Vector Addition System Reachability Problem

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**Definition**

A **vector addition system (VAS)** is a finite set \( A \subseteq \mathbb{Z}^d \).

\( A \) is a set of actions.

\( \mathbb{N}^d \) is the set of markings.

A **run** is a non-empty word \( \rho = m_0 \ldots m_k \) of markings such that:

\[
\forall j \in \{1, \ldots, k\} \quad m_j \in m_{j-1} + A
\]

In this case, \( m_k \) is said to be **reachable** from \( m_0 \).

**Theorem (Mayr 1981, Kosaraju 1982)**

*The reachability problem is decidable.*
Example

\[ A = \{ \nabla, \nabla \} \]

\[ \rho = (0, 2) (1, 3) (2, 4) (3, 5) (4, 6) (3, 4) (2, 2) (1, 0) \]

\( n \) is reachable from \( m \).
**Example**

\[ A = \{ \begin{array}{c} \uparrow \hspace{1cm} \downarrow \end{array} \} \]

**n** is not reachable from **m**.

\[ \phi(x_1, x_2) := 0 \leq x_1 \land 0 \leq x_2 \land x_2 \leq x_1 + 2 \]
Vector addition systems are equivalent to other models:

- Vector addition systems with states
- Petri nets.
A vector addition system with states (VASS) is a graph \( G = (Q, \Delta) \) where:

- \( Q \) is a non-empty finite set of control states
- \( \Delta \subseteq Q \times \mathbb{Z}^d \times Q \) is a finite set of transitions.

\( Q \times \mathbb{N}^d \) set of configurations

A run is a non-empty word \((q_0, m_0) \ldots (q_k, m_k)\) of configurations such that \((q_{j-1}, m_j - m_{j-1}, q_j) \in \Delta\) for every \( j \in \{1, \ldots, k\} \). In this case \((q_k, m_k)\) is said to be reachable from \((q_0, m_0)\).
Let $A$ be a VAS.

We introduce the VASS $G = (\{q\}, \Delta)$ with $\Delta = \{q\} \times A \times \{q\}$.

**Lemma**

$n$ is reachable from $m$ in the VAS $A$, if and only if $(q, n)$ is reachable from $(q, m)$ in the VASS $G$. 

Assume that $G = (Q, \Delta)$ is a VASS without any self loop and such that $Q = \{1, \ldots, k\}$.

We introduce the unitary vector $e_i$:

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)_{\uparrow i}$$

We introduce the VAS $A = \{(e_j - e_i, z) \mid (i, z, j) \in \Delta\}$.

**Lemma**

$(j, n)$ is reachable from $(i, m)$ in the VASS $G$ if and only if $(e_j, n)$ is reachable from $(e_i, m)$ in the VAS $A$. 

Reductions : VASS 2 VAS
The Hopcroft-Pansiot 1979 Example

Configurations reachable from \((p, (1, 0, 0))\)

\[
\{p\} \times \{(x, y, z) \in \mathbb{N}^3 \mid x + y \leq 2^z\} \\
\cup \{q\} \times \{(x, y, z) \in \mathbb{N}^3 \mid x + 2y \leq 2^{z+1}\}
\]
Equivalence Problem

Definition (Equivalence Problem)

INPUT : \((A_1, m_1)\) and \((A_2, m_2)\) two vector addition systems equipped with initial markings.

OUTPUT : Decide the equality of the reachability sets.

Theorem (Hack 1976)

*The equivalence problem is undecidable.*

\[ \Rightarrow \text{No decidable logic for denoting reachability sets.} \]
Subconclusion

Some equivalent models:
- Vector addition systems (ideal for proofs)
- Vector addition systems with states (ideal for examples)
- Petri nets (ideal for modeling parallel processes)

No decidable logic for denoting reachability sets. In the sequel, we show that there is a decidable logic for geometrical properties asymptotically verified by these sets:

Example

\[
\{p\} \times \{(x, y, z) \in \mathbb{N}^3 \mid x + y \leq 2^z\}
\cup \{q\} \times \{(x, y, z) \in \mathbb{N}^3 \mid x + 2y \leq 2^{z+1}\}
\]

\[\implies x \text{ and } y \text{ can be very large compared to } z.\]
Outline

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2. Dense Sets
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4. Almost Semilinear Sets
5. Precise Approximations
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Definition (Vector Spaces)

A set $V \subseteq \mathbb{Q}^d$ is called a vector space if $0 \in V$, $V + V \subseteq V$ and $\mathbb{Q}V \subseteq V$.

Example

The vector spaces $V$ included in $\mathbb{Q}^2$ are exactly:

- The whole set $\mathbb{Q}^2$,
- The line vector spaces $\mathbb{Q}v$ with $v \neq (0,0)$, or
- The zero vector space $\{(0,0)\}$. 
Lemma

For every vector space \( V \subseteq \mathbb{Q}^d \) there exists at most \( d \) vectors \( v_1, \ldots, v_r \in V \) satisfying:

\[
V = \mathbb{Q}v_1 + \cdots + \mathbb{Q}v_r
\]

Definition (Rank)

The rank of a vector space \( V \) is the minimal \( r \in \mathbb{N} \) denoted by \( \text{rank}(V) \) such that there exists a sequence \( v_1, \ldots, v_r \) of vectors in \( V \) satisfying:

\[
V = \mathbb{Q}v_1 + \cdots + \mathbb{Q}v_r
\]

Example

The vector spaces \( V \) included in \( \mathbb{Q}^2 \) are exactly:

- \( \text{rank}(V) = 2 \): The whole set \( \mathbb{Q}^2 \),
- \( \text{rank}(V) = 1 \): The line vector spaces \( \mathbb{Q}v \) with \( v \neq (0, 0) \), or
- \( \text{rank}(V) = 0 \): The zero vector space \( \{(0, 0)\} \).
Lemma (Strict Monotonic Property)

\[ \text{rank}(V) < \text{rank}(W) \text{ for every vector spaces } V \subseteq W. \]
Definition

A set $C \subseteq \mathbb{Q}^d$ is said to be **conic** if $0 \in C$, $C + C \subseteq C$ and $\mathbb{Q}_{\geq 0} C \subseteq C$. A conic set $C$ is said to be **finitely generated** if there exist $c_1, \ldots, c_k \in C$ such that:

$$C = \mathbb{Q}_{\geq 0} c_1 + \cdots + \mathbb{Q}_{\geq 0} c_k$$

\[ \mathbb{Q}_{\geq 0} (1, 1) + \mathbb{Q}_{\geq 0} (1, 0) \]
Lemma

The set $V = C - C$ is a vector space for every conic set $C$. This vector space is the unique minimal one that contains $C$.

