original clause C^i belongs to the intersection of the chosen sets.

$$\text{E.g.}\quad \left.\begin{array}{r} p = \top \\ q = \top \\ r = \bot \end{array}\right\} \Rightarrow \neg p \wedge \neg q \wedge r$$



ATL model checking over ATSs

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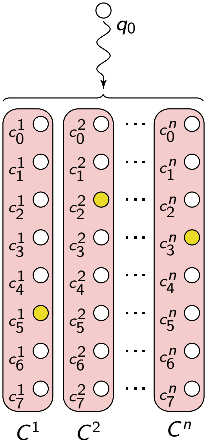
1 extra player chooses one set among $\{c_0^1, ..., c_7^1\}$ to $\{c_0^n, ..., c_7^n\}$



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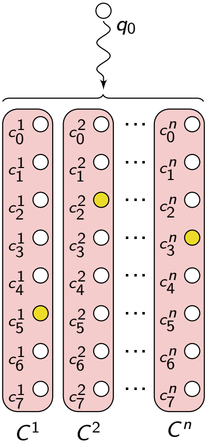

Lemma

The 3-SAT instance is true iff

$$q_0 \models \langle\!\langle P_1, \ldots, P_k \rangle\!\rangle \mathbf{X} \neg \bullet$$

Conclusion

Expressiveness

$$\mathrm{ATL_W} > \mathrm{ATL}$$

Complexity results

	CGS_{fixed}	CGS	ATS
ATL	PTIME	Δ_3	Δ_2
ATL^+	Δ_3	Δ_3	Δ_3

Future Work

- Fairness constraints
- Timed models