The Complexity of Nash Equilibria in Simple Stochastic Multiplayer Games

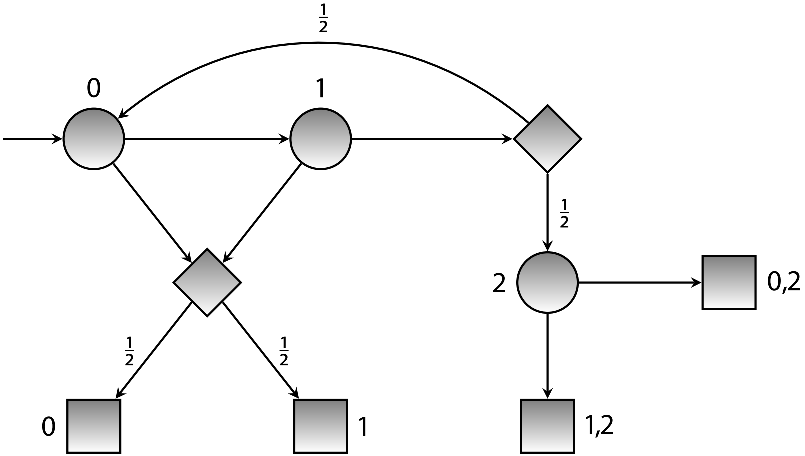
Michael Ummels
ummels@logic.rwth-aachen.de

GASICS 2009

(Joint Work with Dominik Wojtczak)

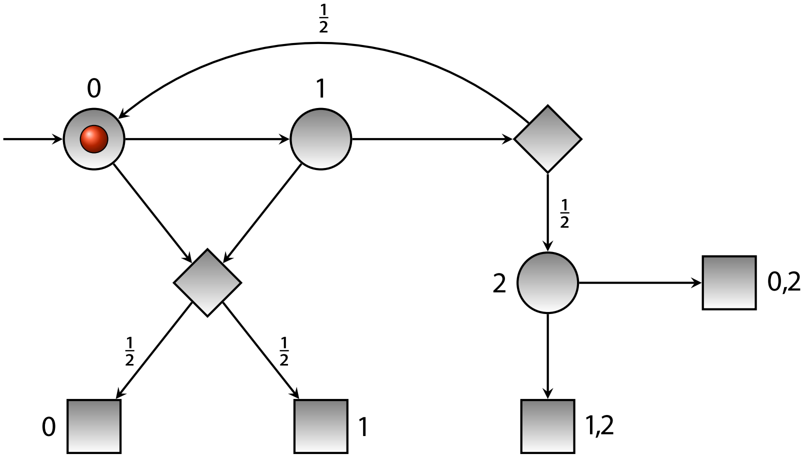
Simple Stochastic Games

What is a simple multiplayer stochastic game (SSMG)? Let's play!



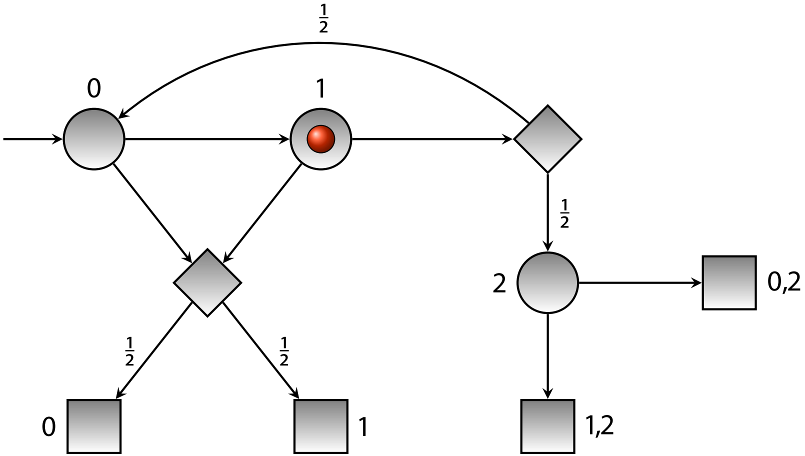
Simple Stochastic Games

What is a simple multiplayer stochastic game (SSMG)? Let's play!



