# Graph Planning with Expected Finite Horizon

KRISHNENDU CHATTERJEE, IST Austria, Austria
LAURENT DOYEN, CNRS & LMF, ENS Paris-Saclay, France

Graph planning gives rise to fundamental algorithmic questions such as shortest path, traveling salesman problem, etc. A classical problem in discrete planning is to consider a weighted graph and construct a path that maximizes the sum of weights for a given time horizon $T$. However, in many scenarios, the time horizon is not fixed, but the stopping time is chosen according to some distribution such that the expected stopping time is $T$. If the stopping time distribution is not known, then to ensure robustness, the distribution is chosen by an adversary, to represent the worst-case scenario.

A stationary plan for every vertex always chooses the same outgoing edge. For fixed horizon or fixed stopping-time distribution, stationary plans are not sufficient for optimality. Quite surprisingly we show that when an adversary chooses the stopping-time distribution with expected stopping time $T$, then stationary plans are sufficient. While computing optimal stationary plans for fixed horizon is NP-complete, we show that computing optimal stationary plans under adversarial stopping-time distribution can be achieved in polynomial time. Consequently, our polynomial-time algorithm for adversarial stopping time also computes an optimal plan among all possible plans.

CCS Concepts: • **Mathematics of computing** → **Graph theory**; **Expectation maximization**; • **Theory of computation** → *Shortest paths*.

Additional Key Words and Phrases: Graph planning, shortest path, finite horizon, expected stopping time

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## 1 INTRODUCTION

*Graph search algorithms.* Reasoning about graphs is a fundamental problem in computer science, which is studied widely in logic (such as to describe graph properties with logic [4, 9]) and artificial intelligence [13, 17]. Graph search/planning algorithms are at the heart of such analysis, and gives rise to some of the most important algorithmic problems in computer science, such as shortest path, traveling salesman problem, etc.

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Authors' addresses: Krishnendu Chatterjee, IST Austria, Am Campus 1, Klosterneuburg, Austria, 3400, Krishnendu.Chatterjee@ist.ac.at; Laurent Doyen, CNRS & LMF, ENS Paris-Saclay, 4 avenue des Sciences, Gif-sur-Yvette, France, 91190, doyen@lsv.fr.


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Table 1. Summary of the results.

| Graph planning with... | Stationary plan always exists | Complexity: Arbitrary plan | Complexity: Stationary plan |
|---|---|---|---|
| Specified distribution | No | PTIME | NP-complete |
| Unknown distribution (best-case) | No | PTIME | NP-complete |
| Unknown distribution (adversarial) | Yes | PTIME | |

*Finite-horizon planning.* A classical problem in graph planning is the *finite-horizon* planning problem [13], where the input is a directed graph with weights assigned to every edge and a time horizon $T$. The weight of an edge represents the reward/cost of the edge. A *plan* is an infinite path, and for finite horizon $T$ the utility of the plan is the sum of the weights of the first $T$ edges. An *optimal* plan maximizes the utility. The computational problem for finite-horizon planning is to compute the optimal utility and an optimal plan, which has applications in artificial intelligence and robotics [17, Chapter 10, Chapter 25], and in control theory and game theory [7, Chapter 2.2], [15, Chapter 6].

*Solutions for finite-horizon planning.* For finite-horizon planning the classical solution approach is dynamic programming (or Bellman equations), which corresponds to backward induction [7, 11]. This approach not only works for graphs, but also for other models (e.g., Markov decision processes [16]). A *stationary* plan is a path where for every vertex always the same choice of edge is made. For finite-horizon planning, stationary plans are not sufficient for optimality, and in general, optimal plans are quite involved. Represented as transducers, optimal plans require a number of states proportional to at least $T$ (see later Example 1). Since in general optimal plans are involved, a related computational question is to compute effective simple plans, i.e., plans that are optimal among stationary plans.

*Expected finite-horizon planning.* A natural variant of the finite-horizon planning problem is to consider expected time horizon, instead of the fixed time horizon. In the finite-horizon problem the allowed stopping time of the planning problem is a Dirac distribution at time $T$. In expected finite-horizon problem the *expected* stopping time is $T$. A well-known example where the fixed finite-horizon and the expected finite-horizon problems are fundamentally different is playing Prisoner's Dilemma: if the time horizon is fixed, then defection is the only dominant strategy, whereas for expected finite-horizon problem cooperation is feasible [14, Chapter 5]. Another classical example of expected finite horizon that is well-studied is the notion of *discounting*, where at each time step the stopping probability is $\lambda$, and this corresponds to an expected stopping time equal to $1/\lambda$ [7].

*Specified vs. unknown distribution.* For the expected finite-horizon problem there are two variants: (a) *specified distribution:* the stopping-time distribution with finite support is specified; and (b) *unknown distribution:* the stopping-time distribution is unknown, and either resolved as the best-case scenario, or resolved as the worst-case scenario by an adversary. The expected finite-horizon problem with adversarial distribution represents the robust version of the planning problem, where the distribution is unknown and the adversary represents the worst-case scenario.

*Motivation.* We now present some motivation to study the expected stopping-time problem with adversarial distribution. As mentioned before, the well-studied discounted-sum problem is a specific example of stopping-time distribution. In comparison, our general framework is relevant in the following scenarios: first, in many scenarios the discount factor is not known precisely, and for robust analysis the discount factor is chosen adversarially; second, the discounted-sum model makes an assumption on the shape of the stopping-time distribution. A weaker assumption is to consider time-varying discount factors [5]. If the discount factors are not known, then robust solutions require the adversarial choice of the discount factors. The above scenarios suggest that complex stopping-time distributions are required to model realistic scenarios, and if the precise parameters are unknown, then robust solutions require adversarial choices. Moreover, in all cases when the stopping-time distribution is important yet unknown, a conservative estimate (i.e., lower bound) of the optimal value is obtained using the adversarial choice. Thus the problems we consider present robust extensions of the classical finite-horizon planning that has a wide range of applications.

*Results.* In this work, we consider the expected finite-horizon planning problems in graphs. To the best of our knowledge this problem has not been studied in the literature.

- Our first simple result is that for the specified distribution problem, the optimal value can be computed in polynomial time (Theorem 1). However, since the specified distribution generalizes the fixed finite-horizon problem, the optimal plan description as an explicit transducer is of size $T$. Hence the output complexity is not polynomial in general. Second, we consider the decision problem whether there is a stationary plan to ensure a given utility. We show that this problem is NP-complete (Theorem 2). We establish the same results (Theorem 6 and Theorem 7) for the best-case scenario of unknown distributions.

Our most interesting results are for the adversarial unknown distribution problem, which we describe below:

- We show that stationary plans suffice for optimality (Theorem 3).
- We show that the optimal value and an optimal stationary plan can be computed in polynomial time (Theorem 4).

We highlight the surprising aspects and novelty of the above results.

- First, the result about optimality of stationary plans for adversarial distribution is surprising and counter-intuitive. In the classical finite-horizon problem (and in the specified-distribution problem), the adversary does not have any choice, and in the best-case scenario the choice of the distribution is made favorably. In terms of the choice of plans and the choice of stopping-time distributions, in the first case there is only one quantification (over the choice of plans), and in the second case there are two quantifications, but no quantifier alternation. In the above cases, stationary plans *do not* suffice for optimality. In contrast, we show that in the presence of an adversary the simpler class of stationary plans suffices for optimality. The adversarial case represents a quantifier alternation between the choice of plans and stopping-time distribution. Quite surprisingly our results establish that simpler plans suffice for optimality in the quantifier alternation case as compared to the cases with no quantifier alternation, or only one quantifier.
- For the expected finite-horizon problem with adversarial distribution, the backward induction approach does not work, as there is no a-priori bound on the stopping time. We develop new algorithmic ideas to establish polynomial-time complexity. Note that our algorithm also computes stationary optimal plans (which are as well optimal

among all plans) in polynomial time, whereas computing stationary optimal plans for fixed finite horizon, or specified distribution, is NP-complete. Thus again our algorithm establishes a surprising result: a problem with quantifier alternation can be solved in polynomial-time, whereas the same problem without quantifier alternation is NP-complete.

Our results are summarized in Table 1 and are relevant for synthesis of robust plans for expected finite-horizon planning.

## 2 PRELIMINARIES

*Weighted graphs.* A *weighted graph* $G = \langle V, E, w\rangle$ consists of a finite set $V$ of vertices, a set $E \subseteq V \times V$ of edges, and a function $w\colon E \to \mathbb{Z}$ that assigns a weight to each edge of the graph.

*Plans and utilities.* A *plan* is an infinite *path* in $G$ from a vertex $v_0$, that is a sequence $\rho = e_0 e_1 \dots$ of edges $e_i = (v_i, v_i') \in E$ such that $v_i' = v_{i+1}$ for all $i \geq 0$. A path induces a sequence of utilities $u_0, u_1, \dots$ where $u_i = \sum_{0 \leq k \leq i} w(e_k)$ for all $i \geq 0$. We denote by $U_G$ the set of all sequences of utilities induced by the paths of $G$. For finite paths $\rho = e_0 e_1 \dots e_k$ (i.e., finite prefixes of paths), we denote by $\mathsf{start}(\rho) = v_0$ and $\mathsf{end}(\rho) = v_k'$ the initial and last vertex of $\rho$, and by $|\rho| = k + 1$ the length of $\rho$.

*Plans as transducers.* A plan is described by a *transducer* (Mealy machine or Moore machine [10]) that given a prefix of the path (i.e., a finite sequence of edges) chooses the next edge. A *stationary* plan is a path where for every vertex the same choice of edge is made always. We define the *size* of a Mealy or Moore machine to be its number of states. A stationary plan as a Mealy machine has one state, and as a Moore machine has at most $|V|$ states. Given a graph $G$ we denote by $S_G$ the set of all sequences of utilities induced by stationary plans in $G$.

*Distributions and stopping times.* A *sub-distribution* is a function $\delta\colon \mathbb{N} \to [0, 1]$ such that $p_\delta = \sum_{t\in\mathbb{N}} \delta(t) \in (0, 1]$. The value $p_\delta$ is the probability mass of $\delta$. Note that $p_\delta \neq 0$. The support of $\delta$ is $\mathsf{Supp}(\delta) = \{t \in \mathbb{N} \mid \delta(t) \neq 0\}$, and we say that $\delta$ is the sum of two sub-distributions $\delta_1$ and $\delta_2$, written $\delta = \delta_1 + \delta_2$, if $\delta(t) = \delta_1(t) + \delta_2(t)$ for all $t \in \mathbb{N}$. A *stopping-time distribution* (or simply, a distribution) is a sub-distribution with probability mass equal to 1. We denote by $\Delta$ the set of all stopping-time distributions, and by $\Delta^{\upuparrows}$ the set of all distributions $\delta$ with $|\mathsf{Supp}(\delta)| \leq 2$, called the bi-Dirac distributions.

*Expected utility and expected time.* The *expected utility* of a sequence $u = u_0, u_1, \dots$ of utilities under a sub-distribution $\delta$ is $\mathbb{E}_\delta(u) = \frac{1}{p_\delta} \cdot \sum_{t\in\mathbb{N}} u_t \cdot \delta(t)$. In particular, the expected utility of the identity sequence $0, 1, 2, \dots$ is called the *expected time*, denoted by $\mathbb{E}_\delta$.

## 3 EXPECTED FINITE-HORIZON: SPECIFIED DISTRIBUTION

Given a stopping-time distribution $\delta$ with finite support, we show that the optimal expected utility can be computed in polynomial time. This result is straightforward.