Definition

The vector space $V = C - C$ is called the vector space generated by the conic set $C$. 
Theorem (Duality)

Let \( V \) be a vector space. A conic set \( C \subseteq V \) is finitely generated if and only if there exists a finite set \( H \subseteq V \setminus \{0\} \) such that:

\[
C = \bigcap_{h \in H} \left\{ c \in V \mid \sum_{i=1}^{d} h(i)c(i) \geq 0 \right\}
\]

\( V = \mathbb{Q}^2 \)

\( C = \mathbb{Q}_{\geq 0}(1, 1) + \mathbb{Q}_{\geq 0}(1, 0) \)

\( H = \{h_1, h_2\} \)
A conic set that is **not** finitely generated:

\[ \{ (0, 0) \} \cup \{ (c_1, c_2) \in \mathbb{Q}_\geq 0 \mid 0 < c_2 \leq c_1 \} \]
Definition

A conic set $C$ is said to be definable if there exists a formula in $\text{FO}(\mathbb{Q}, +, \leq, 0)$ denoting $C$.

$$\phi(x_1, x_2) = (x_1 = 0 \land x_2 = 0) \lor ((\neg x_2 \leq 0) \land x_2 \leq x_1)$$
Lemma

*Every finitely generated conic set is definable.*

Proof.

The conic set \( C = \mathbb{Q}_{\geq 0} c_1 + \cdots + \mathbb{Q}_{\geq 0} c_k \) is denoted by the formula \( \phi(x_1, \ldots, x_d) \) equals to:

\[
\exists \lambda_1 \ldots \exists \lambda_k \left( \bigwedge_{j=1}^{k} 0 \leq \lambda_j \right) \land \left( \bigwedge_{i=1}^{d} x_i = \sum_{j=1}^{k} \lambda_j c_j(i) \right)
\]
**Definition**

The topological closure of $\mathbf{X} \subseteq \mathbb{Q}^d$ is the set $\bar{\mathbf{X}}$ of vectors $\mathbf{y} \in \mathbb{Q}^d$ such that for all $\varepsilon \in \mathbb{Q}_{>0}$ the following intersection is non empty:

$$\mathbf{X} \cap (\mathbf{y} + (\mathbf{-\varepsilon, \varepsilon})^d) \neq \emptyset$$

Let $\mathbf{X} = (1, 5) \times (1, 5)$. Then $\bar{\mathbf{X}} = [1, 5] \times [1, 5]$. 

![Graphical representation of the definition](image)
Lemma
\[ \overline{X \cup Y} = \overline{X} \cup \overline{Y} \]
\[ X \subseteq \overline{X} \]
\[ \overline{X + Y} \subseteq \overline{X} + \overline{Y} \]
\[ \overline{Q \geq 0 X} \subseteq \overline{Q \geq 0} \overline{X} \]

Example
\[ \overline{X + Y} \neq \overline{X} + \overline{Y} \] with:
\[ X = \left\{ x \in \mathbb{Q}^2_{>0} \mid x(2) = \frac{1}{x(1)} \right\} \] and
\[ Y = \mathbb{Q}_{\geq 0}(0, -1). \]

Example
\[ \overline{Q \geq 0 X} \neq \overline{Q \geq 0} \overline{X} \] with:
\[ X = \left\{ x \in \mathbb{Q}^2_{>0} \mid x(2) = \frac{1}{x(1)} \right\} \]

Corollary

The topological closure of a conic set is a conic set.

Proof.
\[ 0 \in \mathbb{C} \subseteq \overline{\mathbb{C}} \]
\[ \overline{\mathbb{C}} + \overline{\mathbb{C}} \subseteq \overline{\mathbb{C} + \mathbb{C}} \subseteq \overline{\mathbb{C}} \]
\[ \overline{Q \geq 0 \mathbb{C}} \subseteq \overline{Q \geq 0 \mathbb{C}} \subseteq \overline{\mathbb{C}} \]
**Lemma**

The topological closure of a set definable in \( \text{FO}(\mathbb{Q}, +, \leq, 0) \) is a finite union of finitely generated conic sets.

**Example**

\[
X = X_1 \cup X_2 \cup X_3 \quad \text{with:}
\]

\[
X_1 = \{(x, y) \in \mathbb{Q}^2 \mid 2x + 3y > 0 \land x - y \geq 0\}
\]

\[
X_2 = \{(x, y) \in \mathbb{Q}^2 \mid x > 0 \land x - y > 0\}
\]

\[
X_3 = \{(x, y) \in \mathbb{Q}^2 \mid x > 0 \land y > 0 \land -x - y > 0\}
\]

Then

\[
\overline{X} = \overline{X}_1 \cup \overline{X}_2 \cup \overline{X}_3 \quad \text{with:}
\]

\[
\overline{X}_1 = \{(x, y) \in \mathbb{Q}^2 \mid 2x + 3y \geq 0 \land x - y \geq 0\}
\]

\[
\overline{X}_2 = \{(x, y) \in \mathbb{Q}^2 \mid x \geq 0 \land x - y \geq 0\}
\]

\[
\overline{X}_3 = \emptyset
\]
Lemma

The topological closure of a definable conic set is a finitely generated conic set.

Lemma

Let $\mathbf{C}$ be a definable conic set.

Since $\mathbf{C}$ is a conic set then $\overline{\mathbf{C}}$ is a conic set.

Since $\mathbf{C}$ is definable then $\overline{\mathbf{C}} = \bigcup_{j=1}^{k} \mathbf{C}_j$ with $\mathbf{C}_j$ a finitely generated conic set.

Just observe that in this case:

$$\overline{\mathbf{C}} = \sum_{j=1}^{k} \mathbf{C}_j$$
Definition

A conic set $\mathbf{C} \subseteq \mathbb{Q}^d$ is said to be locally finitely generated if for every vector space $\mathbf{V} \subseteq \mathbb{Q}^d$ the conic set $\mathbf{C} \cap \mathbf{V}$ is finitely generated.

Theorem

A conic set is definable if and only if it is locally finitely generated.

Example:

- With $\mathbf{V} = \mathbb{Q}^2$ we have $\mathbf{C} \cap \mathbf{V} = \mathbb{Q}_{\geq 0}(1, 1) + \mathbb{Q}_{\geq 0}(1, 0)$.
- With $\mathbf{V} = \mathbb{Q}\mathbf{v}$ then $\mathbf{C} \cap \mathbf{V}$ is $\{(0, 0)\}$, $\mathbb{Q}_{\geq 0}\mathbf{v}$, or $-\mathbb{Q}_{\geq 0}\mathbf{v}$.
- With $\mathbf{V} = \{(0, 0)\}$ then $\mathbf{C} \cap \mathbf{V} = \{(0, 0)\}$. 
Proof: The simple way

Assume that \( C \) is a definable conic set.
For every vector space \( V \) the conic set \( C \cap V \) is definable.
From the previous lemma \( C \cap V \) is finitely generated.
Thus \( C \) is locally finitely generated.
Lemma

Let $C$ be a conic set such that $\overline{C}$ is finitely generated and $C \cap V$ is definable for every vector space $V \subset C - C$. Then $C$ is definable.