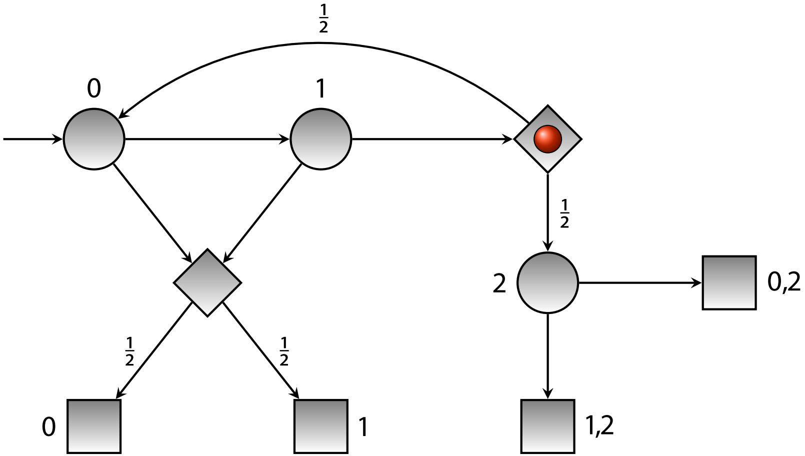
Simple Stochastic Games

What is a simple multiplayer stochastic game (SSMG)? Let's play!



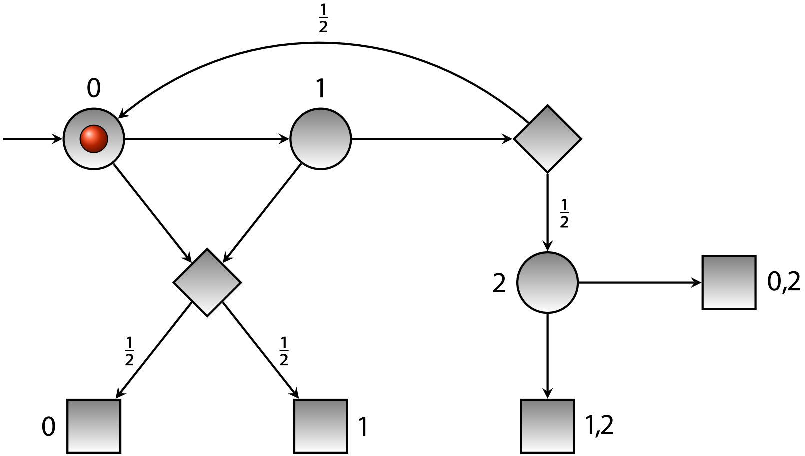
Simple Stochastic Games

What is a simple multiplayer stochastic game (SSMG)? Let's play!

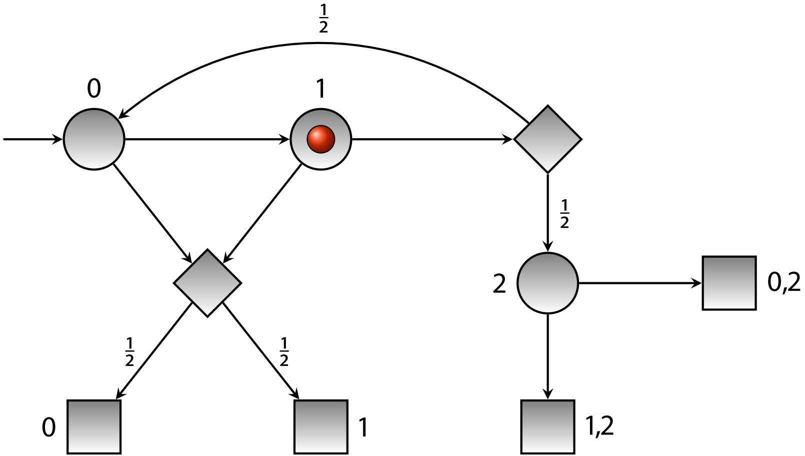


Simple Stochastic Games

What is a simple multiplayer stochastic game (SSMG)? Let's play!