THEOREM 1. *Let $G$ be a weighted graph. Given a stopping-time distribution $\delta = \{(t_1, p_1), \dots, (t_k, p_k)\} \subseteq \mathbb{N} \times \mathbb{Q}$, with all numbers encoded in binary, the optimal expected utility $\sup_{u\in U_G} \mathbb{E}_\delta(u)$ can be computed in polynomial time.*

A special case of the problem in Theorem 1 is the fixed-length optimal path problem, which is to find an optimal path (that maximizes the total utility) of fixed length $T$, corresponding to the distribution $\delta = \{(T, 1)\}$. A pseudo-polynomial time solution is known for this problem, based on a value-iteration algorithm [13, Section 2.3]. The algorithm runs

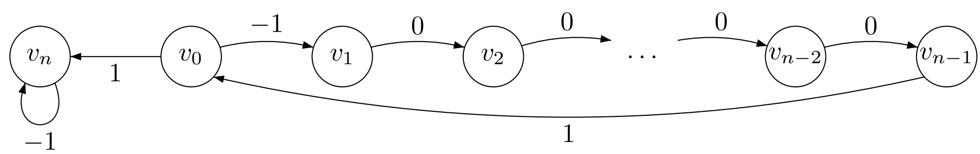



Fig. 1. A weighted graph (with $n+1$ vertices) where the optimal path (of length $T = k \cdot n + 1$) is not simple: at $v_0$, the optimal plan chooses $k$ times the edge $(v_0, v_1)$, and then the edge $(v_0, v_n)$.

in time $O(T \cdot |V|^2)$ (where $T$ is encoded in binary), and relies on the following recursive relation, where $A_t(v)$ is the optimal value among the paths of length $t$ that start in $v$:

$$A_t(v) = \max_{v' \in V} \; w(v, v') + A_{t-1}(v').$$

A polynomial algorithm running in $O(\log(T) \cdot |V|^3)$ to obtain $A_T(v)$ is to compute, in the max-plus algebra[1], the $T$-th power of the transition matrix $M$ of the weighted graph, where $M_{ij} = w(i,j)$ if $(i,j) \in E$, and $M_{ij} = -\infty$ otherwise. The power $M^T$ can be computed in time $O(\log(T) \cdot |V|^3)$ by successive squaring of $M$ and summing up according to the binary representation of $T$. This gives a polynomial algorithm to compute $A_T(v)$, which is the largest element in the row of $M^T$ corresponding to $v$. Note that the entries of the matrix $M^T$ are bounded by $T \cdot W$, where $W$ is the largest absolute weight in the graph. We now present the proof of Theorem 1.

Proof of Theorem 1. Given the weighted graph $G = \langle V, E, w \rangle$ and the distribution $\delta = \{(t_1, p_1), \dots, (t_k, p_k)\}$, we reduce the problem to finding an optimal path of length $k$ in a layered graph $G'$ where the transitions between layer $i$ and layer $i+1$ mimic sequences of $t_{i+1} - t_i$ transitions in the original graph. For $t \geq 2$, define the $t$-th power of $E$ recursively by $E^t = \{(v_0, v_2) \mid \exists v_1 : (v_0, v_1) \in E \wedge (v_1, v_2) \in E^{t-1}\}$ where $E^1 = E$. Let $M$ be the transition matrix of the original weighted graph. We construct the graph $G' = \langle V', E', w' \rangle$ where

- $V' = V \times \{0, \dots, k\}$,
- $E' = \{(\langle v, i \rangle, \langle v', i+1 \rangle) \mid (v, v') \in E^{t_{i+1} - t_i} \wedge 0 \leq i < k\}$ where $t_0 = -1$, and
- $w'(\langle v, i \rangle, \langle v', i+1 \rangle) = (p_{i+1} + p_{i+2} + \cdots + p_k) \cdot (M^{t_{i+1} - t_i})_{v,v'}$.

The optimal expected utility $\sup_{u \in U_G} \mathbb{E}_\delta(u)$ is the same as the optimal fixed-length path value for length $k$ in $G'$. The correctness of this reduction relies on the fact that the probability of not stopping before time $t_{i+1}$ is $p_{i+1} + p_{i+2} + \cdots + p_k$ and the largest utility of a path of length $t_{i+1} - t_i$ from $v$ to $v'$ is $(M^{t_{i+1} - t_i})_{v,v'}$. Given a path $(v_0, v_1)(v_1, v_2) \dots (v_{k-1}, v_k)$ of length $k$ in $G'$ (that induces a sequence $w'_0 \dots w'_{k-1}$ of weights), we can construct a path of length $t_k + 1$ in $G$ (visiting $v_i$ at time $t_i$ and inducing a sequence $u$ of utilities), and we show that the value of the path of length $k$ in $G'$ is the same as the expected utility of the corresponding path in $G$ with stopping time distributed according to $\delta$, as follows (where

[1] In the max-plus algebra, the matrix product $C = A \cdot B$ is defined by $C_{ij} = \max_k A_{ik} + B_{kj}$.

$u_{t_0} = 0$):

$$\sum_{i=0}^{k-1} w'_i = \sum_{i=0}^{k-1} \left( \sum_{j=i+1}^{k} p_j \right) \cdot (u_{t_{i+1}} - u_{t_i})$$
$$= \sum_{j=1}^{k} p_j \cdot \sum_{i=0}^{j-1} (u_{t_{i+1}} - u_{t_i})$$
$$= \sum_{j=1}^{k} p_j \cdot u_{t_j}.$$

Conversely, given an arbitrary path in $G$, let $v_i$ be the vertex visited at time $t_i$, and consider the path $(\langle v_0, 0\rangle, \langle v_1, 1\rangle)(\langle v_1, 1\rangle, \langle v_2, 2\rangle) \dots (\langle v_{k-1}, k-1\rangle, \langle v_k, k\rangle)$ in $G'$, which has a total utility at least the same as the expected utility of the given path in $G$.

Therefore, the problem can be solved by finding the optimal fixed-length path value for length $k$ in $G'$, which can be done in polynomial time (see the remark after Theorem 1). □

In the fixed-horizon problem with $\delta = \{(T, 1)\}$, the optimal plan need not be stationary. The example below shows that in general the transducer for optimal plans require $O(T/|V|)$ states as Mealy machine, and $O(T)$ states as Moore machine.

Example 1. *Consider the graph of Figure 1 with $|V| = n+1$ vertices, and time bound $T = k \cdot n + 1$ (for some parameter $k$). The optimal plan from $v_0$ is to repeat $k$ times the cycle $v_0, v_1, \dots, v_{n-1}$ and then switch to $v_n$. This path has value 1, and all other paths have lower value: if only the cycle $v_0, v_1, \dots, v_{n-1}$ is used, then the value is at most 0, and the same holds if the cycle on $v_n$ is ever used before time $T$. The optimal plan can be represented by a Mealy machine of size $O(T/|V|)$ that counts the number $k \in O(T)$ of cycle repetitions before switching to $v_n$. A Moore machine requires size $O(T)$ as it needs a new memory state at every step of the plan.*

Example 2. *In the example of Figure 2 the optimal plan needs to visit several different cycles, not just repeating a single cycle and possibly switching only at the end. The graph consists of three loops on $v_0$ with weight 0 and respective length 6, 10, and 15, and an edge to $v_1$ with weight 1. For expected time $T = 6 + 10 + 15 + 1$, the optimal plan has value 1 and needs to stop exactly when reaching $v_1$ (to avoid the negative self-loop on $v_1$). It is easy to show that the remaining length $T - 1 = 31$ can only be obtained by visiting each cycle once: as 31 is not an even number, the path has to visit a cycle of odd length, thus the cycle of length 15; analogously, as 31 is not a multiple of 3, the path has to visit the cycle of length 10, etc. This example can be easily generalized to an arbitrary number of cycles by using more prime numbers.*

We now consider the complexity of computing optimal plans among stationary plans.

Theorem 2. *Let $G$ be a weighted graph and $\lambda$ be a rational utility threshold. Given a stopping-time distribution $\delta$, whether $\sup_{u \in S_G} \mathbb{E}_\delta(u) \geq \lambda$ (i.e., whether there is a stationary plan with utility at least $\lambda$) is NP-complete. The NP-hardness holds for the fixed-horizon problem $\delta = \{(T, 1)\}$, even when $T$ and all weights are in $O(|V|)$, and thus expressed in unary.*

Proof. The NP upper bound is easily obtained by guessing a stationary plan (i.e., one edge for each vertex of the graph) and checking that the value of the induced path is at least $\lambda$.

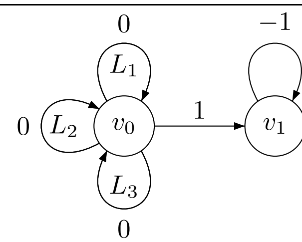



Fig. 2. Three loops of respective length $L_1 = 6 = 2 \cdot 3$, $L_2 = 10 = 2 \cdot 5$, and $L_3 = 15 = 3 \cdot 5$. For $T = 32 = 6 + 10 + 15 + 1$, the optimal plan needs to visit each cycle once.

The NP-hardness follows from a result of [8] where, given a directed graph $\mathcal{G}$ and four vertices $w, x, y, z$, the problem of deciding the existence of two vertex-disjoint simple paths (one from $w$ to $x$ and the other from $y$ to $z$) is shown to be NP-complete. It easily follows that given a directed graph, and two vertices $v_1, v_2$, the problem of deciding the existence of a simple cycle that contains $v_1$ and $v_2$ is NP-complete. We present a reduction from the latter problem, illustrated in Figure 3. We construct a weighted graph from $\mathcal{G}$, by adding two vertices $\mathsf{start}$ and $\mathsf{sink}$, with an edge from $\mathsf{start}$ to each successor of $v_1$, an edge $(v_1, \mathsf{sink})$, and a self-loop on $\mathsf{sink}$. All edges have weight 0 except those from $v_2$ with weight 1, and the edge $(v_1, \mathsf{sink})$ with weight $n + 1$ where $n$ is the number of vertices in $\mathcal{G}$. Let $T = n + 1$ and the utility threshold $\lambda = n + 2$.

If there exists a simple cycle containing $v_1$ and $v_2$ in $\mathcal{G}$, then there exists a stationary plan from $\mathsf{start}$ that visits $v_2$ then $v_1$ in at most $n$ steps. This plan can be prolonged to a plan of $n + 1$ steps by going to $\mathsf{sink}$ and using the self-loop. The total weight is $n + 2 = \lambda$.

If there is no simple cycle containing $v_1$ and $v_2$ in $\mathcal{G}$, then no stationary plan can visit first $v_2$ then $v_1$. We show that every stationary plan has value at most $n + 1 < \lambda$. First if a stationary plan uses the edge $(v_1, \mathsf{sink})$, then $v_2$ is not visited and all weights are 0 except the weight $n + 1$ from $v_1$ to $\mathsf{sink}$. Otherwise, if a stationary plan does not use the edge $(v_1, \mathsf{sink})$, then all weights are at most 1, and the total utility is at most $n+1$. In both cases, the utility is smaller than $\lambda$, which establishes the correctness of the reduction. $\square$

## 4 EXPECTED FINITE-HORIZON: ADVERSARIAL DISTRIBUTION

Our main result is the computation of the following *optimal* values under adversarial distributions[2]. Given a weighted graph $G$ and an expected stopping time $T \in \mathbb{Q}$, we define the following:

- *Optimal values of plans.* For a plan $\rho$ that induces the sequence $u$ of utilities, let

$$val(\rho, T) = val(u, T) = \inf_{\delta \in \Delta : \mathbb{E}_\delta = T} \mathbb{E}_\delta(u).$$

- *Optimal value.* The optimal value is the supremum value over all plans:

$$val(G, T) = \sup_{u \in U_G} val(u, T).$$

Our two main results are related to the plan complexity and a polynomial-time algorithm.

[2]Adversarial distributions may have finite or infinite support.

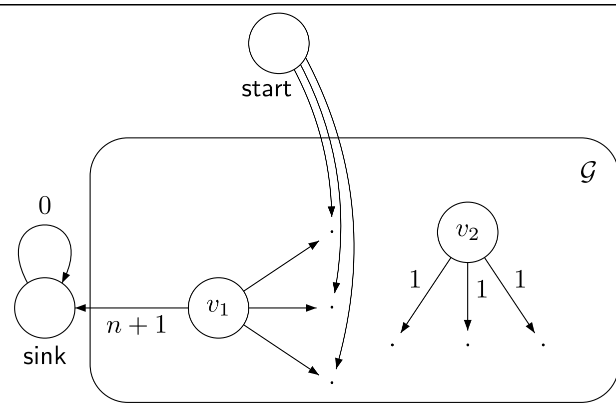



Fig. 3. The NP-hardness reduction of Theorem 2.

