Transforming Game Specifications into Winning Strategies: A General Study

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Outline

1. Church's Problem and the Büchi-Landweber Theorem
2. Regular Winning Conditions
3. Context-Free Winning Conditions

Church's Problem

Problem of Controller Synthesis.

- Input $\alpha \in \Sigma_1^\omega$ Output $\beta \in \Sigma_2^\omega$
- $\Sigma = (\Sigma_1 \times \Sigma_2)$ $\binom{\alpha}{\beta} \in \Sigma^\omega$

Given: Specification $L \subseteq \Sigma^\omega$

Question: Is there a letter-by-letter transducer, that transforms every input $\alpha \in \Sigma_1^\omega$ into an output $\beta \in \Sigma_2^\omega$, such that $\binom{\alpha}{\beta} \in L$? If yes, construct one.

Infinite Games

- Infinite game with two players.
- They pick letters from their alphabet Σ_1 resp. Σ_2 in alternation.
- Concatenation of these letter pairs forms an infinite word.

Example (of a play)

Player 1:	a	b	a	a	a	a	a	a	a	a	…
Player 2:	b	c	a	a	a	a	a	a	a	a	…

Identify this play with the infinite word $\gamma = \binom{a}{b}\binom{b}{c}\binom{a}{a}\binom{a}{a}\binom{a}{a}\ldots$.

- Winning condition: Given by an ω-language L over Σ.
  - If $\gamma \in L$, Player 2 wins the play.
  - If $\gamma \notin L$, Player 1 wins the play.
- In "Gale-Stewart games" take the interleaving of the two sequences, in example: $\gamma = abbcaa\ldots$.

The Büchi-Landweber Theorem

Description of Church's Problem

1. Given (a finite presentation of) L, does Player 2 have a winning strategy in the game defined by L?
2. If yes, compute a winning strategy!

Standard case: L regular ω-language

Basic Result (Büchi-Landweber 1969)

For a regular ω-language L,

1. can be decided, and
2. for construction finite automata (with output) suffice.

Connection winning condition – winning strategy

- Usually winning conditions and strategies reside in different domains.
- What does it means that a strategy is FO-definable?
- We want to establish a connection between winning conditions and winning strategies on the same conceptual level.

Strategies as Tuples of Languages

- Given a strategy f_1
- For $c \in \Sigma_1$ introduce

$$K_c = \{w \in \Sigma^* \mid f_1(w) = c\}$$

- Represent f_1 by a tuple of $*$-languages $(K_c)_{c \in \Sigma_1}$
- Similarly for f_2:

$$K_c = \{w \in (\Sigma)^*(\Sigma_1 \times \{\Box\}) \mid f_2(w) = c\}$$

Example

- Strategy for Player 2: (K_a, K_b)

$K_a = \Sigma^*(\binom{b}{a}\binom{a}{\Box} + \binom{a}{b}\binom{a}{\Box} + \binom{a}{a}\binom{b}{\Box} + \binom{b}{b}\binom{b}{\Box})$

$K_b = \Sigma^*(\binom{a}{a}\binom{a}{\Box} + \binom{a}{b}\binom{b}{\Box} + \binom{b}{a}\binom{b}{\Box} + \binom{b}{b}\binom{a}{\Box})$

Logical Refinements

Theorem (Selivanov 2007, Rabinovich-Thomas 2007)

"Every X-game is determined with X winning strategies."
holds for

- $X =$ *MSO-definable*
- $X =$ *FO($<$)-definable*
- $X =$ *FO(S)-definable*
- $X =$ *FO($<$)+MOD-definable*

But the statement fails for

- $X =$ FO(S)+$\exists^\omega$
- $X =$ FO(S)+MOD

Locally Testable Languages

Definition

A $*$-language L is k-locally testable, if membership of w in L only depends on

- the set of factors of w of length k,
- the prefix of w of length $k - 1$
- and the suffix of w of length $k - 1$.

Refinement: k-locally r-threshold testable languages:
distinguish occurrences by number, up to threshold value r

Locally Testable ω-Languages

Definition

An ω-language L is *k-locally testable*, if membership of α in L only depends on

- the set of factors of α of length k
- and the prefix of α of length $k - 1$.

In the literature: “finitely locally testable”

Definition

An ω-language is called strongly locally testable, if it is of the form

$$L = \bigcup_{i=1}^{n} U_i V_i^{\omega}$$

where U_i, V_i are locally testable.

Example

Example (A 3-locally testable language ω-language)

$$L = \{\alpha \in (\{0,1\}^2)^\omega \mid \text{there is a 1 in the first component or the second component is } (001)^\omega\}$$

- L is 3-locally testable
- Consider typical case that Player 1 always chooses 0
Player 1:	0	0	0	0	0	0	0	0	0	0	…
Player 2:	0	0	1	0	0	1	0	0	1	0	…
- Player 2 has a 3-locally testable winning strategy: if $\binom{0}{0}\binom{0}{0}\binom{0}{\square}$ is a factor of the current play prefix $\rightsquigarrow$ 1 else $\rightsquigarrow$ 0
- There is no 2-locally testable winning strategy: when seeing $\binom{0}{0}\binom{0}{\square}$ choose a 0 or a 1?

Results on Regular Games

Theorem

Every X-game is determined with X winning strategies for the following cases of X:

- *locally testable*
- *k-locally r-threshold testable*
- *piecewise testable*
- *k-piecewise r-threshold testable*

The statement fails for

$X =$ strongly locally testable ω-languages.