Proof.

Let $W = C - C$. There exists a finite set $H \subseteq W \setminus \{0\}$ such that:

$$\overline{C} = \bigcap_{h \in H} \left\{ c \in W \mid \sum_{i=1}^{d} h(i)c(i) \geq 0 \right\}$$

We prove that $X \subseteq C$ where $X = \bigcap_{h \in H} \left\{ c \in W \mid \sum_{i=1}^{d} h(i)c(i) > 0 \right\}$.

Observe that $C = X \cup \bigcup_{h \in H} (C \cap V_h)$ where:

$$V_h = \left\{ v \in W \mid \sum_{i=1}^{d} h(i)v(i) = 0 \right\}$$
Proof: The other way

$H_k$: Locally finitely generated conic sets $C$ such that $\text{rank}(C - C) \leq k$ are definable.

$H_0$ is clearly true since $\text{rank}(C - C) = 0$ implies $C = \{0\}$. Assume $H_k$ true and let $C$ be a locally definable conic set such that $\text{rank}(W) = k + 1$ where $W = C - C$. We observe that $\overline{C}$ is finitely generated and for every vector space $V \subset W$ the conic set $C \cap V$ is locally finitely generated. Since $\text{rank}(V) < \text{rank}(W) \leq k + 1$ we can apply $H_k$. We deduce that $C \cap V$ definable. From the previous lemma we deduce that $C$ is definable. Thus $H_{k+1}$ is true.
A conic set that is not definable:

$$\mathbf{C} = \{(c_1, c_2) \in \mathbb{Q}_\geq 0^2 \mid \sqrt{2} c_2 \leq c_1\}$$

The conic set $\mathbf{C}$ is not finitely generated. Let $\mathbf{V} = \mathbb{Q}^2$. Since $\overline{\mathbf{C} \cap \mathbf{V}} = \mathbf{C}$ we deduce that $\mathbf{C}$ is not definable.
We have introduced the class of definable conic sets and provided an algebraic criterion for membership of conic sets in this class.

**Theorem (Algebraic Criterion)**

A conic set $\mathbf{C} \subseteq \mathbb{Q}^d$ is definable in $\text{FO} (\mathbb{Q}, +, \leq, 0)$ if and only if the conic set $\overline{\mathbf{C} \cap \mathbf{V}}$ is finitely generated for every vector space $\mathbf{V} \subseteq \mathbb{Q}^d$. 
**Definition**

A lattice is a subset $L \subseteq \mathbb{Z}^d$ such that $0 \in L$, $L + L \subseteq L$ and $-L \subseteq L$.

**Lemma**

*For every lattice $L$ there exists a sequence $l_1, \ldots, l_k \in L$ such that:*

$$L = \mathbb{Z}l_1 + \cdots + \mathbb{Z}l_k$$

$L = \mathbb{Z}(1, 1)$
Definition

A set $P \subseteq \mathbb{Z}^d$ is said to be periodic if $0 \in P$ and $P + P \subseteq P$. A periodic set $P$ is said to be finitely generated if there exist $p_1, \ldots, p_k \in P$ such that:

$$P = \mathbb{N}p_1 + \cdots + \mathbb{N}p_k$$

$$P = \mathbb{N}(1, 1) + \mathbb{N}(2, 0)$$
Lattices And Periodic Sets

**Lemma**

The set \( \mathbf{L} = \mathbf{P} - \mathbf{P} \) is a lattice for every periodic set \( \mathbf{P} \). This lattice is the unique minimal one that contains \( \mathbf{P} \).

**Definition**

The lattice \( \mathbf{L} = \mathbf{P} - \mathbf{P} \) is called the lattice generated by the periodic set \( \mathbf{P} \).

\[
\mathbf{P} = \mathbb{N}(1, 1) + \mathbb{N}(2, 0) \quad \mathbf{L} = \mathbf{P} - \mathbf{P}
\]
Lemma
The set $\mathbb{C} = \mathbb{Q}_{\geq 0} \mathbb{P}$ is a conic set for every periodic set $\mathbb{P}$. This conic set is the unique minimal one that contains $\mathbb{P}$.

Definition
The conic set $\mathbb{C} = \mathbb{Q}_{\geq 0} \mathbb{P}$ is called the conic set generated by the periodic set $\mathbb{P}$.

$\mathbb{P} = \mathbb{N}(1, 1) + \mathbb{N}(2, 0)$

$\mathbb{C} = \mathbb{Q}_{\geq 0} \mathbb{P}$
Definition

A periodic set $\mathbf{P}$ is said to be asymptotically definable if the conic set $\mathbf{C} = \mathbb{Q}_{\geq 0} \mathbf{P}$ is definable in FO$(\mathbb{Q}, +, \leq, 0)$. 

\[ p(2) \leq p(1) \]
\[ p(1) + 1 \leq 2p(2) \]

\[ c(2) \]
\[ c(1) \]
**Lemma**

The class of asymptotically definable periodic sets is stable by intersection.

**Proof.**

Let $P_1, P_2$ be two periodic sets. We have:

$$Q_{\geq 0}(P_1 \cap P_2) = (Q_{\geq 0}P_1) \cap (Q_{\geq 0}P_2)$$

Assume that:

- $Q_{\geq 0}P_1$ is denoted by $\phi_1(x)$.
- $Q_{\geq 0}P_2$ is denoted by $\phi_2(x)$.

Then $Q_{\geq 0}(P_1 \cap P_2)$ is denoted by $\phi_1(x) \land \phi_2(x)$. 

\hfill \Box
Lemma

The class of asymptotically definable periodic relations is stable by composition.

Proof.

Let $R_1, R_2 \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ be two periodic relations. We have:

$$Q_{\geq 0}(R_1 \circ R_2) = (Q_{\geq 0}R_1) \circ (Q_{\geq 0}R_2)$$

Assume that:

$Q_{\geq 0}R_1$ is denoted by $\phi_1(x, y)$.

$Q_{\geq 0}R_2$ is denoted by $\phi_2(y, z)$.

Then $Q_{\geq 0}(R_1 \circ R_2)$ is denoted by $\exists y \phi_1(x, y) \land \phi_2(y, z)$. 

\qed
We introduced the class of asymptotically definable periodic sets.

