Wadge Games Between 1-Counter Automata and Models of Set Theory

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Acceptance of infinite words

- **In the sixties**,
  Acceptance of infinite words by finite automata was firstly considered by **Büchi** in order to study the decidability of the monadic second order theory S1S of one successor over the integers.
- Since then ω-regular languages accepted by Büchi automata and their extensions have been much studied and used for **specification and verification of non terminating systems**.

Büchi acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of final states $F \subseteq K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff there is some state $q_f \in F$ appearing infinitely often during the reading of σ.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

An ω-language $L \subseteq \Sigma^\omega$ is accepted by $\mathcal{A}$ if it is the set of infinite words $\sigma \in \Sigma^\omega$ accepted by $\mathcal{A}$.

Muller acceptance condition

An automaton $\mathcal{A}$ reading infinite words over the alphabet Σ is equipped with a finite set of states K and a set of accepting sets of states $\mathcal{F} \subseteq 2^K$.

A run of $\mathcal{A}$ reading an infinite word $\sigma \in \Sigma^\omega$ is said to be accepting iff the set of states appearing infinitely often during this run is an accepting set $F \in \mathcal{F}$.

An infinite word $\sigma \in \Sigma^\omega$ is accepted by $\mathcal{A}$ if there is (at least) one accepting run of $\mathcal{A}$ on σ.

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Context free or regular ω-languages

(Cohen and Gold 1977; Linna 1976)

Let $L \subseteq \Sigma^\omega$. Then the following propositions are equivalent :

- L is accepted by a Büchi pushdown automaton.
- L is accepted by a Muller pushdown automaton.
- $L = \bigcup_{1 \leq i \leq n} U_i.V_i^\omega$,
  for some context free finitary languages U_i and V_i.
- L is a context free ω-language.

A similar theorem holds if we:

- omit the pushdown stack and replace context free by regular,
- or replace pushdown and context-free by 1-counter.

Languages of infinite words

An ω-language over the alphabet Σ is a subset of Σ^ω.

An ω-language is regular iff it is accepted by a Büchi automaton.

An ω-language is context free iff it is accepted by a Büchi pushdown automaton.

A 1-counter ω-language is an ω-language which is accepted by a 1-counter Büchi automaton.

Complexity of ω-languages

The question naturally arises of the complexity of ω-languages accepted by various kinds of automata.

A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Wadge Reducibility

Definition (Wadge 1972)

For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, $L \leq_W L'$ iff there exists a continuous function $f : X^\omega \rightarrow Y^\omega$, such that $L = f^{-1}(L')$.

L and L' are Wadge equivalent ($L \equiv_W L'$) iff $L \leq_W L'$ and $L' \leq_W L$. .

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. The equivalence classes of $\equiv_W$ are called Wadge degrees.

Intuitively $L \leq_W L'$ means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function.

Hence the Wadge degree of an ω-language is a measure of its topological complexity.

Wadge degrees were firstly studied by Wadge for Borel sets using Wadge games.

There is a close relationship between Wadge reducibility and games:

Definition (Wadge 1972)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'.

The two players alternatively write letters a_n of X for Player 1 and b_n of Y for player 2.
Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps.

After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written $b \in Y^\omega$.
Player 2 wins the play iff $[a \in L \leftrightarrow b \in L']$, i.e. iff :

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L')].$$

Wadge Games

Theorem (Wadge)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. Then $L \leq_W L'$ iff Player 2 has a winning strategy in the game $W(L, L')$.

By Martin's Theorem, the Wadge game $W(L, L')$, for Borel sets L and L', is determined: One of the two players has a winning strategy.

$\longrightarrow$ Study of the Wadge hierarchy on Borel sets.

Borel Hierarchy

Σ_1^0 is the class of open subsets of Σ^ω,

Π_1^0 is the class of closed subsets of Σ^ω,

for any integer $n \geq 1$:

Σ_{n+1}^0 is the class of countable unions of Π_n^0-subsets of Σ^ω.

Π_{n+1}^0 is the class of countable intersections of Σ_n^0-subsets of Σ^ω.

Π_{n+1}^0 is also the class of complements of Σ_{n+1}^0-subsets of Σ^ω.

Borel Hierarchy

The Borel hierarchy is also defined for levels indexed by countable ordinals.

For any countable ordinal $\alpha \geq 2$:

Σ^0_α is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.

Π^0_α is the class of complements of Σ^0_α-sets

$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$.