THEOREM 3. *For all weighted graphs $G$ and for all $T$ we have*

$$val(G,T) = \sup_{u \in U_G} val(u,T) = \sup_{u \in S_G} val(u,T),$$

*i.e., optimal stationary plans exist for expected finite-horizon under adversarial distribution.*

REMARK 1. *Note that in contrast to the fixed finite-horizon problem, where stationary plans do not suffice, we show in the presence of an adversary, the simpler class of stationary plans are sufficient for optimality in expected finite-horizon. Moreover, while optimal plans require $O(T/|V|)$-size Mealy (resp., $O(T)$-size Moore) machines for fixed-length plans, our results show that under adversarial distribution optimal stationary plans exist (Theorem 3) and thus require $O(1)$-size Mealy (resp., $O(|V|)$-size Moore) machines.*

THEOREM 4. *Given a weighted graph $G$ and expected finite-horizon $T$, deciding whether $val(G,T) \geq 0$ can be done in $O(|V|^{16} \cdot \log(T))$ time, and computing $val(G,T)$ can be done in $O(|V|^{16} \cdot \log(W \cdot |V|) \cdot \log(T))$ time (where $W$ is the largest absolute weight in the graph $G$).*

### 4.1 Theorem 3: Plan Complexity

In this section we prove Theorem 3. We start with the notion of sub-distributions. Two sub-distributions $\delta, \delta'$ are *equivalent* if they have the same probability mass, and the same expected time, that is $p_\delta = p_{\delta'}$ and $\mathbb{E}_\delta = \mathbb{E}_{\delta'}$. The following result is straightforward.

LEMMA 1. *If $\delta_1, \delta_1'$ are equivalent sub-distributions, and $\delta_1 + \delta_2$ is a sub-distribution, then $\delta_1 + \delta_2$ and $\delta_1' + \delta_2$ are equivalent sub-distributions.*

*Bi-Dirac distributions are sufficient.* By Lemma 1, we can decompose distributions as the sum of two sub-distributions, and we can replace one of the two sub-distributions by a simpler (yet equivalent) one to obtain an equivalent distribution. We show that, given a sequence $u$ of utilities, for all sub-distributions with three time points $t_1, t_2, t_3$ in their support (see Figure 4 where $t_1 < t_2 < T < t_3$), there exists an equivalent sub-distribution with only two time points in its support that gives a lower expected value for $u$. Intuitively, if one has to distribute a fixed probability mass (say 1) among three time points with a fixed expected time $T$, assigning probability $p_i$ at time $t_i$, then we have $p_1 + p_2 + p_3 = 1$, which corresponds to the set of convex combinations of the three points $(t_i, u_i)$ (see the triangle

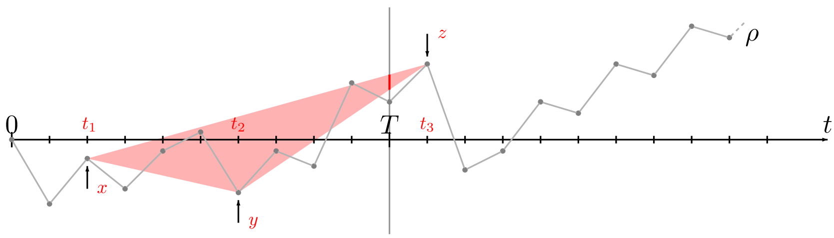



Fig. 4. Bi-Dirac distributions are sufficient.

in Figure 4), and we have $p_1 \cdot t_1 + p_2 \cdot t_2 + p_3 \cdot t_3 = T$, which corresponds to those convex combinations whose first coordinate is $T$ (see the vertical segment at $T$ within the triangle in Figure 4). Finally, the expected utility (to be minimized) is $p_1 \cdot u_{t_1} + p_2 \cdot u_{t_2} + p_3 \cdot u_{t_3}$, which is the second coordinate of the convex combinations. The least expected utility can be obtained for either $p_1 = 0$ or $p_2 = 0$ if $t_1, t_2 < T$ (for $p_1 = 0$ in Figure 4), and for either $p_2 = 0$ or $p_3 = 0$ if $T < t_2, t_3$. In both cases, bi-Dirac distributions are sufficient to compute the optimal expected value.

Lemma 2 (Bi-Dirac distributions are sufficient). *For all sequences $u$ of utilities, for all time bounds $T$, the following holds:*

$$\inf\{\mathbb{E}_\delta(u) \mid \delta \in \Delta \wedge \mathbb{E}_\delta = T\} =$$
$$\inf\{\mathbb{E}_\delta(u) \mid \delta \in \Delta^{\upuparrows} \wedge \mathbb{E}_\delta = T\},$$

*i.e., the set $\Delta^{\upuparrows}$ of bi-Dirac distributions suffices for the adversary.*

Proof. First, we show that for all distributions $\delta \in \Delta$ with $\mathbb{E}_\delta = T$,

$(i)$ there exists an equivalent distribution $\delta' \in \Delta$ such that $|\mathsf{Supp}(\delta') \cap [0, T-1]| \leq 1$ and $\mathbb{E}_{\delta'}(u) \leq \mathbb{E}_\delta(u)$, i.e., only one point before $T$ in the support is sufficient, and

$(ii)$ there exists an equivalent distribution $\delta' \in \Delta$ such that $|\mathsf{Supp}(\delta') \cap [0, T-1]| \leq 1$, and $|\mathsf{Supp}(\delta') \cap [T, \infty)| \leq 1$, and $\mathbb{E}_{\delta'}(u) \leq \mathbb{E}_\delta(u)$, i.e., only one point before $T$ and one point after $T$ in the support are sufficient.

The result of the lemma follows from these two claims.

To prove claim $(i)$, first consider an arbitrary *sub-distribution* $\delta$ with $\mathsf{Supp}(\delta) = \{t_1, t_2, t_3\}$ where $t_1 < t_2 < t_3$. Then $t_1 < \mathbb{E}_\delta < t_3$ and either $\mathbb{E}_\delta \leq t_2$, or $t_2 \leq \mathbb{E}_\delta$.

We show that among the sub-distributions $\delta'$ equivalent to $\delta$ and with $\mathsf{Supp}(\delta') \subseteq \{t_1, t_2, t_3\}$, the smallest expected utility of $u$ is obtained for $\mathsf{Supp}(\delta') \subsetneq \{t_1, t_2, t_3\}$. We present below the argument in the case $t_2 \leq \mathbb{E}_\delta$, and show that either $\delta'(t_1) = 0$, or $\delta'(t_2) = 0$. A symmetric argument in the case $\mathbb{E}_\delta \leq t_2$ shows that either $\delta'(t_2) = 0$, or $\delta'(t_3) = 0$.

Let $x = \delta'(t_1)$, $y = \delta'(t_2)$, and $z = \delta'(t_3)$. Since $\delta'$ and $\delta$ are equivalent, we have

$$x + y + z = p_\delta$$
$$x \cdot t_1 + y \cdot t_2 + z \cdot t_3 = p_\delta \cdot \mathbb{E}_\delta$$

Hence

$$z = p_\delta - x - y$$
$$\underbrace{x \cdot (t_1 - t_3)}_{x'} + \underbrace{y \cdot (t_2 - t_3)}_{y'} = p_\delta \cdot (\mathbb{E}_\delta - t_3)$$

The expected utility of $u$ under $\delta'$ is

$$\begin{aligned}\mathbb{E}_{\delta'}(u) &= x \cdot u_{t_1} + y \cdot u_{t_2} + z \cdot u_{t_3} \\ &= x \cdot (u_{t_1} - u_{t_3}) + y \cdot (u_{t_2} - u_{t_3}) + u_{t_3} \cdot p_\delta \\ &= x' \cdot \frac{u_{t_1} - u_{t_3}}{t_1 - t_3} + y' \cdot \frac{u_{t_2} - u_{t_3}}{t_2 - t_3} + u_{t_3} \cdot p_\delta \end{aligned} \tag{1}$$

Since $x' + y'$ is constant and $x', y' \leq 0$, the least value of $\mathbb{E}_{\delta'}(u)$ is obtained either for $x' = 0$ (if $\frac{u_{t_1} - u_{t_3}}{t_1 - t_3} \leq \frac{u_{t_2} - u_{t_3}}{t_2 - t_3}$), or for $y' = 0$ (otherwise), thus either for $x = 0$, or for $y = 0$. Note that for $x = 0$, we have $y = \frac{p_\delta \cdot (\mathbb{E}_\delta - t_3)}{t_2 - t_3}$ and $z = \frac{p_\delta \cdot (t_2 - \mathbb{E}_\delta)}{t_2 - t_3}$, which is a feasible solution as $0 \leq y \leq 1$ and $0 \leq z \leq 1$ since $t_2 \leq \mathbb{E}_\delta \leq t_3$, and $0 < p_\delta \leq 1$. Symmetrically, for $y = 0$ we have a feasible solution.

As an intermediate remark, note that for $p_\delta = 1$ and $\mathbb{E}_\delta = T$, we get (for $y = y' = 0$, and symmetrically for $x = x' = 0$)

$$\mathbb{E}_{\delta'}(u) = u_{t_3} + \frac{T - t_3}{t_1 - t_3} \cdot (u_{t_1} - u_{t_3}). \tag{2}$$

To complete the proof of Claim $(i)$, given an arbitrary distribution $\delta$ with $\mathbb{E}_\delta = T$, we use the above argument to construct a distribution equivalent[3] to $\delta$ with smaller expected utility and one less point in the support. We repeat this argument until we obtain a distribution $\delta'$ with support that contains at most two points in the interval $[0, k]$ where $k$ is such that $\sum_{i \leq k} \delta(i) \cdot i > T - 1$. Such a value of $k$ exists since $\mathbb{E}_\delta = \sum_{i \in \mathbb{N}} \delta(i) \cdot i = T$. By the construction of $\delta'$, we have $\sum_{i \leq k} \delta'(i) \cdot i > T - 1$ and therefore at most one point in the support of $\delta'$ lies in the interval $[0, T - 1]$, which completes the proof of Claim $(i)$.

To prove claim $(ii)$, consider a distribution $\delta$ from $(i)$ with $\mathbb{E}_\delta = T$, thus we can assume that $\delta(t_0) \neq 0$ for some $t_0 < T$, and $\delta(t) = 0$ for all $t < T$ with $t \neq t_0$. Let $\nu = \inf_{t \geq T} \frac{u_t - u_{t_0}}{t - t_0}$, and we consider two cases:

- if for all $t \geq T$ such that $t \in \mathsf{Supp}(\delta)$, we have $\frac{u_t - u_{t_0}}{t - t_0} = \nu$, then by an analogous of Equation (1), we get

$$\begin{aligned}\mathbb{E}_\delta(u) &= u_{t_0} + \sum_{t \geq T} \delta(t) \cdot (t - t_0) \cdot \frac{u_t - u_{t_0}}{t - t_0} \\ &= u_{t_0} + \nu \cdot \sum_{t \geq 0} \delta(t) \cdot (t - t_0) \\ &= u_{t_0} + \nu \cdot (T - t_0)\end{aligned}$$

  which is the expected utility of $u$ under a bi-Dirac distribution with support $\{t_0, t\}$ where $t \geq T$ is any element of $\mathsf{Supp}(\delta)$ (see Equation (2));

[3] Equivalence follows from Lemma 1.

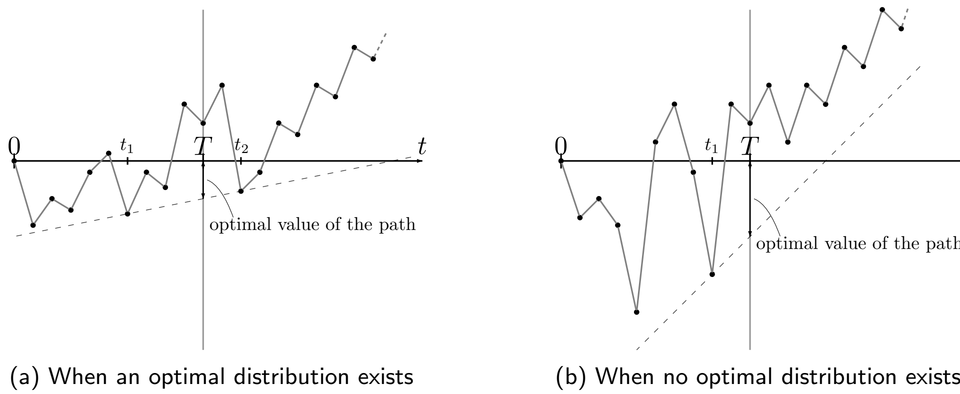



Fig. 5. Geometric interpretation of the value of a path.