From an asymptotically definable periodic set $P$, we can extract two properties:

- the “repeated motif”, i.e. the lattice $L = P - P$ denoted by a finite sequence of vectors in $L$.
- the “asymptotic direction”, i.e. the conic set $C = \mathbb{Q}_{\geq 0}P$ denoted by a formula in FO($\mathbb{Q}, +, \leq, 0$).

Stability properties:

- asymptotically definable periodic sets are stable by intersection.
- asymptotically definable periodic relations are stable by composition.
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Presburger Sets

Definition
A set $X \subseteq \mathbb{Z}^d$ is said to be Presburger if it can be denoted by a formula in FO ($\mathbb{Z}, +, \leq, 0, 1$).

Theorem (Ginsburg and Spanier - 1966)
A set $X \subseteq \mathbb{Z}^d$ is Presburger if and only if it is semilinear, i.e. a finite union of sets $b + P$ where $b \in \mathbb{Z}^d$ and $P \subseteq \mathbb{Z}^d$ is a finitely generated periodic set.
Almost Semilinear Sets

**Definition**

A set $X \subseteq \mathbb{Z}^d$ is said to be **almost semilinear** if for every Presburger set $S \subseteq \mathbb{Z}^d$, the set $X \cap S$ is a finite union of sets $b + P$ where $b \in \mathbb{Z}^d$ and $P \subseteq \mathbb{Z}^d$ is an asymptotically definable periodic set.

**Example**

\[
x(1) \in 2^{\mathbb{N}} - 1 \land x(2) = 1
\]
Almost Semilinear Sets

**Definition**

A set \( X \subseteq \mathbb{Z}^d \) is said to be *almost semilinear* if for every Presburger set \( S \subseteq \mathbb{Z}^d \), the set \( X \cap S \) is a finite union of sets \( b + P \) where \( b \in \mathbb{Z}^d \) and \( P \subseteq \mathbb{Z}^d \) is an asymptotically definable periodic set.

**Example**

\[ x(1) \in 2^\mathbb{N} - 1 \land x(2) = 1 \]
We introduced the class of almost semilinear sets.

In the sequel we show that this class:

- Contains VAS reachability relations.
- Is sufficient to deduce inductive invariants in the Presburger arithmetic.
Let $P \subseteq \mathbb{Z}^d$ be an asymptotically definable periodic set.

**Definition**

The linearization of $P$ is:

$$\text{lin}(P) = (P - P) \cap \overline{Q_{\geq 0}P}$$

**Lemma**

$\text{lin}(P)$ is a finitely generated periodic set.

**Proof.**

$L = P - P$ is a lattice.

$\overline{Q_{\geq 0}P}$ is a finitely generated conic set.
The linearization $\text{lin}(P)$ provides an over-approximation of $P$.
Let $P_1, P_2 \subseteq \mathbb{Z}^d$ be two asymptotically definable periodic sets and $b_1, b_2 \in \mathbb{Z}^d$ be two vectors such that:

$$(b_1 + P_1) \cap (b_2 + P_2) = \emptyset$$

In general:

$$(b_1 + \text{lin}(P_1)) \cap (b_2 + \text{lin}(P_2)) \neq \emptyset$$
Definition

The dimension \( \dim(X) \) of a non-empty set \( X \subseteq \mathbb{Z}^d \) is the minimal integer \( r \in \{0, \ldots, d\} \) such that:

\[
X \subseteq \bigcup_{j=1}^{k} b_j + V_j
\]

where \( b_j \in \mathbb{Z}^d \) and \( V_j \) is a vector space satisfying \( \text{rank}(V_j) \leq r \).

\( \dim(\emptyset) = -1 \) by convention.

Example

\[
\begin{align*}
\dim(\mathbb{N}) &= 1 \\
\dim(\{(0, 1), (1, 0)\}) &= 0 \\
\dim(\{(x, y) \in \mathbb{N}^2 \mid x \leq y\}) &= 2
\end{align*}
\]
Theorem

Let $P_1, P_2 \subseteq \mathbb{Z}^d$ be two asymptotically definable periodic sets and $b_1, b_2 \in \mathbb{Z}^d$ such that:

$$(b_1 + P_1) \cap (b_2 + P_2) = \emptyset$$

In this case, the set

$$X = (b_1 + \text{lin}(P_1)) \cap (b_2 + \text{lin}(P_2))$$

satisfies:

$$\dim(X) < \max\{\dim(b_1 + P_1), \dim(b_2 + P_2)\}$$
We introduced a way to over-approximate asymptotically definable periodic sets into finitely generated ones. The approximation is proved precise in some sense.
Let $R \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ be a relation definable in FO ($\mathbb{Z}, +, \leq, 0, 1$). Decide the membership in the reflexive and transitive closure $R^*$.

**Example**

Let $A$ be a VAS. We introduce:

$$R = \{(m, n) \in \mathbb{N}^d \times \mathbb{N}^d \mid n - m \in A\}$$

Then $R^*$ is the reachability relation.

In general undecidable since the one step reachability relation $R$ of a Minsky machine is definable in FO ($\mathbb{Z}, +, \leq, 0, 1$).
Inductive Invariants

Let \( R \subseteq \mathbb{Z}^d \times \mathbb{Z}^d \).

**Definition**

The forward image \( \text{post}_R(X) \) of a set \( X \subseteq \mathbb{Z}^d \) by \( R \) is defined by:

\[
\text{post}_R(X) = \bigcup_{x \in X} \{ y \in \mathbb{Z}^d \mid (x, y) \in R \}
\]

If \( \text{post}_R(X) \subseteq X \) then \( X \) is called a forward inductive invariant for \( R \).

**Definition**

The backward image \( \text{pre}_R(Y) \) of a set \( Y \subseteq \mathbb{Z}^d \) by \( R \) is defined by:

\[
\text{pre}_R(Y) = \bigcup_{y \in Y} \{ x \in \mathbb{Z}^d \mid (x, y) \in R \}
\]

If \( \text{pre}_R(Y) \subseteq Y \) then \( Y \) is called a backward inductive invariant for \( R \).
Definition (Separators)

A separator for a binary relation \( R \subseteq \mathbb{Z}^d \times \mathbb{Z}^d \) is a pair \((X, Y)\) of subsets of \( \mathbb{Z}^d \) such that \( \text{post}_{R^*}(X) \cap \text{pre}_{R^*}(Y) = \emptyset \). The set \( D = \mathbb{Z}^d \setminus (X \cup Y) \) is called the domain. A separator is said to be closed if its domain is empty.

If \((X, Y)\) is a closed separator for \( R \) then \( X \) is a forward invariant and \( Y \) is a backward invariant.