Context-Free Languages

Definition

A pushdown automaton (PDA) $\mathcal{A} = \langle Q, \Sigma, \Gamma, \delta, q_{\text{in}}, \bot \rangle$ consists of

- finite set of states Q, initial state q_{in},
- input alphabet Σ,
- stack alphabet Γ, bottom stack symbol $\bot$,
- transition function $\delta\colon Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to \mathcal{P}(Q \times \Gamma^*)$

A PDA $\mathcal{A}$ is deterministic, if $\forall\ q, a, Z$ holds $|\delta(q, a, Z)| + |\delta(q, \varepsilon, Z)| \leq 1$.

Definition

CFL = class of languages recognized by a PDA with final state set
CFL_ω = class of ω-languages recognized by a PDA with Muller set
DCFL and DCFL_ω the corresponding classes for det. PDAs

Context-Free Specifications

Theorem

Church's Problem for $L \in CFL_\omega$ is undecidable.

Reduction: $\textsc{Universality}(\mathrm{CFL}_\omega) \leq \textsc{Church}(\mathrm{CFL}_\omega)$
For $L_1 \in \mathrm{CFL}_\omega$ define

$$L := \left\{ \binom{\alpha}{\beta} \in \binom{\Sigma_1}{\Sigma_2}^\omega \,\middle|\, \alpha \in L_1, \beta \in \Sigma_2^\omega \right\}$$

Then $L_1 = \Sigma_1^\omega \iff$ Player 2 has a winning strategy in the game defined by L. □

Theorem (Finkel 2001)

Even for the class of closed CFL_ω's it is undecidable which player has a winning strategy in the Gale-Stewart game defined by L.

DCFL

Theorem (Walukiewicz 1996)

Parity games on deterministic pushdown graphs are determined with deterministic pushdown winning strategies.

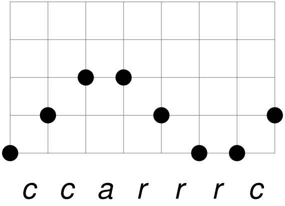
This result can be adapted easily to pushdown winning conditions.

Theorem

Games with winning conditions $L \in DCFL_\omega$ are determined with winning strategies in DCFL.

Visibly Pushdown Languages

- A VPA is a PDA with alphabet $\Sigma = \Sigma_c \cup \Sigma_r \cup \Sigma_{int}$
  - call - VPA pushes exactly one symbol: $\Delta_1 \subseteq Q \times \Sigma_c \times Q \times (\Gamma \setminus \{\bot\})$
  - return - VPA pops one symbol: $\Delta_2 \subseteq Q \times \Sigma_r \times \Gamma \times Q$
  - internal - stack is preserved: $\Delta_3 \subseteq Q \times \Sigma_{int} \times Q$



- ε-transitions are not allowed
- defines VPL and VPL_ω in the usual way
- nice closure properties

Visibly Pushdown Specifications

Theorem (Löding-Madhusudan-Serre 2004)

Visibly pushdown games are decidable and a pushdown winning strategy can be computed effectively.

Developed an equivalent deterministic automata model:

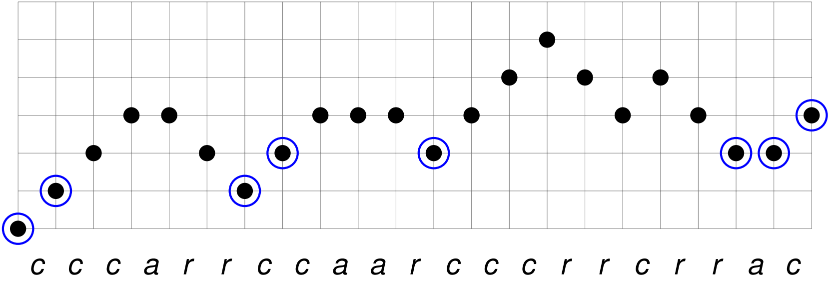
- Stair Visibly Pushdown Automata (StVPA)

$$\mathrm{VPL}_\omega = \mathrm{StVPL}_\omega = \mathrm{StDVPL}_\omega$$

Theorem

Visibly pushdown games are determined with visibly pushdown winning strategies.

Stair Pushdown Automata



$$\text{Steps}_\alpha = \{n \in \mathbb{N} \mid \forall m \geq n\colon sh(\alpha \upharpoonright m) \geq sh(\alpha \upharpoonright n)\}$$

Evaluation of the acceptance condition only at the steps.
Deterministic PDA with Parity condition $\rightsquigarrow$ StDCFL_ω

Theorem

StDCFL_ω-games are determined with DCFL winning strategies.

Results on Context-Free Games

Theorem

Every X-game is determined with X winning strategies for the following cases of X:

- *DCFL*
- *VPL*
- *StDCFL_ω with DCFL winning strategies*
- *real-time-DCFL*

The statement fails for

$X = \text{CFL}$

Summary

- Game specifications and strategies can be expressed by (tuples of) languages.
- Locally (threshold) testable games are determined with locally (threshold) testable winning strategies.
- Piecewise (threshold) testable games are determined with piecewise (threshold) testable winning strategies.

- Outlook
  - Consider further language classes, e.g. 1-counter PDAs.
  - Formulate a general result covering the known cases (work in progress).
    Can we find a method to come from X-winning conditions to Y-winning strategies?

Summary