- otherwise there exists $t \geq T$ such that $t \in \mathsf{Supp}(\delta)$ and $\frac{u_t - u_{t_0}}{t - t_0} > \nu$. By an analogous of Equation (1), we have

$$\mathbb{E}_\delta(u) - u_{t_0} = \sum_{t \geq T} \delta(t) \cdot (t - t_0) \cdot \frac{u_t - u_{t_0}}{t - t_0}$$
$$\text{where } \sum_{t \geq T} \delta(t) \cdot (t - t_0) = T - t_0,$$

that is $\frac{\mathbb{E}_\delta(u) - u_{t_0}}{T - t_0}$ is a convex combination of elements greater than or equal to $\nu$, among which one is greater than $\nu$. It follows that $\frac{\mathbb{E}_\delta(u) - u_{t_0}}{T - t_0} > \nu$, and thus there exists $\epsilon > 0$ such that $\frac{\mathbb{E}_\delta(u) - u_{t_0}}{T - t_0} > \nu + \epsilon$.
Consider $t_1 \geq T$ such that $\frac{u_{t_1} - u_{t_0}}{t_1 - t_0} < \nu + \epsilon$ (which exists by definition of $\nu$), and let $\delta'$ be the bi-Dirac distribution $\delta'$ with support $\{t_0, t_1\}$ and expected time $T$. By an analogous of Equation (2), we have

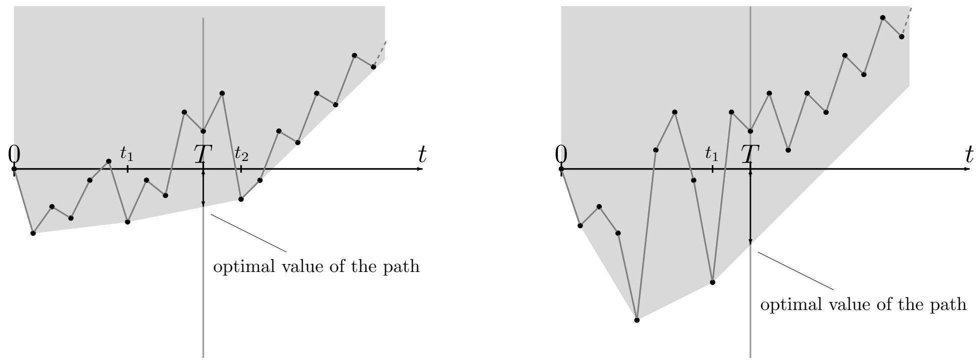
$$\begin{aligned}\mathbb{E}_{\delta'}(u) - u_{t_0} &= \frac{T - t_0}{t_1 - t_0} \cdot (u_{t_1} - u_{t_0}) \\ &< (T - t_0) \cdot (\nu + \epsilon) < \mathbb{E}_\delta(u) - u_{t_0}\end{aligned}$$

Therefore, $\mathbb{E}_{\delta'}(u) < \mathbb{E}_\delta(u)$ which concludes the proof since $\delta'$ is a bi-Dirac distribution with $\mathbb{E}_{\delta'} = T$. □

*Geometric interpretation.* It follows from the proof of Lemma 2 (and Equation (2)) that the value of the expected utility of a sequence $u$ of utilities under a bi-Dirac distribution with support $\{t_1, t_2\}$ (where $t_1 < T < t_2$) and expected time $T$ is

$$u_{t_1} + \frac{T - t_1}{t_2 - t_1} \cdot (u_{t_2} - u_{t_1}).$$

In Figure 5a, this value is obtained as the intersection of the vertical axis at $T$ and the line that connects the two points $(t_1, u_{t_1})$ and $(t_2, u_{t_2})$. Intuitively, the optimal value of a path is obtained by choosing the two time points $t_1$ and $t_2$ such that the connecting line intersects the vertical axis at $T$ as down as possible.




(a) For the example of Fig. 5a. (b) For the example of Fig. 5b.

Fig. 6. Convex hull interpretation of the value of a path.

LEMMA 3. *For all sequences $u$ of utilities, if $u_t \geq a \cdot t + b$ for all $t \geq 0$, then the value of the sequence $u$ is at least $a \cdot T + b$.*

PROOF. By Lemma 2, it is sufficient to consider bi-Dirac distributions, and for all bi-Dirac distributions $\delta$ with arbitrary support $\{t_1, t_2\}$ the value of $u$ under $\delta$ is

$$\begin{aligned} & u_{t_1} + \frac{T - t_1}{t_2 - t_1} \cdot (u_{t_2} - u_{t_1}) \\ = & \frac{u_{t_1} \cdot (t_2 - T) + u_{t_2} \cdot (T - t_1)}{t_2 - t_1} \\ \geq & \frac{(a \cdot t_1 + b) \cdot (t_2 - T) + (a \cdot t_2 + b) \cdot (T - t_1)}{t_2 - t_1} \\ \geq & \, a \cdot T + b \end{aligned}$$

□

It is always possible to fix an optimal value of $t_1$ (because $t_1 \leq T$ is to be chosen among a finite set of points), but the optimal value of $t_2$ may not exist, as in Figure 5b. The value of the path is then obtained as $t_2 \to \infty$. In general, there exists $t_1 \leq T$ such that it is sufficient to consider bi-Dirac distributions with support containing $t_1$ to compute the optimal value. We say that $t_1$ is a *left-minimizer* of the expected value in the path. Given such a value of $t_1$, let $\nu = \inf_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$, and we show in Lemma 4 that $u_t \geq u_{t_1} + (t - t_1) \cdot \nu$, for all $t \geq 0$. This motivates the following definition.

*Line of equation $f_u(t)$.* Given a left-minimizer $t_1$, we define the line of equation $f_u(t)$ as follows:

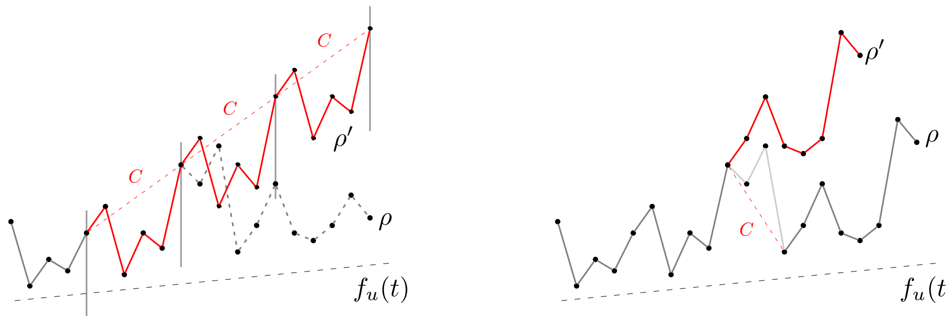
$$f_u(t) = u_{t_1} + (t - t_1) \cdot \nu.$$

Note that the optimal expected utility is

$$\min_{0 \leq t_1 \leq T} \inf_{t_2 \geq T} u_{t_1} + \frac{T - t_1}{t_2 - t_1} \cdot (u_{t_2} - u_{t_1}) = \min_{0 \leq t_1 \leq T} u_{t_1} + (T - t_1) \cdot \nu = f_u(T).$$

In other words, $f_u(T)$ is the optimal value.

LEMMA 4 (GEOMETRIC INTERPRETATION). *For all sequences $u$ of utilities, we have $u_t \geq f_u(t)$ for all $t \geq 0$, and the expected value of $u$ is $f_u(T)$.*




(a) Repeating a good cycle (Lemma 6). (b) Removing a bad cycle (Lemma 7).

Fig. 7. Constructing a lasso without decreasing the value (Lemma 6 and Lemma 7).

Proof. The result holds by definition of $\nu$ for all $t \geq T$. For $t < T$, assume towards contradiction that $u_t < u_{t_1} + (t - t_1) \cdot \nu$. Let $\varepsilon > 0$ be such that $u_t = u_{t_1} + (t - t_1) \cdot \nu - \varepsilon$. We obtain a contradiction by showing that there exists a bi-Dirac distribution under which the expected value of $u$ is smaller than the optimal value of $u$. Consider a bi-Dirac distribution with support $\{t, t_2\}$ where the value $t_2$ is defined later.

We need to show that

$$u_t + \frac{T - t}{t_2 - t} \cdot (u_{t_2} - u_t) < u_{t_1} + (T - t_1) \cdot \nu,$$

that is

$$\frac{u_t \cdot (t_2 - T) + u_{t_2} \cdot (T - t)}{t_2 - t} < u_{t_1} + (T - t_1) \cdot \nu$$

which, since $u_t = u_{t_1} + (t - t_1) \cdot \nu - \varepsilon$, holds if (successively):

$$u_{t_1} \cdot (t_2 - T) + (t - t_1) \cdot (t_2 - T) \cdot \nu + u_{t_2} \cdot (T - t) \leq \varepsilon \cdot (t_2 - T) + u_{t_1} \cdot (t_2 - t) + (t_2 - t) \cdot (T - t_1) \cdot \nu,$$
$$u_{t_1} \cdot (t - T) + u_{t_2} \cdot (T - t) + \nu \cdot (t \cdot t_2 + t_1 \cdot T - t_2 \cdot T - t \cdot t_1) \leq \varepsilon \cdot (t_2 - T),$$
$$(u_{t_2} - u_{t_1}) \cdot (T - t) + \nu \cdot (t_2 - t_1) \cdot (t - T) - \varepsilon \cdot (t_2 - T) \leq 0,$$
$$(T - t) \cdot \left( \tfrac{u_{t_2} - u_{t_1}}{t_2 - t_1} - \nu \right) \cdot (t_2 - t_1) - \varepsilon \cdot (t_2 - T) \leq 0.$$

We consider two cases: $(i)$ if the infimum $\nu$ is attained, then we have $\nu = \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$ for some $t_2 \geq T$, and the inequality holds; $(ii)$ otherwise, we can choose $t_2$ arbitrarily, and large enough to ensure that $(T - t) \cdot \left( \frac{u_{t_2} - u_{t_1}}{t_2 - t_1} - \nu \right)$ is smaller than $\frac{\varepsilon}{2}$, so that the inequality holds. □

A corollary of the geometric interpretation lemma is that the value of a path can be obtained as the intersection of the vertical line at time $T$ with the boundary of the convex hull of the region above the sequence of utilities, namely $convexHull(\{(t, y) \in \mathbb{N} \times \mathbb{R} \mid y \geq u_t\})$. This result is illustrated in Figure 6.

*Simple lassos are sufficient.* A *lasso* is a path of the form $AC^\omega$ where $A$ and $C$ are finite paths (with $C$ a nonempty cycle), where $AC^\omega$ is $A$ followed by infinite repetition of the cycle $C$. A lasso is *simple* if all strict prefixes of the finite path $AC$ are acyclic. In other words, simple lassos correspond to stationary plans.

We show that there is always a simple lasso with optimal value. Our proof has four steps. Given a path $\rho$ that gives the utility sequence $u$, let $\nu$ be the slope of $f_u(t)$. Given a cycle $C$ in the path $\rho$, let $S_C$ be the sum of the weights in $C$ and let $M_C = \frac{S_C}{|C|}$ be the average weight of the cycle edges. The cycle $C$ is *good* if $M_C \geq \nu$, i.e., the average weight of the cycle is at least $\nu$, and *bad* otherwise.

- First, we show (in Lemma 5) that every path contains a good cycle.
- Second, we show (in Lemma 6) that if the first cycle in a path is good, then repeating the cycle cannot decrease the value of the path.
- Third, we show (in Lemma 7) that removing a bad cycle from a path cannot decrease the value of the path.
- Finally, we show (in Lemma 8) that given any path, using the above two operations of removal of bad cycles and repetition of good cycles, we obtain a simple lasso that does not decrease the value of the original path.

Thus we establish that simple lassos (or stationary plans) are sufficient for optimality. To formalize the ideas we consider the notion of cycle decomposition.

*Cycle decomposition.* The *cycle decomposition* of a path $\rho = e_0 e_1 \dots$ is an infinite sequence of simple cycles $C_1, C_2, \dots$ obtained as follows: push successively $e_0, e_1, \dots$ onto a stack, and whenever we push an edge that closes a (simple) cycle, we remove the cycle from the stack and append it to the cycle decomposition. Note that the stack content is always a prefix of a path of length at most $|V|$.

LEMMA 5. *Let $T \in \mathbb{N}$. Given a path $\rho$ that induces a sequence $u$ of utilities, let $\nu = \min_{0 \leq t_1 \leq T} \inf_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$. Then, in the cycle decomposition of $\rho$ there exists a simple cycle $C$ with $M_C \geq \nu$.*

PROOF. Towards contradiction, assume that all cycles $C$ in the cycle decomposition of $\rho$ are such that $M_C < \nu$. Let $t_1$ be a left-minimizer of $\rho$. Since all cycles in $\rho$ have average weight smaller than $\nu$, we have:

$$\liminf_{t_2 \to \infty} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1} < \nu$$

Since the infimum is bounded by the liminf, it follows that

$$\min_{0 \leq t_1 \leq T} \inf_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1} < \nu$$

which is in contradiction with the definition of $\nu$. □

We show that repeating a good cycle, and removing a bad cycle from a path cannot decrease the value of the path.