Example

Separators \((X, Y)\) are included in closed separators, for instance:

\[
(post_{R^*}(X), \mathbb{Z}^d \setminus post_{R^*}(X))
\]

\[
(\mathbb{Z}^d \setminus pre_{R^*}(Y), pre_{R^*}(Y))
\]
Main result of this section:

**Theorem**

Let \( R \subseteq \mathbb{Z}^d \times \mathbb{Z}^d \) be a binary relation such that its reflexive and transitive closure \( R^* \) is an almost semilinear relation. Presburger separators are included in closed Presburger separators.

**Corollary**

Let \( R \subseteq \mathbb{Z}^d \times \mathbb{Z}^d \) be a binary relation such that its reflexive and transitive closure \( R^* \) is an almost semilinear relation. For every \((x, y) \notin R^* \) there exists a Presburger forward invariant \( I \) such that \( x \in I \) and \( y \notin I \).

**Proof.**

Observe that \((\{x\}, \{y\})\) is a Presburger separator. There exists a closed Presburger separator \((I, J)\) such that: \( \{x\} \subseteq I \) and \( \{y\} \subseteq J \). Since \( I \cap J = \emptyset \) we get \( y \notin I \).
Assume that $R^*$ is almost semilinear.

**Lemma**

$\text{post}_{R^*}(X)$ and $\text{pre}_{R^*}(Y)$ are almost semilinear sets for every Presburger sets $X, Y \subseteq \mathbb{Z}^d$.

**Proof.**

Let $S \subseteq \mathbb{Z}^d$ be a Presburger set. We have:

$$\text{post}_{R^*}(X) \cap S = \{ y \in \mathbb{Z}^d \mid \exists (x, y) \in R^* \cap (X \times S) \}$$
Induction

\[(X_0, Y_0)\] Presburger Separator

\[\text{with } \dim(D_0) > \dim(D)\]
$(X_0, Y_0)$ Presburger Separator
This is a finite union $\bigcup_i (b_i + P_i)$. 
\[ S := X_0 \cup (\bigcup_i (b_i + \text{lin}(P_i))) \]

\( S \) is an over-approximation of \( \text{post}_{R^*}(X_0) \).

\( S \cap Y_0 \) is not necessary empty.
\( Y := Y_0 \cup (\mathbb{N}^d \setminus S) \)

\((X_0, Y)\) is a Presburger separator such that \( Y_0 \subseteq Y \)
$(X_0, Y)$ Presburger Separator
This is a finite union $\bigcup_j (c_j + Q_j)$.
\( T := \mathbf{Y} \cup (\bigcup_{j}(c_j + \text{lin}(Q_j))) \)

\( T \) is an over-approximation of \( \text{pre}_{R^*}(\mathbf{Y}) \).

\( T \cap X_0 \) is not necessary empty.
\( X := X_0 \cup (\mathbb{N}^d \setminus T) \)

\((X, Y)\) is a Presburger separator such that \(X_0 \subseteq X\)
Induction

The domain $D$ of $(X, Y)$ satisfies $D = D_0 \cap (\bigcup_{i,j} D_{i,j})$ where:

$$D_{i,j} = (b_i + \text{lin}(P_i)) \cap (c_j + \text{lin}(Q_j))$$

As $(b_i + P_i) \cap (c_j + Q_j) = \emptyset$ we get:

$$\dim(D_{i,j}) < \max\{\dim(b_i + P_i), \dim(c_j + Q_j)\}$$

As $b_i + P_i$ and $c_j + Q_j$ are both included in $D_0$, we get:

$$\dim(b_i + P_i) \leq \dim(D_0) \quad \text{dim}(c_j + Q_j) \leq \dim(D_0)$$

Thus:

$$\dim(D) < \dim(D_0)$$
Assume that $R$ is denoted by a Presburger formula and $R^*$ is almost semilinear. The non-membership in $R^*$ can be proved with formulas in the Presburger arithmetic denoting forward inductive invariants for $R$. 
Outline

1. Introduction
2. Dense Sets
3. Discrete Sets
4. Almost Semilinear Sets
5. Precise Approximations
6. Inductive Invariants
7. Well Orders
8. Production Relations
9. Reachability Relations
10. One More Thing...
11. Conclusion
Well Orders

Definition

An order $\sqsubseteq$ over a set $S$ is said to be well if for every sequence $(s_n)_{n \in \mathbb{N}}$ of elements $s_n \in S$ there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of indexes $n_k \in \mathbb{N}$ such that $(s_{n_k})_{k \in \mathbb{N}}$ is non decreasing for $\sqsubseteq$.

Example

The ordered set $(\mathbb{N}, \leq)$ is well but $(\mathbb{Z}, \leq)$ is not well.

Example (Pigeon Hole Principle)

An ordered set $(S, =)$ is well if and only if $S$ is finite.
Dickson’s Lemma

**Definition**
Let \((S, \sqsubseteq)\) be an ordered set.
We introduce the ordered set \((S^d, \sqsubseteq^d)\) where \(\sqsubseteq^d\) is defined component-wise by
\[
(s_1, \ldots, s_d) \sqsubseteq^d (t_1, \ldots, t_d) \text{ if } s_i \sqsubseteq t_i \quad \forall i
\]

**Lemma (Dickson’s Lemma)**
The order set \((S^d, \sqsubseteq^d)\) is well for every well ordered set \((S, \sqsubseteq)\).

**Example**
\((\mathbb{N}^d, \leq)\) is well.
Higmann’s Lemma

Definition
Let \((S, \sqsubseteq)\) be an ordered set.
We introduce the ordered set \((S^*, \sqsubseteq^*)\) where \(\sqsubseteq^*\) is defined by \(u \sqsubseteq^* v\) if \(u\) and \(v\) can be decomposed as follows:

\[
\begin{align*}
u &= w_0 \; t_1 \; w_1 \; \cdots \; t_d \; w_d \\
u &= w_0 \; t_1 \; w_1 \; \cdots \; t_d \; w_d
\end{align*}
\]

where \(s_j, t_j \in S\).

Lemma (Higmann’s Lemma)
The ordered set \((S^*, \sqsubseteq^*)\) is well for every well ordered set \((S, \sqsubseteq)\).
**Definition**

Let $\rho = m_0 \ldots m_k$ be a run of a VAS $A$. We introduce the action $a_j = m_j - m_{j-1}$ for each $j \in \{1, \ldots, k\}$.

- $\text{src}(\rho) = m_0$ the **source**.
- $\text{tgt}(\rho) = m_k$ the **target**.
- $\text{lab}(\rho) = a_1 \ldots a_k$ the **label**.

Let $w \in A^*$ be a word of actions.

**Definition**

The binary relation $\xrightarrow{w} \mathbin{\text{over }} \mathbb{N}^d$ is defined by $m \xrightarrow{w} n$ if there exists a run $\rho$ such that $\text{src}(\rho) = m$, $\text{lab}(\rho) = w$ and $\text{tgt}(\rho) = n$.