Borel Hierarchy

Below an arrow $\rightarrow$ represents a strict inclusion between Borel classes.

$$
\begin{tikzcd}
 & \Pi^0_1 \arrow[dr] & & & & \Pi^0_\alpha \arrow[dr] & & \Pi^0_{\alpha+1} & \\
\Delta^0_1 \arrow[ur] \arrow[dr] & & \Delta^0_2 \arrow[ur] \arrow[dr] & \cdots\ \cdots & \Delta^0_\alpha \arrow[ur] \arrow[dr] & & \Delta^0_{\alpha+1} \arrow[ur] \arrow[dr] & & \cdots \\
 & \Sigma^0_1 \arrow[ur] & & & & \Sigma^0_\alpha \arrow[ur] & & \Sigma^0_{\alpha+1} &
\end{tikzcd}
$$

A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ where ω_1 is the first uncountable ordinal.

Beyond the Borel Hierarchy

There are some subsets of Σ^ω which are not Borel. Beyond the Borel hierarchy is the projective hierarchy.

The class of Borel subsets of Σ^ω is strictly included in the class Σ^1_1 of analytic sets which are obtained by projection of Borel sets.

A set $E \subseteq \Sigma^\omega$ is in the class Σ^1_1 iff :

$\exists F \subseteq (\Sigma \times \{0, 1\})^\omega$ such that F is Π^0_2 and

E is the projection of F onto Σ^ω

A set $E \subseteq \Sigma^\omega$ is in the class Π^1_1 iff $\Sigma^\omega - E$ is in Σ^1_1.

Suslin's Theorem states that : Borel sets = $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Complete Sets

A set $E \subseteq \Sigma^\omega$ **is $\mathcal{C}$-complete**, where $\mathcal{C}$ is a Borel class Σ^0_α or Π^0_α

or the class Σ^1_1, **for reduction by continuous functions** iff :

$\forall F \subseteq \Gamma^\omega \quad F \in \mathcal{C}$ **iff** $F \leq_W E$.

Example : $\{\sigma \in \{0,1\}^\omega \mid \exists^\infty i \;\; \sigma(i) = 1\}$ is a Π^0_2-complete-set and it is accepted by a deterministic Büchi automaton.

More Examples of Complete Sets

Examples :

$\{\sigma \in \{0,1\}^\omega \mid \exists i \;\; \sigma(i) = 1\}$ is a Σ^0_1**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \forall i \;\; \sigma(i) = 1\} = \{1^\omega\}$ is a Π^0_1**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \exists^{<\infty} i \;\; \sigma(i) = 1\}$ is a Σ^0_2**-complete-set**.

All these ω-languages are ω-regular.

Complexity of ω-languages of deterministic machines

deterministic finite automata (Landweber 1969)

- ω-**regular languages** accepted by **deterministic Büchi automata** are Π^0_2-sets.
- ω-**regular languages** are boolean combinations of Π^0_2-sets hence Δ^0_3-sets.

deterministic Turing machines

- ω-**languages** accepted by **deterministic Büchi Turing machines** are Π^0_2-sets.
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Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi or Muller Turing machines accept **effective analytic sets** (Staiger). The class **Effective-Σ^1_1** of **effective analytic sets** is obtained as the class of **projections of arithmetical sets** and **Effective-Σ^1_1** $\subsetneq \Sigma^1_1$.

Let ω_1^{CK} be the first non recursive ordinal.

Topological Complexity of Effective Analytic Sets

- There are some Σ^1_1-complete sets in **Effective-Σ^1_1**.
- For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in the class **Effective-Σ^1_1**.
- **(Kechris, Marker and Sami 1989)**
  The supremum of the set of Borel ranks of **Effective-Σ^1_1-sets** is a countable ordinal $\gamma^1_2 > \omega_1^{\mathrm{CK}}$.

Let $1 - CL_\omega$ be the class of real-time 1-counter ω-languages.

Let $\mathcal{C}$ be a class of ω-languages such that:

$$1 - CL_\omega \subseteq \mathcal{C} \subseteq \textbf{ Effective-}\Sigma^1_1.$$

(a) **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class** $\mathcal{C}$.

(b) **(F. 2005)** The Borel hierarchy of **the class** $\mathcal{C}$ is equal to the Borel hierarchy of **the class Effective-**Σ^1_1.

(c) γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class** $\mathcal{C}$.

(d) For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class** $\mathcal{C}$.

Topological complexity of 1-counter or context free ω-languages

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(c) γ_2^1 is the supremum of the set of Borel ranks of ω-languages in **the class $\mathcal{C}$.**

(d) For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ_α^0-complete and some Π_α^0-complete ω-languages in **the class $\mathcal{C}$.**

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Let $\mathcal{C}$ be a class of ω-languages such that:

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- **(a)** **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class** $\mathcal{C}$.
- **(b)** **(F. 2005)** The Borel hierarchy of **the class** $\mathcal{C}$ is equal to the Borel hierarchy of **the class Effective-**Σ^1_1.
- **(c)** γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class** $\mathcal{C}$.
- **(d)** For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class** $\mathcal{C}$.