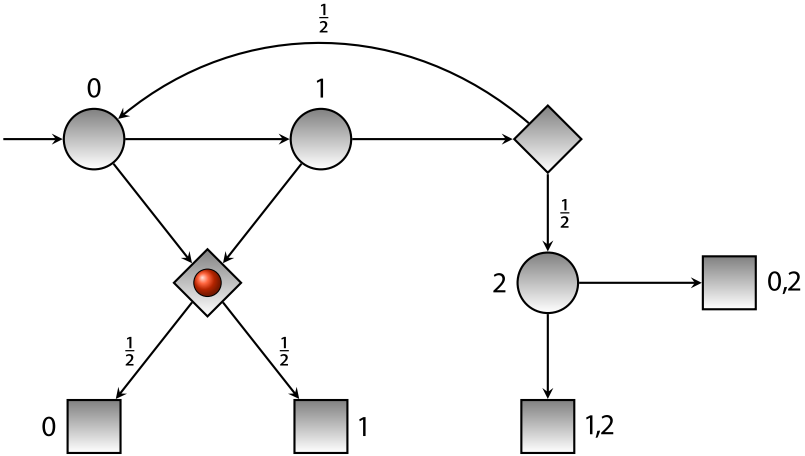


What is a simple multiplayer stochastic game (SSMG)? Let's play!

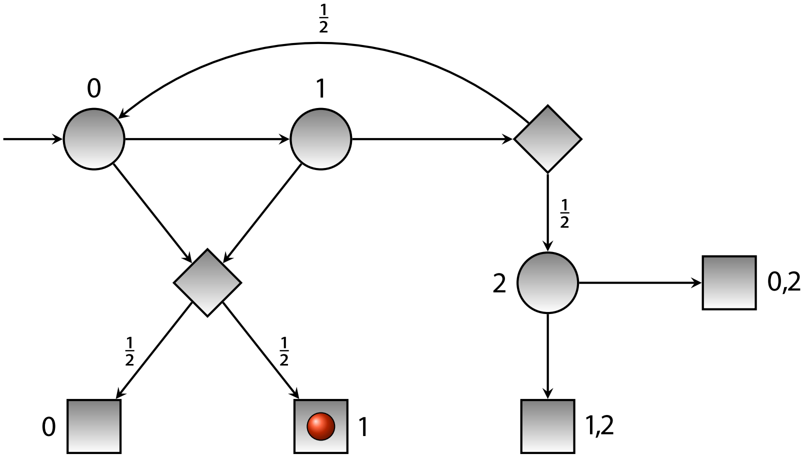


Simple Stochastic Games

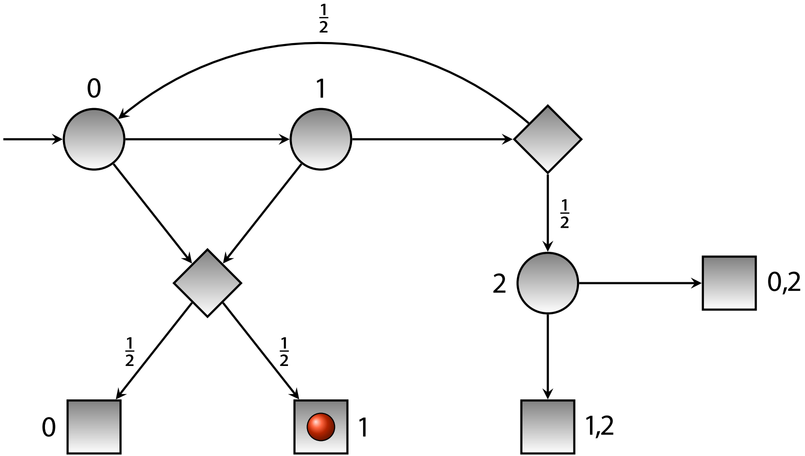
What is a simple multiplayer stochastic game (SSMG)? Let's play!



What is a simple multiplayer stochastic game (SSMG)? Let's play!



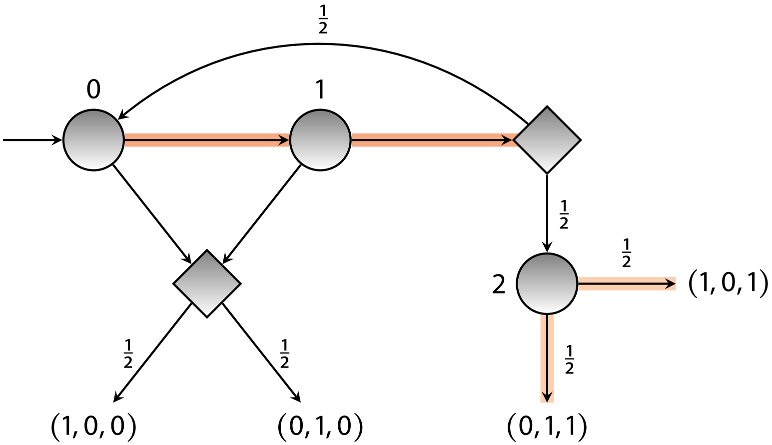
What is a simple multiplayer stochastic game (SSMG)? Let's play!