LEMMA 6. *Let $T \in \mathbb{N}$. If the first cycle $C$ in the cycle decomposition of a path $\rho$ is good, i.e., $M_C \geq \nu$ where $\nu = \min_{0 \leq t_1 \leq T} \inf_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$, then there exists a lasso $\rho'$ such that $val(\rho', T) \geq val(\rho, T)$.*

PROOF. Let $u$ be the sequence of utilities induced by $\rho$. Since $C$ is the first cycle in $\rho$, there is a prefix of $\rho$ of the form $AC$ where $A$ is a finite path. Consider the lasso $\rho' = AC^\omega$ and its induced sequence of utilities $u'$.

We show that the value of $\rho'$ is at least the value of $\rho$. By Lemma 4, the optimal value of $u$ is $f_u(T)$, and the sequence $u$ is above the line $f_u(t)$ (which has slope $\nu$), i.e., $u(t) \geq f_u(t)$ for all $t \geq 0$. By Lemma 3 it is sufficient to show that $u'$ is above the line $f_u(t)$ to establish that the optimal value of $u'$ is at least $f_u(T)$, that is $val(\rho', T) \geq val(\rho, T)$, and conclude the proof (the argument is illustrated in Figure 7a).
We show that $u'(t) \geq f_u(t)$ for all $t \geq 0$:

- either $t \leq |A| + |C|$, and then $u'(t) = u(t) \geq f_u(t)$,
- or $t > |A| + |C|$, and then let $k \in \mathbb{N}$ such that $|A| \leq t - k \cdot |C| \leq |A| + |C|$, and we have

$$\begin{aligned}
u'(t) &= u(t - k \cdot |C|) + k \cdot S_C && (\rho' = AC^\omega) \\
&\geq f_u(t - k \cdot |C|) + k \cdot M_C \cdot |C| \\
& && (u \text{ is above } f_u(t) \text{ and } S_C = M_C \cdot |C|) \\
&\geq f_u(t) - \nu \cdot k \cdot |C| + k \cdot M_C \cdot |C| \\
& && (f_u(t) \text{ is linear with slope } \nu) \\
&\geq f_u(t) + k \cdot |C| \cdot (M_C - \nu) \\
&\geq f_u(t). && (M_C \geq \nu) \quad \square
\end{aligned}$$

LEMMA 7. *Let $T \in \mathbb{N}$. If a path $\rho$ contains a bad cycle $C$, that is such that $M_C < \nu$ where $\nu = \min_{0 \leq t_1 \leq T} \inf_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$, then removing $C$ from $\rho$ gives a path $\rho'$ such that $val(\rho', T) \geq val(\rho, T)$.*

PROOF. Let $u, u'$ be the sequences of utilities induced by respectively $\rho$ and $\rho'$, By the same argument as in the proof of Lemma 6 (using Lemma 3 and Lemma 4), it is sufficient to show that $u'$ is above the line $f_u(t)$. Since $C$ is a cycle in $\rho$, there is a prefix of $\rho$ of the form $AC$ where $A$ is a finite path, and for all $t \geq 0$ we have (the argument is illustrated in Figure 7b): either $t \leq |A|$, then $u'(t) = u(t) \geq f_u(t)$, or $t > |A|$, and then

$$\begin{aligned}
u'(t) &= u(t + |C|) - S_C \\
& && (C \text{ is removed from } \rho \text{ to get } \rho') \\
&\geq f_u(t + |C|) - M_C \cdot |C| \\
& && (u \text{ is above } f_u(t) \text{ and } S_C = M_C \cdot |C|) \\
&\geq f_u(t) + \nu \cdot |C| - M_C \cdot |C| \\
& && (f_u(t) \text{ is linear with slope } \nu) \\
&\geq f_u(t) + |C| \cdot (\nu - M_C) \\
&\geq f_u(t). && (M_C < \nu) \quad \square
\end{aligned}$$

Now we can show how to construct a simple lasso with value at least the value of a given arbitrary path, and it follows that simple lassos are sufficient for optimality.

LEMMA 8. *Let $T \in \mathbb{N}$. There exists a simple lasso $AC^\omega$ such that $val(AC^\omega, T) = val(G, T)$.*

PROOF. Given an arbitrary path $\rho$, we construct a simple lasso with at least the same value as $\rho$. It follows that the optimal value is obtained by stationary plans. The construction repeats the following steps:

(1) Let $C$ be the first cycle in the cycle decomposition of $\rho$;

(2) if $C$ is a bad cycle for the original path $\rho$, then we remove it to obtain a new path $\rho'$. We continue the procedure with $\rho'$ (go to step 1.);
(3) otherwise $C$ is a good cycle for the original path $\rho$. Let $A$ be the prefix of $\rho$ until $C$ starts, and we construct the lasso $AC^{\omega}$.

First, note that if the above procedure terminates, then the constructed lasso has a value at least the value of the original path $\rho$ (by Lemma 6 and Lemma 7), and it is a simple lasso by definition of the cycle decomposition.

Now we show that the procedure always terminates. By Lemma 5, there always exists a good cycle in the cycle decomposition of $\rho$, and thus eventually a good cycle becomes the first cycle in the path constructed by the above procedure, which then terminates. □

Theorem 3 follows from the above lemmas.

### 4.2 Theorem 4: Algorithm and Complexity Analysis

In this section we present our algorithm and then the complexity analysis (Theorem 4).

*Algorithm.* The key challenges to obtain an algorithm are as follows. First, while for the fixed-horizon problem backward induction or powering of transition matrix leads to an algorithm, for expected time horizon with an adversary, there is no a-priori bound on the number of steps, and hence the backward induction approach is not applicable. Second, stationary optimal plans suffice, and as shown in Theorem 2 computing optimal stationary plans for the fixed horizon problem is NP-hard. We present an algorithm that iteratively constructs the *most promising* candidate paths according to a partial order of the paths, and the key is to define the partial order.

It follows from the geometric interpretation lemmas (Lemma 3 and Lemma 4) that the value of a path is at least 0 if its sequence of utilities is above some line that contains the point $(T, 0)$.

Lemma 9. *The value of a sequence $u$ of utilities is at least 0 if and only if there exists a slope $M \in \mathbb{R}$ such that $u_t \geq M \cdot (t - T)$ for all $t \geq 0$.*

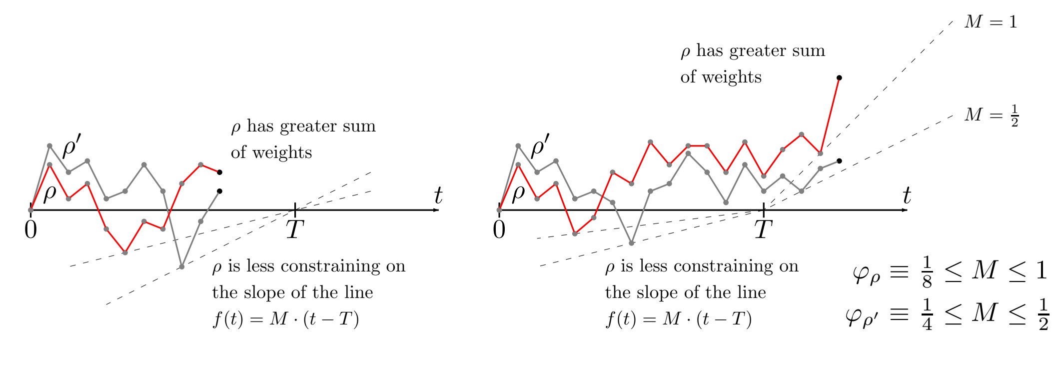
Proof. If the value of $u$ is at least 0, then $f_u(T) \geq 0$ and by Lemma 4 we have $u_t \geq f_u(t)$ for all $t \geq 0$. Then $u_t \geq f_u(t) - f_u(T)$ (which is a linear function of $t$) and we can take for $M$ the value of the coefficient of $t$ in the expression $f_u(t) - f_u(T)$.

To prove the other direction, consider the line of equation $f(t) = M \cdot (t - T)$, and by Lemma 3, the value of the sequence $u$ is at least $f(T) = 0$. □

The expression $u_t - M \cdot (t - T)$ that appears in the condition of Lemma 9 can be obtained by subtracting $M$ to each weight of the graph, and shifting the sum of the weights by the constant $T \cdot M$. Since $M$ is unknown, we can define the following symbolic constraint on $M$ (associated with a path $\rho$) that ensures, if it is satisfiable, that the sequence of utilities of $\rho = e_0 e_1 \dots e_k$ is above the line of equation $f(t) = M \cdot (t - T)$ :

$$\varphi_\rho \equiv \bigwedge_{0 \leq i \leq k} (u_i \geq M \cdot (i - T))$$

Note that $k = |\rho| - 1$, and the constraint $\varphi_\rho$ represents an interval (possibly empty, possibly unbounded) of values for $M$. Intuitively, a finite path is more promising (thus preferred) in order to be prolonged to an infinite path with value at least 0 if the total sum of weights is large and the constraint $\varphi_\rho$ is weak (see Figure 8a and Figure 8b). To each finite path $\rho$, we associate a pair $\langle u, \psi \rangle$ consisting of the sum $u$ of the weights in $\rho$, and the constraint $\psi = \varphi_\rho$.




(a) The path length is smaller than $T$.

(b) The path length is greater than $T$.

Fig. 8. The path $\rho$ is preferred to $\rho'$.

Given two pairs $\langle u, \psi \rangle$, $\langle u', \psi' \rangle$ (associated with paths $\rho$ and $\rho'$ respectively), we write $\langle u, \psi \rangle \succeq \langle u', \psi' \rangle$ if $u \ge u'$ and $\psi'$ implies $\psi$, and we say that $\rho$ is *preferred* to $\rho'$ (this is a partial order). Given a set $S$ of such pairs, denote by $\lceil S \rceil = \{z_1 \in S \mid \forall z_2 \in S : z_2 \succeq z_1 \to z_1 \succeq z_2\}$ the set of $\succeq$-maximal elements of $S$. Note that the elements of $\lceil S \rceil$ are pairwise $\succeq$-incomparable.

Intuitively, if $\rho$ and $\rho'$ end in the same vertex, and $\rho$ is preferred to $\rho'$, then it is easier to extend $\rho$ than $\rho'$ to obtain an (infinite) path with expected value at least 0. Formally, for all infinite paths $\pi$ with $\mathsf{start}(\pi) = \mathsf{end}(\rho) = \mathsf{end}(\rho')$ we have $val(\rho \cdot \pi, T) \ge val(\rho' \cdot \pi, T)$. We use this result in the following form.

Lemma 10. *Let $\rho_1$, $\rho_A$ be two paths of the same length with the same end state, i.e., $\mathsf{end}(\rho_1) = \mathsf{end}(\rho_A)$. If $\rho_1$ is preferred to $\rho_A$, then for all paths $\rho_C$ with $\mathsf{start}(\rho_C) = \mathsf{end}(\rho_A)$, the path $\rho_1 \cdot \rho_C$ is preferred to the path $\rho_A \cdot \rho_C$.*

Proof sketch. Let $\rho_{1C} = \rho_1 \cdot \rho_C$ an $\rho_{AC} = \rho_A \cdot \rho_C$. Denote by $u_1$, $u_A$, $u_{1C}$, and $u_{AC}$ the sum of the weights of the paths $\rho_1$, $\rho_A$, $\rho_1 \cdot \rho_C$, and $\rho_A \cdot \rho_C$ respectively.

Since $u_1 \ge u_A$ and $\varphi_{\rho_A} \to \varphi_{\rho_1}$, it is easy to see that $u_{1C} \ge u_{AC}$, and that for every length $|\rho_1| \le k \le |\rho_1| + |\rho_C|$, the sum of the weights of the prefix of length $k$ of $\rho_1 \cdot \rho_C$ is at least as large as the sum of the weights of the prefix of length $k$ of $\rho_A \cdot \rho_C$. It follows that $\varphi_{\rho_{AC}} \to \varphi_{\rho_{1C}}$ as well, hence $\rho_1 \cdot \rho_C$ is preferred to $\rho_A \cdot \rho_C$. □

Our algorithm uses the procedure $\mathsf{BestPaths}(t_0, v_0, u_0, \psi_0)$ (shown as Algorithm 1) that iteratively computes the $\succeq$-maximal pairs $\langle u, \psi \rangle$ corresponding to the paths $\rho_1$ of length $1, 2, \dots, |V|$ that start at time $t_0$ in vertex $v_0$ (see Figure 9), and that prolong a path $\rho_\sharp$ with sum of weight $u_0$ and constraint $\psi_0$ on $M$ (where $u$ is the sum of weights along $\rho_\sharp \cdot \rho_1$, and $\psi \equiv \varphi_{\rho_\sharp \cdot \rho_1}$). We give a precise statement of this result in Lemma 11.