**Definition**

We denote by $\xrightarrow{\star}$ the reachability relation.
Definition (Inspired from Hauschildt 1990)

The production relation of a marking $m$ is the binary relation $\rightarrow_m$ defined over the markings by:

$$r \rightarrow_m s \quad \text{if} \quad m + r \rightarrow m + s$$

Example

$\rightarrow_m$ is equal to $\rightarrow$ when $m = 0$. 
Lemma
Production relations are periodic.

Proof.

\[ r_1 \xrightarrow{\text{m}} s_1 \text{ and } r_2 \xrightarrow{\text{m}} s_2 \text{ implies } r_1 + r_2 \xrightarrow{\text{m}} s_1 + s_2 \]
Application : Iterate

\[ m + r \] \[ m + s \] \[ m + 4r \] \[ m + 4s \]
Main result of this section:

**Theorem**

*Production relations are asymptotically definable.*

I.e. the following relation is definable in FO \((\mathbb{Q}, +, \leq, 0)\):

\[
Q_{\geq 0} \xrightarrow{\ast_m} = \{(\lambda r, \lambda s) \mid \lambda \in Q_{\geq 0} \text{ and } r \xrightarrow{\ast_m} s\}
\]
We introduce an element $\infty \not\in \mathbb{N}$ and we let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

**Definition**

A vector $\mathbf{x} \in \mathbb{N}_\infty^d$ is called an extended marking. The set $I = \{i \in \{1, \ldots, d\} \mid \mathbf{x}(i) = \infty\}$ is called the set of relaxed components.

Let $\mathbf{m} \in \mathbb{N}^d$ and $I \subseteq \{1, \ldots, d\}$. The extended marking $\mathbf{m}^I$ obtained from $\mathbf{m}$ by relaxing components in $I$ is defined by:

$$m^I(i) = \begin{cases} \infty & \text{if } i \in I \\ m(i) & \text{if } i \notin I \end{cases}$$

**Example**

Let $\mathbf{m} = (1, 2, 1000)$ and $I = \{3\}$ then $\mathbf{m}^I = (1, 2, \infty)$. 
Definition

We introduce the binary relations $\rightarrow_a$ over the set of extended markings relaxed over the same set of components $I$ by $x \rightarrow_a y$ if:

$$\forall i \not\in I \quad y(i) = x(i) + a(i)$$

An extended run is a non-empty word $\rho = x_0 \ldots x_k$ of extended markings relaxed over the same set $I$ such that for every $j \in \{1, \ldots, d\}$ there exists $a_j \in A$ such that $x_{j-1} \rightarrow_a x_j$.

Let $\rho = m_0 \ldots m_k$ and $I \subseteq \{1, \ldots, d\}$. The extended run $\rho^I$ obtained from $\rho$ by relaxing components in $I$ is defined by $\rho^I = m^I_0 \ldots m^I_k$.

Example

Let $\rho = (0, 0, 100)(0, 1, 99) \ldots (0, 100, 0)$ be a run.
Let $I = \{2, 3\}$.
The extended run $\rho^I$ is $\rho^I = (0, \infty, \infty) \ldots (0, \infty, \infty)$.
Recall that $\rightarrow_m$ is asymptotically definable if and only for every vector space $V \subseteq \mathbb{Q}^d \times \mathbb{Q}^d$ the following conic set is finitely generated:

$$\mathbb{Q}_{\geq 0} \rightarrow_m, V$$

where:

$$\rightarrow_m, V = \{ (r, s) \in V \mid r \rightarrow_m s \}$$
Definition

Let $\Omega_{m,V}$ be the set of runs of the following form with $(r,s) \in V$:

$$\Omega_{m,V} = \bigcup_{(r,s) \in \overrightarrow{m,V}} \{ \text{runs } \rho \mid \text{src}(\rho) = m + r \land \text{tgt}(\rho) = m + s \}$$
Reachability Graphs

**Definition**

We introduce:

\[
Q_{m,V} = \bigcup_{\rho \in \Omega_{m,V}} \{ q \in \mathbb{N}^d \mid q \text{ occurs in } \rho \}
\]

\[
Q_{m,V}(i) = \{ q(i) \mid q \in Q_{m,V} \}
\]

\[
l_{m,V} = \{ i \in \{1, \ldots, d\} \mid Q_{m,V}(i) \text{ is infinite} \}
\]

We introduce the finite graph \( G_{m,V} = (X, \Delta) \) defined by:

- \( X = \{ q^{m,V} \mid q \in Q_{m,V} \} \).

- \( \Delta \) is the set of triples \((x, a, y) \in X \times A \times X \) such that \( x \xrightarrow{a} y \).
We introduce an approximation of $\vec{\rightarrow}_{m,V}$

**Definition**

We introduce the relation $R_{m,V}$ of couples $(r,s) \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V$ such that (1) $r(i) = 0$ and $s(i) = 0$ for every $i \notin I_{m,V}$, and (2) there exist a cycle in $G_{m,V}$ on the state $m^{I_{m,V}}$ labeled by a word $a_1 \ldots a_k$ such that:

$$r + \sum_{j=1}^{k} a_j = s$$
Lemma

We have:

\[ \rightarrow_{m,V}^* \subseteq R_{m,V} \]

Proof.

Let \((r, s)\) in \(\rightarrow_{m,V}^*\). There exists a run \(\rho = m_0 \ldots m_k\) in \(\Omega_{m,V}\) such that \(m_0 = m + r\) and \(m_k = m + s\).

Since \(m + \mathbb{N}r\) and \(m + \mathbb{N}s\) are included in \(Q_{m,V}\) we deduce that \(r(i) > 0\) or \(s(i) > 0\) implies \(i \in I_m, V\). Hence:

\[ m_0^{l_{m,V}} = m^{l_{m,V}} \quad m_k^{l_{m,V}} = m^{l_{m,V}} \]

We deduce that \((r, s) \in R_{m,V}\) from the following cycle where \(a_j = m_j - m_{j-1}\):

\[ m_0^{l_{m,V}} \xrightarrow{a_1} \ldots \xrightarrow{a_k} m_k^{l_{m,V}} \]
In general the other inclusion is wrong but let us try proving it:

\[ R_{m, V} \subseteq \rightarrow^*_m \]

Let \((r, s) \in R_{m, V} \). Then \((r, s) \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V \) and there exist a cycle in \(G_{m, V} \) on the state \( m^{l_m, V} \) labeled by a word \( a_1 \ldots a_k \) such that:

\[
    r + \sum_{j=1}^{k} a_j = s
\]

We deduce that:

\[
    (m + r)^{l_m, V} \xrightarrow{a_1 \ldots a_k} (m + s)^{l_m, V}
\]

However in general we do not have

\[
    m + r \xrightarrow{a_1 \ldots a_k} m + s
\]

since components in \( l_{m, V} \) relaxed in the first case are integers in the second case.
Definition

An intraproduction for \((m, V)\) is a tuple \((r, x, s)\) such that:

\[
\begin{align*}
  r &\xrightarrow{m} x \xrightarrow{m} s \\
\end{align*}
\]

and such that \((r, s) \in V\).

with \(V = \{(u, v) \mid u(1) = v(1) = 0\}\).
\( m + \mathbb{N}x \subseteq Q_{m,V} \) for every intraproduction \((r, x, s)\) for \((m, V)\).
Lemma

For every $i \in I_{m,V}$ there exists an intraproduction $(r, x, s)$ for $(m, V)$ such that $x(i) > 0$.