Winners are decided at terminal vertices. Infinite plays are *bad*.

Strategies and Probabilities

A strategy profile induces a probability distribution on sets of plays.



Payoff of this strategy profile: $(\frac{1}{2}, \frac{1}{2}, 1)$

Nash Equilibria

Definition: A strategy profile is a Nash equilibrium if no player can gain from unilaterally switching to a different strategy.

Question: Do Nash equilibria always exist?

Theorem (Chatterjee & al., 2004)

Any SSMG has a Nash equilibrium (in pure finite-state strategies).

Next Question: Can we compute one?

Proposition

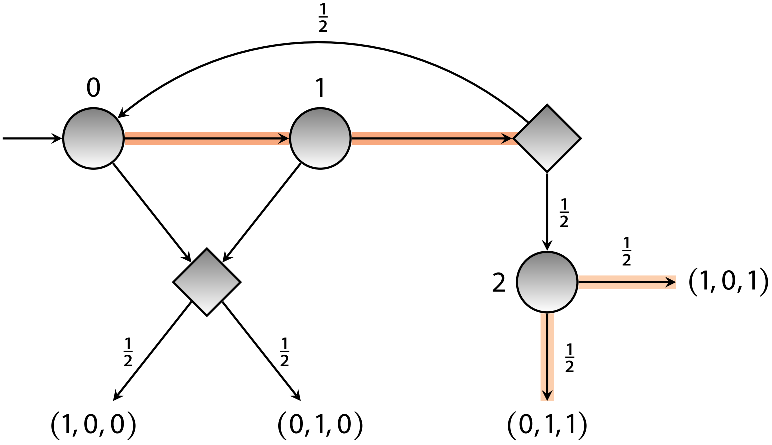
The problem of computing a (pure finite-state) Nash equilibrium of an SSMG is in FNP.

Open Problem: Can one compute a Nash equilibrium in polynomial time?

But there may be many Nash equilibria (with different payoffs)...

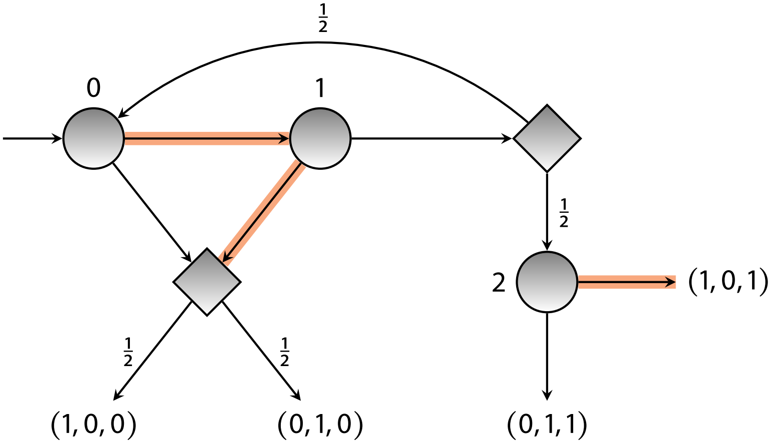
Example

Nash Equilibrium where Player 2 wins (with probability 1):



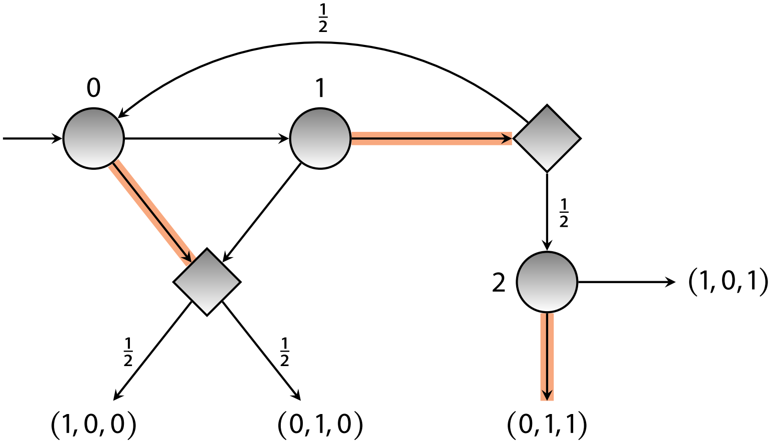
Example

Nash Equilibrium where Player 2 loses:



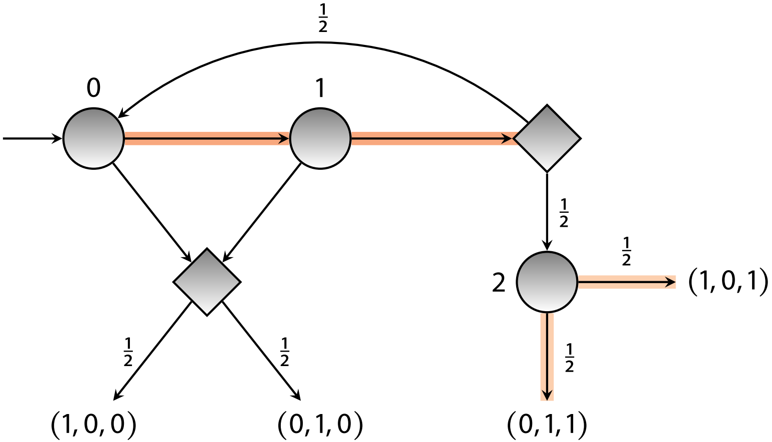
Example

Nash Equilibrium where Player 2 loses:



Example

Nash Equilibrium where Player 2 wins (with probability 1):



Observation: Mixed strategies are more powerful...

The Problem NE

Goal: Compute a Nash equilibrium that meets certain requirements on the payoff.

The problem NE: Given an SSMG $\mathcal{G}$, two payoff thresholds $\overline{x}, \overline{y} \in [0,1]^k$, decide whether the game has a Nash equilibrium with payoff $\geq \overline{x}$ and $\leq \overline{y}$.

Special case: Given an SSMG $\mathcal{G}$, decide whether the game has a Nash equilibrium where Player 0 wins almost surely.

Variants of the Problem:

- Mixed strategies
- Pure strategies
- Mixed, memoryless strategies
- Pure, memoryless strategies

Our Results

The Complexity of NE (and the special case):

	Pure	Mixed
Arbitrary	Undecidable	?
Memoryless	NP-complete	PSPACE *

* NP-hard and at least as hard as SQRT-SUM

Our Results

The Complexity of NE (and the special case):

	Pure	Mixed
Arbitrary	Undecidable	?
Memoryless	NP-complete	PSPACE *

* NP-hard and at least as hard as SQRT-SUM

Pure, memoryless Nash equilibria

Deciding the existence of a pure, memoryless NE with payoff $\geq \overline{x}$ and $\leq \overline{y}$:

- Guess a pure, memoryless strategy profile $\overline{\sigma}$.
- For each player i:

1. Compute the payoff r_i of $\overline{\sigma}$ for player i.
2. Compute the maximal payoff z_i player i can achieve by herself.
3. Check whether $x_i \leq r_i \leq y_i$ and $z_i \leq r_i$.

1\. and 2. are doable in polynomial time (via linear programming).

Theorem

NE for pure, memoryless strategies is in NP.

Remark: NP-hardness is shown by a reduction from SAT and even holds for games with only two players (and also for mixed strategies).

Our Results

The Complexity of NE (and the special case):

	Pure	Mixed
Arbitrary	Undecidable	?
Memoryless	NP-complete	PSPACE *

* NP-hard and at least as hard as SQRT-SUM

Memoryless Nash Equilibria

Deciding the existence of a memoryless NE with payoff $\geq \overline{x}$ and $\leq \overline{y}$:

- Guess the support S of a memoryless strategy profile $\overline{\sigma}$.
- For each player i, compute the set R_i of vertices from where the set of winning terminal vertices is reachable when playing $\overline{\sigma}$.
- Evaluate an existential first-order sentence ψ (which is polynomial-time computable from $\mathcal{G}, \overline{x}, \overline{y}, S$ and $(R_i)_{i \in Players}$) over $\mathfrak{R} = (\mathbb{R}, +, \cdot, 0, 1)$.

ψ states that there exists a memoryless Nash equilibrium $\overline{\sigma}$ with payoff $\geq \overline{x}$ and $\leq \overline{y}$ whose support is precisely S.

Note: The existential theory of $\mathfrak{R}$ is decidable in PSPACE.

Theorem

NE is in NPSPACE, and hence in PSPACE, for memoryless strategies.