Lemma 11 (Correctness of $\mathsf{BestPaths}$). *Let $\rho_\sharp$ be a finite path of length $t_0$, that ends in state $\mathsf{end}(\rho_\sharp) = v_0$ with sum of weight $u_0$ and associated constraint $\psi_0$ on $M$. Let $D = \mathsf{BestPaths}(t_0, v_0, u_0, \psi_0)$. Then,*

- *for all $0 \le i \le |V|$, for all $v_1 \in V$, for all pairs $\langle u, \psi \rangle \in D[t_0 + i, v_1]$, there exists a path $\rho_1$ of length $i$ with $\mathsf{start}(\rho_1) = v_0$ and $\mathsf{end}(\rho_1) = v_1$, such that*
  - *$u$ is the sum of weights of the path $\rho_\sharp \cdot \rho_1$, and*

**Algorithm 1** BestPaths$(t_0, v_0, u_0, \psi_0)$

**Input** : $t_0 \in \mathbb{N}$ is an initial time point, $v_0$ is an initial vertex, $u_0$ is the initial sum of weights, and $\psi_0$ is the initial constraint on the slope parameter $M$.

**Output**: The table of $\succeq$-maximal values of paths from $v_0$ with initial values $t_0, u_0, \psi_0$.

**begin**

`/* initialization */`

1 $D[t_0, v_0] \leftarrow \{\langle u_0, \psi_0\rangle\}$

2 **for** $v \in V \setminus \{v_0\}$ **do**

3     $D[t_0, v] \leftarrow \varnothing$

`/* iterations */`

4 **for** $i = 1, \dots, |V|$ **do**

5     **for** $v \in V$ **do**

6         $D[t_0 + i, v] \leftarrow \varnothing$

7         **for** $v_1 \in V$ *and* $\langle u_1, \psi_1\rangle \in D[t_0 + i - 1, v_1]$ **do**

8             **if** $(v_1, v) \in E$ **then**

9                 $u \leftarrow u_1 + w(v_1, v)$

10                 $t \leftarrow t_0 + i - 1$

11                 $\psi \leftarrow \psi_1 \wedge (u \geq M \cdot (t - T))$

12                 $D[t_0 + i, v] \leftarrow D[t_0 + i, v] \cup \{\langle u, \psi\rangle\}$

13         $D[t_0 + i, v] \leftarrow \lceil D[t_0 + i, v]\rceil$

14 **return** $D$

**end**

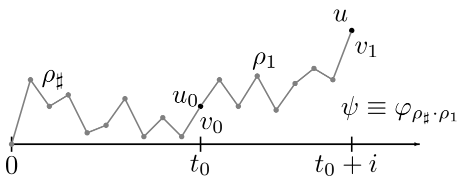
- $\psi \equiv \varphi_{\rho_\sharp \cdot \rho_1}$ *is the constraint on* $M$ *associated with the path* $\rho_\sharp \cdot \rho_1$*;*

- *for all paths* $\rho_1$ *of length* $i \leq |V|$ *such that* $\mathsf{start}(\rho_1) = v_0$ *and* $\mathsf{end}(\rho_1) = v_1$*, there exists a pair* $\langle u', \psi'\rangle \in D[t_0 + i, v_1]$ *such that* $\langle u', \psi'\rangle \succeq \langle u, \psi\rangle$ *where*
  - $u$ *is the sum of weights of the path* $\rho_\sharp \cdot \rho_1$*, and*
  - $\psi \equiv \varphi_{\rho_\sharp \cdot \rho_1}$ *is the constraint on* $M$ *associated with the path* $\rho_\sharp \cdot \rho_1$*.*

Proof. For the first item, the proof is by induction on $i$. The case $i = 0$ holds since $D[t_0, v_1]$ is nonempty only for $v_1 = v_0$ (lines 1-3 of Algorithm 1), and we can take for $\rho_1$ the empty path since then $D[t_0, v_0] = \{\langle u_0, \psi_0\rangle\}$ contains the pair associated with $\rho_\sharp = \rho_\sharp \cdot \rho_1$.

For the inductive case, consider length $i \geq 1$ and assume that the result holds for length $i-1$. Then for all pairs $\langle u_1, \psi_1\rangle \in D[t_0+i-1, v_1]$ where $v_1 \in V$ (see also line 7 of Algorithm 1), there exists a path $\rho_1$ of length $i - 1$ such that $\langle u_1, \psi_1\rangle$ is the pair associated with $\rho_\sharp \cdot \rho_1$. It is easy to see that the pair $\langle u, \psi\rangle$ added to $D[t_0+i, v]$ at line 12 of Algorithm 1 is associated with the path $\rho_\sharp \cdot \rho_1 \cdot (v_1, v)$ where $u = u_1 + w(v_1, v)$ and $\psi \equiv \psi_1 \wedge (u \geq M \cdot (t - T))$ with $t = t_0 + i - 1 = |\rho_\sharp \cdot \rho_1 \cdot (v_1, v)| - 1$. Since the assignment at line 13 of Algorithm 1 can only remove pairs from $D[t_0 + i, v]$, the result follows.

For the second item, the result follows from similar arguments as above, a proof by induction on $i$ using Lemma 10, and the fact that the algorithm explores all successors $v$ of each vertex $v_1$ that ends a path associated with a pair $\langle u_1, \psi_1\rangle \in D[t_0 + i - 1, v_1]$. $\square$




$\langle u, \psi \rangle \in D[t_0 + i, v_1]$
where $D = \mathsf{BestPaths}(t_0, v_0, u_0, \psi_0)$

Fig. 9. The result of the computation of $\mathsf{BestPaths}(t_0, v_0, u_0, \psi_0)$.

As we know that simple lassos are sufficient for optimal value (Lemma 8), our algorithmic solution is to explore finite paths from the initial vertex, until a loop is formed. Thus it is sufficient to explore paths of length at most $|V|$. However, given a simple lasso $\rho_A \cdot \rho_C^\omega$, it is not sufficient that the finite path $\rho_A \cdot \rho_C$ lies above a line $M \cdot (t - T)$ (where $M$ satisfies the constraint $\psi_{AC}$ associated with $\rho_A \cdot \rho_C$) to ensure that the value of the lasso $\rho_A \cdot \rho_C^\omega$ is at least 0. The reason is that by repeating the cycle $\rho_C$ several times, the path may eventually cross the line $M \cdot (t - T)$. We show (in Lemma 12) that this cannot happen if the average weight $M_C$ of the cycle is greater than the slope of the line (i.e., $M_C \geq M$).

Lemma 12. *Given a lasso $\rho_A \cdot \rho_C^\omega$, let $\psi_{AC}$ be the symbolic constraint on $M$ associated with the finite path $\rho_A \cdot \rho_C$, and let $M_C$ be the average weight of the cycle $\rho_C$. The lasso $\rho_A \cdot \rho_C^\omega$ has value at least 0 if and only if the formula $\psi_{AC} \wedge (M_C \geq M)$ is satisfiable.*

Proof. First, if the lasso $\rho_A \cdot \rho_C^\omega$ has value at least 0, then by Lemma 9, there exists a slope $M \in \mathbb{R}$ such that $u_t \geq M \cdot (t - T)$ for all $t \geq 0$ (where $u_t$ is the sum of weights at time $t$ in $\rho_A \cdot \rho_C^\omega$). For such value of $M$, the formula $\psi_{AC}$ holds (by definition), and it is easy to see that $M_C \geq M$ (otherwise, there would exist $t \geq 0$ such that $u_t < M \cdot (t - T)$). Therefore $\psi_{AC} \wedge (M_C \geq M)$ is satisfiable.

Second, if the formula $\psi_{AC} \wedge (M_C \geq M)$ is satisfiable, then let $M$ be a satisfying value, and by Lemma 9 and a similar argument as above, the lasso $\rho_A \cdot \rho_C^\omega$ has value at least 0. $\square$

The algorithm $\mathsf{ExistsPositivePath}(v_0)$ explores the paths from $v_0$, and keeps the $\succeq$-preferred paths, that is those with the largest total weight and weakest constraint on $M$. There may be several $\succeq$-incomparable paths of a given length $i$ that reach a given vertex $\hat{v}$, therefore we need to compute a *set* $A[i, \hat{v}]$ of $\succeq$-incomparable pairs (line 1 of Algorithm 2).

Given a pair $\langle u_1, \psi_1 \rangle \in A[i, \hat{v}]$, the algorithm $\mathsf{ExistsPositivePath}$ further explores (for-loop at line 3 of Algorithm 2) the paths from $\hat{v}$, until a cycle $\rho_C$ of length $j$ is formed around $\hat{v}$, with average weight $M_C = \frac{u_2 - u_1}{j}$ and associated pair $\langle u_2, \psi_2 \rangle \in C[i + j, \hat{v}]$ (line 7 of Algorithm 2) such that $\psi_2 \wedge (M_C \geq M)$ is satisfiable. We claim that there exists such a cycle if and only if there exists a lasso with value at least 0. The claim is established in the following lemma.

Lemma 13 (Correctness of $\mathsf{ExistsPositivePath}$). *There exists an infinite path from $v_0$ with value at least 0 if and only if $\mathsf{ExistsPositivePath}(v_0)$ returns* $\mathsf{true}$.

Proof. **(First part)**

For the first direction of the proof, if there exists an infinite path with value at least 0, then by Lemma 8 there exists a lasso $\rho = \rho_A \cdot \rho_C^\omega$ with value at least 0.

Consider the call $A \leftarrow \mathsf{BestPaths}(t_0, v_0, u_0, \psi_0)$ in $\mathsf{ExistsPositivePath}$ (line 1 of Algorithm 2) where $t_0 = u_0 = 0$ and $\psi_0 \equiv \mathsf{true}$. Let $\hat{v} = \mathsf{end}(\rho_A)$ and let $i$ be the length of $\rho_A$ (note that

**Algorithm 2** $\mathsf{ExistsPositivePath}(v_0)$

**Input** : $v_0$ is an initial vertex.
**Output**: $\mathsf{true}$ iff there exists a path from $v_0$ with expected utility at least 0.
**begin**

1 $A \leftarrow \mathsf{BestPaths}(0, v_0, 0, \mathsf{true})$
2 **for** $i = 0, \dots, |V|$ **do**
3   **for** $\hat{v} \in V$ *and* $\langle u_1, \psi_1 \rangle \in A[i, \hat{v}]$ **do**
4     $C \leftarrow \mathsf{BestPaths}(i, \hat{v}, u_1, \psi_1)$
5     **for** $j = 1, \dots, |V| - i$ **do**
6       **for** $\langle u_2, \psi_2 \rangle \in C[i + j, \hat{v}]$ **do**
7         **if** $\psi_2 \wedge \frac{u_2 - u_1}{j} \geq M$ *is satisfiable* **then return** $\mathsf{true}$
8 **return** $\mathsf{false}$

**end**

$i < |V|$ because $\rho_A$ is acyclic). By the correctness result of $\mathsf{BestPaths}$ (Lemma 11 (item 2), where $\rho_\sharp$ is the empty path), there is a pair $\langle u_1, \psi_1 \rangle \in A[i, \hat{v}]$ such that $\langle u_1, \psi_1 \rangle \succeq \langle u_A, \psi_A \rangle$ where $\langle u_A, \psi_A \rangle$ is the pair associated with $\rho_A$, thus $u_1 \geq u_A$ and $\psi_A \rightarrow \psi_1$ hold. Then by Lemma 11 (item 1), there is a path $\rho_1$ of length $i$ from $v_0$ to $\hat{v}$, and $u_1$ is the sum of weights of $\rho_1$, and $\psi_1 \equiv \varphi_{\rho_1}$ is the constraint on $M$ associated with $\rho_1$ (i.e., $\rho_1$ is preferred to $\rho_A$).

Now consider the call $C \leftarrow \mathsf{BestPaths}(i, \hat{v}, u_1, \psi_1)$ in $\mathsf{ExistsPositivePath}$ (line 4 of Algorithm 2). Let $\rho_\sharp = \rho_1$ in Lemma 11 and note that the assumptions of that lemma are satisfied, namely $\langle u_1, \psi_1 \rangle$ is the pair associated with $\rho_1$, and $\hat{v} = \mathsf{end}(\rho_1)$.