Proof.

There exist $q_1 \leq q_2$ in $Q_{m,V}$ such that $q_1(i) < q_2(i)$.
Let $(r_1, s_1)$ and $(r_2, s_2)$ in $V$ such that:

$$m + r_1 \xrightarrow{u_1} q_1 \xrightarrow{v_1} m + s_1 \quad m + r_2 \xrightarrow{u_2} q_2 \xrightarrow{v_2} m + s_2$$

\[\begin{align*}
    m + (r_1 + r_2) &\xrightarrow{u_2 v_1} m + (r_1 + s_1 + q_2 - q_1) \\
    &\xrightarrow{u_1 v_2} m + (s_1 + s_2)
\end{align*}\]
Lemma (Simultaneously Large Components)

For every \( n \in \mathbb{N} \) there exists \( q_n \in Q_{m,V} \) such that for every \( i \in \{1, \ldots, d\} \):

\[
\begin{align*}
q_n(i) &= m(i) \quad \text{if } i \notin I_{m,V} \\
q_n(i) &\geq m(i) + n \quad \text{if } i \in I_{m,V}
\end{align*}
\]

Proof.

For each \( i \in I_{m,V} \) there exists an intraproduction \((r_i, x_i, s_i)\) such that \( x_i(i) > 0 \). Since \( \rightarrow_{m,V} \) is periodic we deduce that the set of intraproductions is periodic. Hence the following tuple is an intraproduction:

\[
(r, x, s) = \sum_{i \in I_{m,V}} (r_i, x_i, s_i)
\]

Observe that \( x(i) > 0 \) for every \( i \in I_{m,V} \). Moreover since \( m + \mathbb{N}x \subseteq Q_{m,V} \) we deduce that \( x(i) > 0 \) implies that \( i \in I_{m,V} \).

Just consider \( q_n = m + nx \). \( \square \)
Lemma

We have:

\[ R_{m, V} \subseteq Q_{\geq 0}^* \rightarrow m, V \]

Proof.

Let \((r, s) \in R_{m, V}\). Then \((r, s) \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V\) and there exists a cycle in \(G_{m, V}\) on the state \(m^{l_{m, V}}\) labeled by a word \(w = a_1 \ldots a_k\) such that \(r + \sum_{j=1}^k a_j = s\).

We deduce that \((m + r)^{l_{m, V}} \xrightarrow{w} (m + s)^{l_{m, V}}\). There exists \(n \in \mathbb{N}\) large enough such that \(q_n + r \xrightarrow{w} q_n + s\). As \(\xrightarrow{*}_{q_n}\) is periodic we deduce \(q_n + hr \xrightarrow{*} q_n + hs\) for every \(h \in \mathbb{N}\).

As \(q_n \in Q_{m, V}\) we have \(m + r' \xrightarrow{*} q_n \xrightarrow{*} m + s'\) for some \((r', s') \in (\mathbb{N}^d \times \mathbb{N}^d) \cap V\).

Therefore \((m + r' + hr) \xrightarrow{*} m + s' + hs\) and \((r', s') + h(r, s) \subseteq \xrightarrow{*} m, V\).

Hence \((r', s') + \frac{h}{h} (r, s) \in Q_{\geq 0}^* \rightarrow m, V\) for every \(h \in \mathbb{N}_{>0}\).
We have proved:

**Lemma**

\[
\mathbb{Q}_{\geq 0} \xrightarrow{*} m, \nu = \mathbb{Q}_{\geq 0} R_{m, \nu}
\]

We deduce:

**Theorem**

*Production relations are asymptotically definable.*

**Proof.**

Since \( R_{m, \nu} \) is Presburger as the Parikh image of a regular language, we deduce that \( \mathbb{Q}_{\geq 0} R_{m, \nu} \) is finitely generated. Hence \( \mathbb{Q}_{\geq 0} \xrightarrow{*} m, \nu \) is finitely generated for every vector space \( \nu \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \). We have proved that \( \mathbb{Q}_{\geq 0} \xrightarrow{*} m \) is definable.

\( \square \)
We have proved that for every marking $m \in \mathbb{N}^d$ the following relation is definable in $\text{FO}(\mathbb{Q},+,\leq,0)$:

$$\mathbb{Q}_{\geq 0} \overset{*}{\rightarrow}_m = \{(\lambda r, \lambda s) \mid \lambda \in \mathbb{Q}_{\geq 0}, r \overset{*}{\rightarrow}_m s\}$$
Main result of this section:

**Theorem**

*The reachability relation \( \rightarrow^* \) is almost semilinear.*
Let $\rho = m_0 \ldots m_k$ be a run.

\[ \begin{align*}
    r_0 &\rightarrow_{m_0} r_1 \\
    &\rightarrow_{m_1} \cdots \\
    &\rightarrow_{m_k} r_{k+1}
\end{align*} \]
Definition (Inspired from Hauschildt)

The production relation of a run $\rho = m_0 \ldots m_k$ is the binary relation $\rightarrow^*_{\rho}$ defined by:

$$\rightarrow^*_{\rho} = \rightarrow^*_{m_0} \circ \cdots \circ \rightarrow^*_{m_k}$$

The production relations $\rightarrow^*_{\rho}$ are periodic and asymptotically definable.
Lemma

\[(\text{src}(\rho), \text{tgt}(\rho)) \, + \, \overset{*}{\to}_{\rho} \, \subseteq \, \overset{*}{\to}\]

Proof.