The Square-Root-Sum Problem

Square-Root-Sum Problem (SQRT-SUM): Given $d_1, \ldots, d_n \in \mathbb{N}$ and $k \in \mathbb{N}$, decide whether $\sum_{i=1}^{n} \sqrt{d_i} \geq k$.

The precise complexity of SQRT-SUM is not known:

- Known to be in PSPACE (actually in the 4th level of the counting hierarchy).
- No non-trivial lower bounds known.

Open Problem (since 1970s): Does SQRT-SUM lie inside the polynomial hierarchy? Is it in NP?

Theorem

There is a polynomial-time reduction from SQRT-SUM to NE for memoryless strategies.

Our Results

The Complexity of NE (and the special case):

	Pure	Mixed
Arbitrary	Undecidable	?
Memoryless	NP-complete	PSPACE *

* NP-hard and at least as hard as SQRT-SUM

Undecidability

Undecidability is shown by a reduction from the non-halting problem for two-counter machines.

Theorem

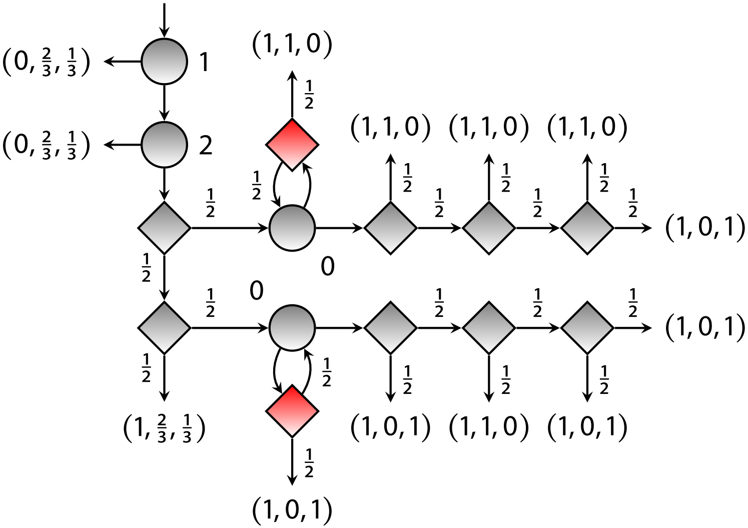
For every two-counter machine $\mathcal{M}$ one can (algorithmically) construct a nine-player SSMG $\mathcal{G}$ such that the computation of $\mathcal{M}$ is infinite iff $\mathcal{G}$ has a pure Nash equilibrium where player 0 wins almost surely.

Corollary

NE is undecidable for pure strategies.

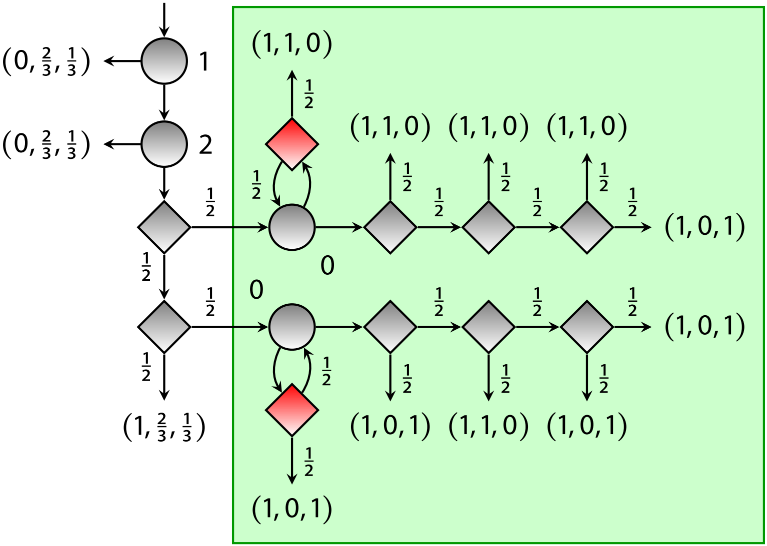
Remark: The reduction is similar to one used by Brázdil et al. (2006) for showing that stochastic games with *branching-time* winning conditions are undecidable.