Since $\rho_A \cdot \rho_C^\omega$ is a lasso, we have $\mathsf{start}(\rho_C) = \mathsf{end}(\rho_C) = \mathsf{end}(\rho_A) = \hat{v}$ and let $j$ be the length of $\rho_C$ (note that $i + j \leq |V|$). By Lemma 11 (item 2), there is a pair $\langle u_2, \psi_2 \rangle \in C[i + j, \hat{v}]$ such that $\langle u_2, \psi_2 \rangle \succeq \langle u_{1C}, \psi_{1C} \rangle$ where $\langle u_{1C}, \psi_{1C} \rangle$ is the pair associated with $\rho_1 \cdot \rho_C$, thus $u_2 \geq u_{1C}$ and $\psi_{1C} \rightarrow \psi_2$ hold, and by Lemma 11 (item 1), there is a path $\rho_2$ of length $j$ such that $\mathsf{start}(\rho_2) = \mathsf{end}(\rho_2) = \hat{v}$ and $u_2$ is the sum of weights of $\rho_1 \cdot \rho_2$, and $\psi_2 \equiv \varphi_{\rho_1 \cdot \rho_2}$ is the constraint on $M$ associated with $\rho_1 \cdot \rho_2$.

Now we show that $\psi_2 \wedge \frac{u_2 - u_1}{j} \geq M$ is satisfiable, and thus $\mathsf{ExistsPositivePath}(v_0)$ returns $\mathsf{true}$ (Line 7 of Algorithm 2). First, by Lemma 12 the formula $\psi_{AC} \wedge (M_C \geq M)$ is satisfiable, and by Lemma 10 we have $\psi_{AC} \rightarrow \psi_{1C}$. We showed above that $\psi_{1C} \rightarrow \psi_2$, thus $\psi_2 \wedge (M_C \geq M)$ is satisfiable. Now, since the length of the cycle $\rho_C$ (and of $\rho_2$) is $j - i$ (i.e., the length of $\rho_A \cdot \rho_C$ minus the length of $\rho_A$), we have $M_C = \frac{S_C}{j}$. Moreover we showed above that $u_2 \geq u_{1C} = u_1 + S_C$, thus $M_C = \frac{S_C}{j} \leq \frac{u_2 - u_1}{j}$, and since $\psi_2 \wedge (M_C \geq M)$ is satisfiable it follows that $\psi_2 \wedge \frac{u_2 - u_1}{j} \geq M$ is satisfiable as well.

**(Second part)**

For the second direction of the proof, if $\mathsf{ExistsPositivePath}(v_0)$ returns $\mathsf{true}$, then there exists $i, j, \hat{v}, \langle u_1, \psi_1 \rangle, \langle u_2, \psi_2 \rangle$ (corresponding to the for-loops in lines 2, 3, 5, 6 of Algorithm 2) such that:

- $0 \leq i \leq |V|$ and $1 \leq j \leq |V| - i$,
- $\hat{v} \in V$,

- $\langle u_1, \psi_1 \rangle \in A[i, \hat{v}]$ and $\langle u_2, \psi_2 \rangle \in C[i+j, \hat{v}]$ where $A = \mathsf{BestPaths}(0, v_0, 0, \mathsf{true})$, and $C = \mathsf{BestPaths}(i, \hat{v}, u_1, \psi_1)$,
- $\psi_2 \wedge \frac{u_2-u_1}{j} \geq M$ is satisfiable.

Therefore, by Lemma 11 (item 1), there exist paths $\rho_A$ and $\rho_C$ such that:

- $\rho_A$ is a path of length $i$ from $v_0$ to $\hat{v}$, such that $u_1$ is the sum of weights of the path $\rho_A$, and $\psi_1 \equiv \varphi_{\rho_A}$;
- $\rho_C$ is a path of length $j$ with $\mathsf{start}(\rho_C) = \mathsf{end}(\rho_C) = \hat{v}$ (thus $\rho_C$ is a cycle), such that $u_2$ is the sum of weights of the path $\rho_A \cdot \rho_C$, and $\psi_2 \equiv \varphi_{\rho_A \cdot \rho_C}$ is the constraint on $M$ associated with the path $\rho_A \cdot \rho_C$.

Therefore, $u_2 - u_1$ is the sum of the weights along $\rho_C$, and thus $M_C = \frac{u_2-u_1}{j}$. Since the formula $\psi_2 \wedge \frac{u_2-u_1}{j} \geq M$ is satisfiable, it follows that $\varphi_{\rho_A \cdot \rho_C} \wedge (M_C \geq M)$ is satisfiable, and by Lemma 12, the lasso $\rho_A \cdot \rho_C^\omega$ has value at least 0. □

*Optimal value.* We can compute the optimal value using the procedure ExistsPositivePath as follows. From Lemma 4, the optimal value is either of the form $\frac{u_{t_1} \cdot (t_2 - T) + u_{t_2} \cdot (T - t_1)}{t_2 - t_1}$, or of the form $u_{t_1} + (T - t_1) \cdot \nu$ where the following bounds hold ($\nu = \inf_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$):

- $0 \leq t_1 \leq t_2 \leq |V|$
- $0 \leq t_2 - t_1 \leq |V|$
- $0 \leq T - t_1 \leq |V|$
- $0 \leq t_2 - T \leq |V|$
- $-W \cdot |V| \leq u_{t_1}, u_{t_2} \leq W \cdot |V|$
- $\nu$ is a rational number $\frac{p}{q}$ where $-W \cdot |V| \leq p \leq W \cdot |V|$ and $1 \leq q \leq |V|$

Therefore, in both cases we get the following result.

LEMMA 14. *The optimal value belongs to the set*

$$\mathsf{ValueSpace} = \left\{ \frac{p}{q} \mid -2W \cdot |V|^2 \leq p \leq 2W \cdot |V|^2 \text{ and } 1 \leq q \leq |V| \right\}.$$

Given a value $\frac{p}{q}$, we can decide if there exists a path with expected value at least $\frac{p}{q}$ by subtracting $\eta = \frac{p}{q \cdot T}$ from all the weights the graphs, and asking if there exists a path with expected value at least 0 in the modified graph. Indeed, if we define $w'(e) = w(e) + \eta$ for all edges $e \in E$, then for all paths $\rho$, if $u$ is the sequence of utilities along $\rho$ according to $w$, and $u'$ is the sequence of utilities along $\rho$ according to $w'$, then

$$\begin{aligned} \sum_i p_i \cdot u'_i &= \sum_i p_i \cdot (u_i + \eta \cdot i) \\ &= \eta \cdot \sum_i p_i \cdot i + \sum_i p_i \cdot u_i \\ &= T \cdot \eta + \sum_i p_i \cdot u_i, \end{aligned}$$

thus the value of the path is shifted by $T \cdot \eta = \frac{p}{q}$. Then it follows from Lemma 14 that the optimal value can be computed by a binary search using $O(|\mathsf{ValueSpace}|) = O(\log(W \cdot |V|))$ calls to ExistsPositivePath.

*Optimal path.* An optimal path can be constructed by a slight modification of the algorithm. In BestPaths, we can maintain a path associated to each pair in $D$ as follows: the empty path is associated to the pair $\langle u_0, \psi_0 \rangle$ added at line 1 of Algorithm 1, and given the path $\rho_1$ associated with the pair $\langle u_1, \psi_1 \rangle$ (line 7 of Algorithm 1), we associate the path $\rho_1 \cdot (v_1, v)$ with the pair $\langle u, \psi \rangle$ added to $D$ at line 12 of Algorithm 1. It is easy to see that for every pair $\langle u, \psi \rangle$ in $D$, the associated path can be used as the path $\rho_1$ in Lemma 11 (item 1). Therefore, when ExistsPositivePath$(v_0)$ returns true (line 7 of Algorithm 2), we can output the path $\rho_1 \cdot \rho_2^\omega$ where $\rho_i$ is the path associated with the pair $\langle u_i, \psi_i \rangle$ $(i = 1, 2)$.

*Complexity analysis.* We present the running-time analysis of ExistsPositivePath (Algorithm 2) and we show that it runs in polynomial time. The key challenge is to bound the number of $\succeq$-incomparable pairs. The number of such pairs corresponds to the number of simple paths in a graph, and hence can be exponential in general. Our main argument is to establish a polynomial bound on the number of $\succeq$-incomparable pairs.

To analyze the complexity of the algorithm, we need to bound the size of the array $D$ computed by BestPaths (Algorithm 1). We show that there cannot be too many different pairs in a given entry $D[t_0 + i, v_1]$. By Lemma 11, to each pair $\langle u, \psi \rangle \in D[t_0 + i, v_1]$ we can associate a path $\rho$ of length $i$ with $\mathsf{start}(\rho) = v_0$ and $\mathsf{end}(\rho) = v_1$, such that (our analysis holds for all paths $\rho_\sharp$ in Lemma 11, and as $\rho_\sharp$ plays no role in the argument, we proceed with empty $\rho_\sharp$ for simplicity of the exposition[4]):

- $u$ is the sum of weights of the path $\rho$, and
- $\psi \equiv \varphi_\rho$ is the constraint on $M$ associated with the path $\rho$.

It is important to note that the constraint $\psi$ is determined by (at most) two points $t_L, t_R$ in $\rho$ (see also Figure 8a and Figure 8b), one before $T$ and one after $T$, namely

$$\psi \equiv \big(u_{t_L} \geq M \cdot (t_L - T)\big) \wedge \big(u_{t_R} \geq M \cdot (t_R - T)\big)$$

where $t_L = \mathrm{argmax}_{0 \leq i \leq T}(\frac{u_i}{i-T})$ and $t_R = \mathrm{argmin}_{T \leq i \leq |\rho|}(\frac{u_i}{i-T})$.

Note that the first constraint in the above expression is a lower bound on $M$ since $t_L \leq T$, and the second constraint (which may not exist, if $|\rho| < T$) is an upper bound on $M$. For simplicity of exposition, we assume that $|\rho| \geq T$. The case $|\rho| < T$ is handled analogously ($t_R$ is undefined in that case).

Define the *down-point* of $\rho = e_0 e_1 \dots e_{|\rho|-1}$ as $\mathsf{downpoint}(\rho) = \langle t_L, v_L, t_R, v_R \rangle$ where $t_L$ and $t_R$ are defined above, and $v_L = \mathsf{end}(e_0 e_1 \dots e_{t_L})$, and $v_R = \mathsf{end}(e_0 e_1 \dots e_{t_R})$ (for $|\rho| < T$, the down-point of $\rho$ is $\mathsf{downpoint}(\rho) = \langle t_L, v_L \rangle$).

Decompose $\rho$ into $\rho_L = e_0 e_1 \dots e_{t_L}$, $\rho_M = e_{t_L+1} e_{t_L+2} \dots e_{t_R}$, and $\rho_R = e_{t_R+1} e_{t_R+2} \dots e_{|\rho|-1}$. We claim that the paths corresponding to two different pairs in $D[t_0 + i, v_1]$ have different down-points, which will give us a polynomial bound on the size of $D[t_0 + i, v_1]$. Intuitively, and towards contradiction, if two down-points are the same in two different paths, then we can select the best pieces among $(\rho_L, \rho_M, \rho_R)$ from the two paths and construct a path that is preferred, and thus whose pair is in $D[t_0 + i, v_1]$ and subsumes some pair in $D[t_0 + i, v_1]$, which is a contradiction since the elements of $D[t_0 + i, v_1]$ are $\succeq$-maximal.

LEMMA 15. *Let $D = \mathsf{BestPaths}(t_0, v_0, u_0, \psi_0)$ and $1 \leq i \leq |V|$. For all pairs $\langle u, \psi \rangle, \langle u', \psi' \rangle \in D[t_0 + i, v_1]$, let $\rho, \rho'$ be their respective associated path; if $\langle u, \psi \rangle \neq \langle u', \psi' \rangle$, then the down-points of $\rho$ and $\rho'$ are different (*$\mathsf{downpoint}(\rho) \neq \mathsf{downpoint}(\rho')$*).*

[4]The proof can be carried out analogously by considering $\rho_\sharp \cdot \rho$ instead of $\rho$ with heavier notation.


(a) Edge $v_0 \to v_1$ is taken too early (the sup-value is at most $0$).

(b) Edge $v_0 \to v_1$ is taken too late (the sup-value is at most $\frac{T}{T+1} < 1$).

Fig. 10. The optimal sup-value requires memory in the example of Fig. 2 (the optimal sup-value is $1$).