Let

\[m_0 \xrightarrow{a_1} m_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} m_k\]

\[r_0 \overset{*}{\to} m_0 \quad r_1 \overset{*}{\to} m_1 \quad \cdots \quad r_k \overset{*}{\to} m_k \quad r_{k+1}\]

There exist \(w_0, \ldots, w_k \in A^*\) such that:

\[m_0 + r_0 \xrightarrow{w_0} m_0 + r_1\quad m_k + r_k \xrightarrow{w_k} m_k + r_{k+1}\]

Hence

\[m_0 + r_0 \xrightarrow{w_0 \ a_1 \ w_1 \ \cdots \ w_k \ a_k \ w_{k+1}} m_k + r_{k+1}\]
Definition

We introduce the order $\preceq$ over the set of runs by $\rho \preceq \rho'$ if:

$$(\text{src}(\rho'), \text{tgt}(\rho')) + \stackrel{*}{\rightarrow}_{\rho'} \subseteq (\text{src}(\rho), \text{tgt}(\rho)) + \stackrel{*}{\rightarrow}_{\rho}$$
Theorem

The order $\preceq$ is well.

Proof.

We associate to every run $\rho = m_0 \ldots m_k$ the following word $\alpha(\rho)$:

$$\alpha(\rho) = (a_1, m_1) \ldots (a_k, m_k)$$

where $a_j = m_j - m_{j-1}$

We introduce the well order $\sqsubseteq$ over $S = A \times \mathbb{N}^d$ defined by $(a, m) \sqsubseteq (b, n)$ if $a = b$ and $m \leq n$. Let $\rho'$ be another run.

Assume $\alpha(\rho) \sqsubseteq^* \alpha(\rho')$:

We have $\alpha(\rho') = w_0(a_1, m_1 + r_1)w_1 \ldots (a_k, m_k + r_k)w_k$.

Assume $\text{src}(\rho) \leq \text{src}(\rho')$: We have $\text{src}(\rho') = m_0 + r_0$.

Assume $\text{tgt}(\rho) \leq \text{tgt}(\rho')$: We have $\text{tgt}(\rho') = m_k + r_{k+1}$.

We deduce that $r_0 \xrightarrow{m_1} r_1 \cdots \xrightarrow{m_k} r_{k+1}$.

$\alpha(\rho) \sqsubseteq^* \alpha(\rho')$, $\text{src}(\rho) \leq \text{src}(\rho')$ and $\text{tgt}(\rho) \leq \text{tgt}(\rho')$ implies $\rho \preceq \rho'$. ☐
Let $\Omega$ be the set of runs. We have:

$$\xrightarrow{*} = \bigcup_{\rho \in \min_{\preceq} \Omega} (\text{src}(\rho), \text{tgt}(\rho)) + \xrightarrow{*}_{\rho}$$
Theorem

\[ \rightarrow^* \text{ is an almost semilinear relation.} \]

Proof.

Let us consider \( b \in \mathbb{N}^d \times \mathbb{N}^d \) and a finitely generated periodic relation \( P \subseteq \mathbb{Z}^d \times \mathbb{Z}^d \). We introduce the set \( \Omega_{b,P} \) of runs \( \rho \) such that 
\( (\text{src}(\rho),\text{tgt}(\rho)) \in b + P \). We introduce an order \( \preceq_P \) over \( \Omega_{b,P} \) defined by 
\( \rho \preceq_P \rho' \) if \( \rho \preceq \rho' \) and \( (\text{src}(\rho'),\text{tgt}(\rho')) \in (\text{src}(\rho),\text{tgt}(\rho)) + P \). Observe that 
\( \preceq_P \) is well over \( \Omega_{b,P} \). Moreover we have:

\[
(\rightarrow^*) \cap (b + P) = \bigcup_{\rho \in \text{min}_{\preceq_P} \Omega_{b,P}} (\text{src}(\rho),\text{tgt}(\rho)) + ((\rightarrow^*_\rho) \cap P)
\]

Thus \( \rightarrow^* \) is an almost semilinear relation. \( \square \)
Theorem

Let $A$ be a VAS and let $n$ be a marking that is not reachable from a marking $m$. There exists a Presburger formula $\phi$ denoting a forward inductive invariant $I$ such that $m \in I$ and $n \not\in I$.

Corollary

The reachability problem is decidable.
Algorithm With an Easy Implementation

1. Reachability \((m, A, n)\)
   
   2. \(k \leftarrow 0\)
   
   3. repeat forever
      
      4. for each word \(\sigma \in A^k\)
         
         5. if \(m \xrightarrow{\sigma} n\)
            
            6. return "reachable"
      
      7. for each Presburger formula \(\phi(x)\) of length \(k\)
         
         8. if \(m \models \phi\), and \(n \models \neg \phi\) and
            
            9. \(\phi(x) \land y - x \in A \land \neg \phi(y)\) unsat
            
            10. return "unreachable"
      
      11. \(k \leftarrow k + 1\)
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Let us recall the following example:

**Example**

Let \( A \) be a VAS. We introduce:

\[
R = \{(m, n) \in \mathbb{N}^d \times \mathbb{N}^d \mid n - m \in A}\]

Then \( R^* \) is the reachability relation.

Thus if \( R \) is the one step reachability relation of a VAS, then \( R^* \) is an almost semilinear relation.
Monotonicity

Definition (Monotonic)

A relation $R \subseteq \mathbb{N}^d \times \mathbb{N}^d$ is said to be monotonic if $(m + v, n + v) \in R$ for every $(m, n) \in R$ and for every $v \in \mathbb{N}^d$.

Example

Let $A$ be a VAS.
We introduce:

$$R = \{(m, n) \in \mathbb{N}^d \times \mathbb{N}^d \mid n - m \in A\}$$

$R$ is a monotonic Presburger relation.
Lemma

For every monotonic Presburger relation $R \subseteq \mathbb{N}^d \times \mathbb{N}^d$ there exist a VASS $G$ and two control states $p, q$ such that $(q, (y_1, y_2))$ is reachable from $(p, (x_1, x_2))$ if and only if:

$$(x_1 + x_2, y_1 + y_2) \in R^*$$

Proof.

Based on the decomposition of a monotonic Presburger relation into a finite union of monotonic linear relations.
Open Problem

Theorem

The reflexive and transitive closure of a monotonic Presburger relation is a monotonic almost semilinear relation.

Open question: Does the class of monotonic almost semilinear relations is stable by reflexive and transitive closure?

Application: reachability problem for VAS with zero tests.
Conclusion

- We presented geometrical properties satisfied by VAS reachability sets.
- We proved that the Presburger arithmetic is sufficient for denoting certificates of non-reachability.

Open problems:

- Size of formulas denoting $\mathbb{Q}_{\geq 0} \rightarrow_{\mathbf{m}}^*$.
- Find new algorithms for deciding the reachability problem (efficient in practice).
- Extension to the VAS + zero tests. Idea: prove that $R^*$ is almost semilinear for every monotonic almost semilinear relation $R$.
- Extension to the Branching VAS. Idea: replace the Higmann’s lemma by the Kruskal’s lemma.
- Close the complexity gap between lower bound and upper bound.
- At least, provide a clear upper bound (in the fast growing hierarchy).