Sketch of the proof

It is well known that every Turing machine can be simulated by a (non real time) 2-counter automaton.

We denote **BCL**$(2)_\omega$ the class of ω-languages accepted by Büchi 2-counter automata.

Thus the topological complexity of ω-languages in the class **BCL**$(2)_\omega$ is equal to the topological complexity of ω-languages accepted by Büchi Turing machines.

Sketch of the proof

First, from a 2-counter automaton A accepting an ω-language $L \subseteq X^\omega$, we construct a real-time 8-counter Büchi automaton B accepting an ω-language of the same topological complexity.

First, we add a storage type called a queue to a 2-counter Büchi automaton in order to read ω-words in real-time.

Then the queue can be simulated by

- two pushdown stacks or
- four counters,
  because each pushdown stack may be simulated by two counters.

Sketch of the proof

This simulation is not done in real-time but one can bound the number of transitions needed to simulate the queue. This allows to pad the strings in L with enough extra letters so that the new language $\theta_S(L)$ will be read in real-time by a 8-counter Büchi automaton.

The padding is obtained via the function $\theta_S : X^\omega \rightarrow (X \cup \{E\})^\omega$, where $S = (3k)^3$, with $k = card(X) + 2$, and for all $x \in X^\omega$:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

The ω-language $\theta_S(L)$ is accepted in real time by a Büchi automaton with $2 + 4 + 2 = 8$ counters.

Sketch of the proof

The next step is to simulate a *real-time* 8-counter Büchi automaton $\mathcal{A}$, by a *real-time* 1-counter Büchi automaton $\mathcal{B}$.

The eight first prime numbers are $2; 3; 5; 7; 11; 13; 17; 19$.

We code the content $(c_1, c_2, \ldots, c_8)$ of eight counters by the product $2^{c_1} \times 3^{c_2} \times \ldots \times (17)^{c_7} \times (19)^{c_8}$.

Then we code ω-words in $Y = X \cup \{E\}$ by ω-words in $Z = Y \cup \{A, B, 0\}$.

The new ω-words will have a **special shape** which will allow the propagation of the values of the counters of $\mathcal{A}$.

Sketch of the proof

The product of the eight first prime numbers is:

$$K = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 9699690$$

An ω-word $x \in Y^\omega$ is coded by the ω-word

$$h(x) = A.0^K.x(1).B.0^{K^2}.A.0^{K^2}.x(2).B.\ldots B.0^{K^n}.A.0^{K^n}.x(n).B\ldots$$

If $L(\mathcal{A}) \subseteq Y^\omega$ is accepted by a real time 8-counter Büchi automaton $\mathcal{A}$, then one can construct from $\mathcal{A}$ a 1-counter Büchi automaton $\mathcal{B}$, reading words over $Y \cup \{A, B, 0\}$, such that:

$$\forall x \in Y^\omega \quad h(x) \in L(\mathcal{B}) \longleftrightarrow x \in L(\mathcal{A})$$

Sketch of the proof

The mapping $h : Y^\omega \to (Y \cup \{A, B, 0\})^\omega$ is continuous.

The complement $h(Y^\omega)^-$ of the ω-language $h(Y^\omega)$ is an open subset of $(Y \cup \{A, B, 0\})^\omega$ and is accepted by a real time 1-counter automaton.

Thus the ω-language

$$h(L(\mathcal{A})) \cup h(Y^\omega)^- = L(\mathcal{B}) \cup h(Y^\omega)^-$$

is in the class $\mathbf{BCL}(1)_\omega$ and it has the same topological complexity as the ω-language $L(\mathcal{A})$.

Independence from the Axiomatic System ZFC of Set Theory

ZFC : Zermelo-Fraenkel Axiomatic System ZF + Axiom of Choice AC.

ZFC : commonly accepted Axiomatic System for Set Theory in which all usual mathematics can be developped.

Using some notions of set theory we show:

The topological complexity of a 1-counter ω-language may depend on the models of ZFC.

The axioms of ZFC express some natural facts that we consider to hold in the universe of sets.

These axioms are first-order sentences in the logical language of set theory whose only non logical symbol is the membership binary relation symbol $\in$.

The Axiomatic System ZFC of Set Theory

The *Axiom of Extensionality* states that two sets x and y are equal iff they have the same elements:

$$\forall x \forall y \, [\, x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y) \,].$$

The *Pairing Axiom* states that for all sets x and y there exists a set $z = \{x, y\}$ whose elements are x and y:

$$\forall x \forall y \, [\, \exists z (\forall w (w \in z \leftrightarrow (w = x \vee w = y)))]$$

Similarly the *Powerset Axiom* states the existence of the set of subsets of a set x.