Simulating a Counter



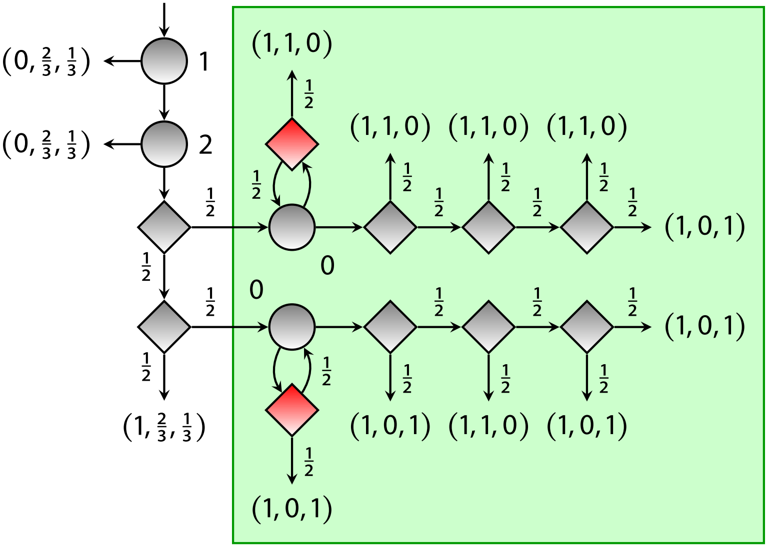
Encoding of the counter: (Maximal) number of visits to the red vertex.

Simulating a Counter



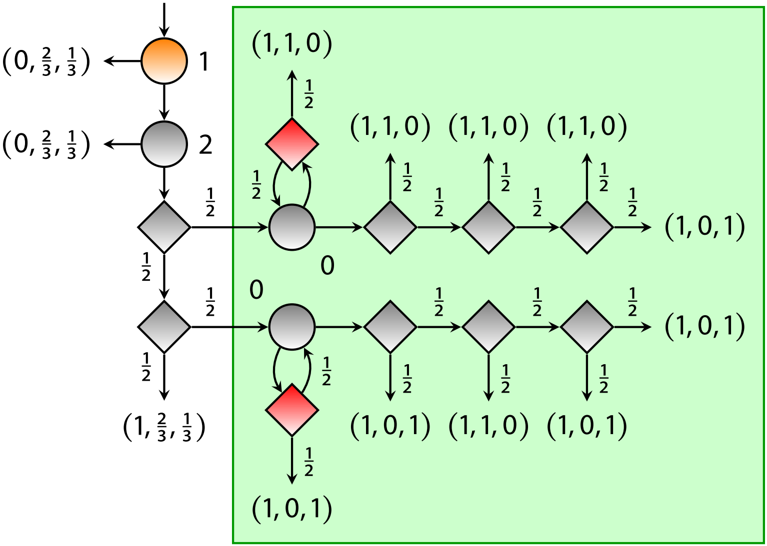
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{4} \cdot \frac{2}{3}$.

Simulating a Counter



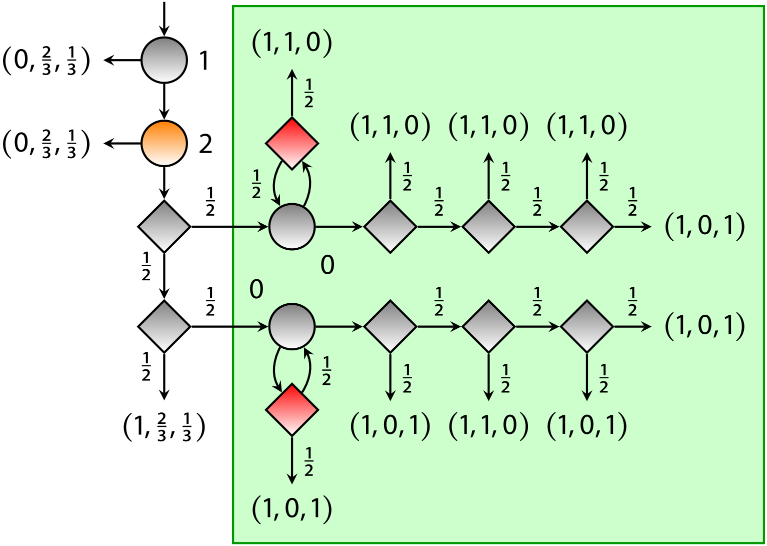
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{6}$.