Models of the Axiomatic System ZFC

A model $(\mathbf{V}, \in)$ of the axiomatic system **ZFC** is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of **ZFC**.

Perfect Sets, Thin Sets

Definition

Let $P \subseteq \Sigma^\omega$, where Σ is a finite alphabet having at least two letters. The set P is a perfect subset of Σ^ω iff it is a non-empty closed set which has no isolated points.

A perfect subset of Σ^ω has cardinality $2^{\aleph_0}$.

Definition

A set $X \subseteq \Sigma^\omega$ is said to be thin iff it contains no perfect subset.

Theorem (Souslin)

*(**ZFC**) An analytic set $X \subseteq \Sigma^\omega$ is either countable or contains a perfect subset. Thus every thin analytic set is countable.*

This result is not true for co-analytic sets in **ZFC**. We need additional axioms like analytic determinacy.

The Largest Thin Effective Coanalytic Set

Theorem (Kechris 1975; Guaspari, Sacks)

(**ZFC**) *Let Σ be a finite alphabet having at least two letters. There exists a thin Π_1^1-set $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ which contains every thin, Π_1^1-subset of Σ^ω. It is called the largest thin Π_1^1-set in Σ^ω.*

Theorem (Kechris 1975; Guaspari, Sacks)

1. *There is a model V_1 of* **ZFC** *in which $\mathcal{C}_1(\Sigma^\omega)$ is countable.*
2. *There is a model V_2 of* **ZFC** *in which $\mathcal{C}_1(\Sigma^\omega)$ is uncountable.*

The Largest Thin Effective Coanalytic Set

Theorem

1. *There is a model V_1 of* **ZFC** *in which the largest thin Π_1^1-set in Σ^ω is countable, hence a Σ_2^0-set.*
2. *There is a model V_2 of* **ZFC** *in which the largest thin Π_1^1-set in Σ^ω is not a Borel set.*

Proof. There is a model V_1 of **ZFC** in which the largest thin Π_1^1-set in Σ^ω is countable. It is a countable union of singletons, and each singleton is a closed set. Thus $\mathcal{C}_1(\Sigma^\omega)$ is a countable union of closed sets, i.e. a Σ_2^0-set.

There is a model V_2 of **ZFC** in which the largest thin Π_1^1-set in Σ^ω is uncountable. But it is thin, hence has no perfect subset. Thus it cannot be a Borel set because Borel sets have the perfect set property: a Borel set is either countable or contains a perfect subset.

From effective coanalytic sets to 1-counter automata

The complement of $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ is an effective analytic set accepted by a Büchi Turing machine $\mathcal{T}$.

We can now use previous constructions to obtain:

- A 2-counter Büchi automaton $\mathcal{A}_1$,
- A real time 8-counter Büchi automaton $\mathcal{A}_2$,
- A real time 1-counter Büchi automaton $\mathcal{A}_3$,

such that $L(\mathcal{T})$, $L(\mathcal{A}_1)$, $L(\mathcal{A}_2)$, and $L(\mathcal{A}_3)$, all have the same toplogical complexity.

The Topological complexity of a 1-counter ω-language depends on the models of ZFC

Theorem (F. 2009)

There exists a 1-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC**.

1. *There is a model V_1 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is an analytic but non Borel set.*
2. *There is a model V_2 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is a Π^0_2-set.*

The ω-language $(0^\star \cdot 1)^\omega \subseteq \{0,1\}^\omega$ is ω-regular, accepted by a Büchi automaton $\mathcal{B}$, and is Π^0_2-complete in every model of **ZFC**. This implies:

Theorem

There exist two 1-counter Büchi automata $\mathcal{A}$ and $\mathcal{B}$ such that $L(\mathcal{A}) \leq_W L(\mathcal{B})$ is independent from **ZFC***:*

(1) There is a model V_1 of **ZFC** *in which Player 2 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.*

(2) There is a model V_2 of **ZFC** *in which Player 2 has no winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.*

A similar result holds for 2-tape Büchi automata.

Theorem

There exist two 1-counter Büchi automata $\mathcal{A}$ and $\mathcal{B}$ such that "$W(L(\mathcal{A}), L(\mathcal{B}))$ is determined" is independent from **ZFC***:*

(1) There is a model V_1 of **ZFC** *in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is determined.*

(2) There is a model V_2 of **ZFC** *in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined.*

A similar result holds for 2-tape Büchi automata.