Simulating a Counter



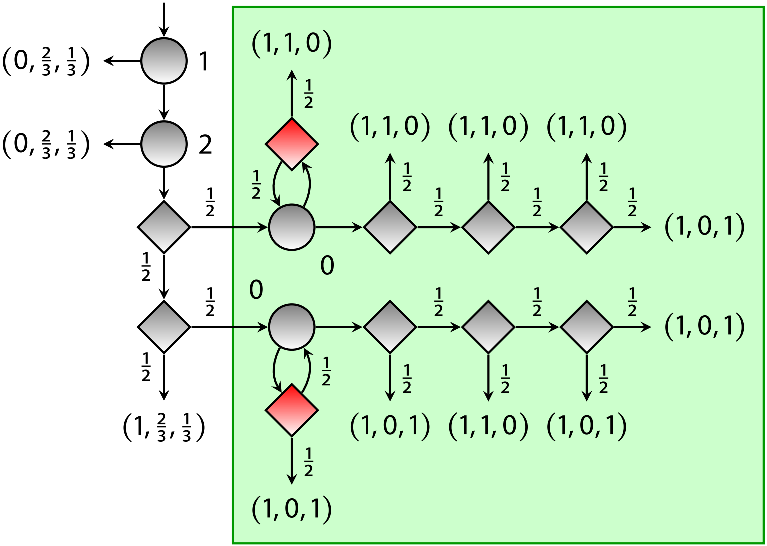
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{6} \geq \frac{2}{3}$.

Simulating a Counter



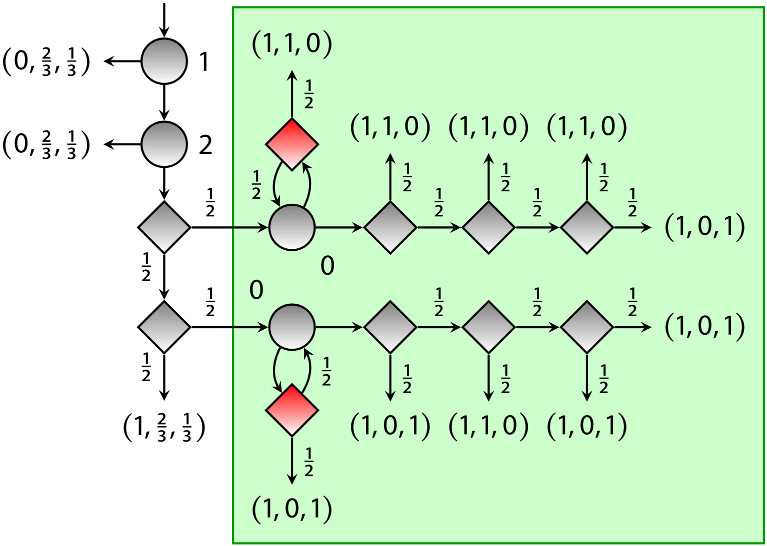
Payoff for player 1 in NE where player 0 wins almost surely: $p + \frac{1}{6} = \frac{2}{3}$.

Simulating a Counter



Necessary for NE where player 0 wins almost surely: $p = \frac{1}{2}$.

Simulating a Counter



Necessary for NE where player 0 wins almost surely: Counter maintained.

Finite-State Nash Equilibria

A consequence of the proof is:

Theorem

There exists an SSMG $\mathcal{G}$ with a pure Nash equilibrium where player 1 wins with probability 1, but in any pure finite-state Nash equilibrium player 1 wins with probability 0.

Question: Is NE decidable when restricted to pure finite-state strategies?

Remark: NE for pure finite-state strategies is recursively enumerable.

Theorem

NE is undecidable for pure finite-state strategies.

The proof is by a reduction from the halting problem for two-counter machines (similar to the one before).

The Qualitative Fragment

Remark: All undecidability results rely on the possibility of having a non-binary payoff.

Question: What happens if we restrict to Nash equilibria with a binary payoff?

The problem QualNE: Given an SSMG $\mathcal{G}$, a payoff vector $\overline{x} \in \{0,1\}^k$, decide whether the game has a Nash equilibrium with payoff $\overline{x}$.

Theorem

QualNE is decidable in polynomial time.

Remark: Decidability holds for games with arbitrary ω-regular winning conditions. Moreover, the problem is invariant under restricting the search space to pure, finite-state Nash equilibria.

Conclusion

What to take home?

Finding good Nash equilibria in simple stochastic games is hard.

Future work:

- Mixed Nash equilibria.
- Games with few players.
- Infinite-state games.