Proof. We prove the contrapositive, for $|\rho| \geq T$ (the case $|\rho| < T$ is simpler, and proved analogously). Assume that $\langle t_L, v_L, t_R, v_R \rangle = \langle t'_L, v'_L, t'_R, v'_R \rangle$ (the down-points are equal), and we show that then $\langle u, \psi \rangle = \langle u', \psi' \rangle$.

First, since $t_L = t'_L$ and $v_L = v'_L$, we claim that the sum of weights at time $t_L$ is the same in $\rho$ and in $\rho'$, that is $u_{t_L} = u'_{t_L}$, and therefore, $\varphi_{\rho_L} \equiv \varphi_{\rho'_L}$ (remember that the constraint $\psi$ associated with $\rho$ and $\rho'$ is determined by $t_L = t'_L$). The proof of this claim is by contradiction. Assume that $u_{t_L} > u'_{t_L}$ (the argument for the case $u_{t_L} < u'_{t_L}$ is analogous). Consider the path $\overline{\rho} = \rho_L \cdot \rho'_M \cdot \rho'_R$, and note that $\overline{\rho}$ is indeed a path[5], as $\mathsf{end}(\rho_L) = v_L = v'_L = \mathsf{start}(\rho'_M)$. Comparing $\overline{\rho}$ and $\rho'$, since $u_{t_L} > u'_{t_L}$ it is easy to see that $\bar{u} > u'$ where $\bar{u}$ is the sum of weights of $\overline{\rho}$, and by the same argument we have $\psi' \to \psi_{\overline{\rho}}$. It follows that $\overline{\rho}$ is preferred to $\rho'$, and by Lemma 11 the set $D[t_0 + i, v_1]$ contains a pair $\langle u^*, \psi^* \rangle \succeq \langle \bar{u}, \varphi_{\overline{\rho}} \rangle \succeq \langle u', \psi' \rangle$. Since $D[t_0 + i, v_1]$ is a set of $\succeq$-maximal elements (line 13 of Algorithm 1), it follows that $\langle u', \psi' \rangle \notin D[t_0 + i, v_1]$, in contradiction with the assumption of the lemma.

Second, by an analogous argument, since $t_R = t'_R$ and $v_R = v'_R$, the sum of weights at time $t_R$ is the same in $\rho$ and in $\rho'$, that is $u_{t_R} = u'_{t_R}$, and therefore, $\varphi_{\rho_R} \equiv \varphi_{\rho'_R}$. Finally $u = u'$ and $\psi \equiv \psi'$, which concludes the proof. ☐

It follows from Lemma 15 that the size of all sets $D[t_0 + i, v_1]$ for $1 \leq i \leq |V|$ and $v_1 \in V$ is at most $|V|^4$, the maximum number of different down-points.

We now show that the worst-case complexity of BestPaths and ExistsPositivePath is polynomial, and thus the optimal expected value problem is solvable in polynomial time.

The worst-case complexity of BestPaths is $O(|V|^{10})$, as there are two nested for-loops over $V$ (line 4 and line 5 in Algorithm 1), in which the dominating operation is the computation of the $\succeq$-maximal elements of $D[t_0+i, v]$ (line 13), which is quadratic in the size of $D[t_0+i, v]$, thus in $O(|V|^8)$.

The worst-case complexity of ExistsPositivePath is $O(|V| \cdot |V| \cdot |V|^4 \cdot |V|^{10}) = O(|V|^{16})$, as a product of the size of the three outermost for-loops, and the dominating call to BestPaths (line 4) in $O(|V|^{10})$. Therefore we obtain Theorem 4.

## 5 EXPECTED FINITE-HORIZON: BEST-CASE DISTRIBUTION

We now consider the problem of maximizing the value of a plan where the value of a plan is computed as the supremum value (instead of the infimum value) over all distributions with

[5]Note that if $\rho$ and $\rho'$ have a common prefix (such as $\rho_\sharp$), then $\overline{\rho}$ also has the same prefix.

expected stopping time $T$. The optimization problem is thus to choose a path as well as a stopping-time distribution in order to maximize the value.

Given a weighted graph $G$ and an expected stopping time $T \in \mathbb{Q}$, we define the following:

- *Optimal sup-value of plans.* For a plan $\rho$ that induces the sequence $u$ of utilities, let
$$val_{\sup}(\rho, T) = val_{\sup}(u, T) = \sup_{\delta \in \Delta : \mathbb{E}_\delta = T} \mathbb{E}_\delta(u).$$
- *Optimal sup-value.* The optimal sup-value is the supremum value over all plans:
$$val_{\sup}(G, T) = \sup_{u \in U_G} val_{\sup}(u, T).$$

Since the distribution is chosen by the maximizer and there is no adversary, the optimal sup-value is at least as large as the optimal (inf-)value defined in Section 4. However, while stationary plans suffice against adversarially chosen distributions, it turns out that optimal plans for the sup-value are in general not stationary (i.e., memory is necessary for optimality).

EXAMPLE 3. *Consider the example of Figure 2 described in Example 2 where $T = 31$. If the edge $(v_0, v_1)$ is used exactly at time $T$ (which requires memory as shown in Example 2), then the sup-value of the path is 1 by choosing $\delta = \{(T, 1)\}$. We show that if the edge $(v_0, v_1)$ is not used exactly at time $T$, then the sup-value is less than 1, and therefore the optimal value is 1 and requires memory. Figure 10a and Figure 10b illustrate the situation when the edge $(v_0, v_1)$ is used before time $T$ or after time $T$. In both cases, the sup-value is less than 1, and if the edge $(v_0, v_1)$ is never used, then the value is 0, thus also less than 1.*

In Example 3, the memory is used before time $T$ to get the peak of utility positioned optimally with respect to $T$. However, we show that after time $T$ memory is no longer necessary. A plan $\rho = e_0 e_1 \dots$ is *stationary after* $T$ if for all $T \leq t_1 < t_2$, if $e_{t_1} = (\cdot, v)$ and $e_{t_2} = (\cdot, v)$, then $e_{t_1+1} = e_{t_2+1}$. We denote by $S_G^{\geq T}$ the set of all sequences of utilities induced by plans in $G$ that are stationary after $T$.

THEOREM 5. *For all weighted graphs $G$ and for all $T$ we have*
$$val_{\sup}(G, T) = \sup_{u \in U_G} val_{\sup}(u, T) = \sup_{u \in S_G^{\geq T}} val_{\sup}(u, T),$$
*i.e., optimal stationary-after-$T$ plans exist for expected finite-horizon under best-case distribution.*

PROOF. By Lemma 2, bi-Dirac distributions are sufficient for optimality (the lemma is stated for inf, but it holds for sup as well by considering the graph with all weights multiplied by $-1$). The geometric interpretation of Lemma 3 and Lemma 4 can be adapted to the sup-value by defining, given a sequence $u$ of utilities, the *optimal line* of equation $f_u(t) = u_{t_1} + (t - t_1) \cdot \nu$ where $t_1$ is a *left-maximizer* (defined analogously to left-minimizers) and $\nu = \sup_{t_2 \geq T} \frac{u_{t_2} - u_{t_1}}{t_2 - t_1}$. The sequence $u$ always lies under the optimal line (i.e., $u_t \leq f_u(t)$ for all $t \geq 0$), and the optimal sup-value of $u$ is $f(T)$.

The argument for the proof of Theorem 5 follows the same line as the proof of Theorem 3, namely to construct, given an arbitrary plan, a plan that is stationary after $T$ and has at least the sup-value of the given plan. This construction proceeds by considering the cycle decomposition of the suffix $e_T e_{T+1} \dots$ of the given plan, and given the first cycle $C$ in the cycle decomposition:

- either $f_u(t) = u_t$ for some $t$ in the cycle (hence $t \leq T + |V|$), and then repeating the cycle $C$ gives a plan that is stationary after $T$ and has better (or equal) sup-value,

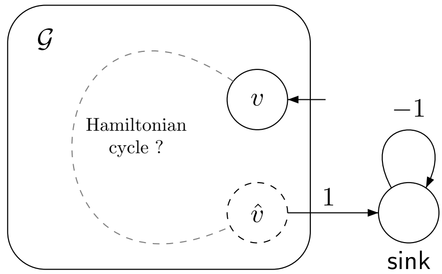



Fig. 11. The NP-hardness reduction of Theorem 6.

- or $f_u(t) > u_t$ for all $t$ in the cycle, and then we can either $(i)$ remove the cycle $C$ if $M_C < \nu$, or $(ii)$ repeat the cycle $C$ forever if $M_C \geq \nu$. Analogous analysis as in Lemma 6 and Lemma 7 shows that the resulting plan has better (or equal) sup-value, and the analysis in the proof Lemma 5 and Lemma 8 shows that a cycle $C$ with $M_C \geq \nu$ exists in $u$, and thus we eventually get by this procedure a plan that is stationary after $T$ and has better (or equal) sup-value than the given plan. □

It follows from Theorem 5 that an optimal plan always exists under best-case distribution (since there are finitely many stationary-after-$T$ plans).

We show that computing optimal plans among *stationary* plans cannot be done in polynomial time unless P = NP. In contrast, the optimal sup-value for arbitrary paths and best-case distribution can be computed in polynomial time.

THEOREM 6. *Given a weighted graph $G$, an integer $T$, and a threshold $\lambda \in \mathbb{Q}$, deciding whether $\sup_{u \in S_G} val_{\sup}(u, T)$ is at least $\lambda$ is NP-complete. The NP-hardness holds for $T$ and all weights expressed in unary.*

PROOF. The NP upper bound is easily obtained by guessing a stationary plan (i.e., one edge for each vertex of the graph) and checking that the value of the induced path is at least $\lambda$.

The NP-hardness is obtained by a reduction from the Hamiltonian cycle problem, which is to decide, given a directed graph $\mathcal{G} = \langle V, E \rangle$, whether $\mathcal{G}$ contains a simple cycle of length $|V|$. The reduction is as follows. Given $\mathcal{G}$, pick a vertex $v \in V$ and create a copy $\hat{v}$ of $v$ with the same incoming neighbors as $v$. Add an edge $(\hat{v}, \mathsf{sink})$ with weight 1 and a self-loop on sink with weight $-1$. All other edges have weight 0. Let $v$ be the initial vertex, and let $T = |V| + 1$ and $\lambda = 1$. The correctness of the reduction is established as follows. If $\mathcal{G}$ contains a Hamiltonian cycle, then from $v$, there is a stationary path to $\hat{v}$ of length $|V|$ that can be extended to sink. The value of the path at time $T$ is 1, and thus the sup-optimal value is at least 1. On the other hand, if $\mathcal{G}$ does not contain a Hamiltonian cycle, then all stationary paths from $v$ to sink have length at most $n - 1$, hence the sup-optimal value is less than 1 (corresponding to the situation in Figure 10a). □

The following basic lemma is useful to construct an optimal plan.

LEMMA 16. *Given a finite path $\rho$ from vertex $v_0$ to vertex $v_1$ of length $\ell$, with sum of weights $u_1$, there exists a path $\rho'$ from $v_0$ to $v_1$ of the same length $\ell$, with sum of weights at least $u_1$, and of the shape $\rho' = AC^xB$ where $C$ is a simple cycle ($x \geq 0$), and the length of $AB$ is at most $|V|^3$.*

Proof. Given a path $\rho$ as in the lemma, if $\ell \leq |V|^3$ we take $A = \rho$, $x = 0$, and $B = \epsilon$. Otherwise, $\ell > |V|^3$ and consider a cycle $C$ in $\rho$ with maximal average value $M_C$. If $\rho$ contains several occurrences of $C$ (say $\rho = A_1 C^x A_2 C^y A_3$), we group them and construct the path $A_1 C^{x+y} A_2 A_3$. We iterate this process until we get a path of the shape $A_1 C^x A_2$ where $C$ does not occur in the cycle decomposition of the path $A_1 A_2$. Note that this new path has the same length and same sum of weights as $\rho$.

In the cycle decomposition of the path $A_1 A_2$, consider for each length $1 \leq k \leq |V|$ the number $n_k$ of cycles of length $k$. If $|A_1 A_2| > |V|^3$, then $n_k \geq |V|$ for some $k$ (indeed a path with $n_k \leq |V|-1$ for all $k$ has length at most $(|V|-1) \cdot \frac{|V|^2+|V|}{2} + |V| - 1 < |V|^3$). Consider such $k$, and let $m = |C|$ the length of $C$. In the path $A_1 A_2$, remove $m$ cycles of length $k$ (note that this is possible since $m \leq |V| \leq n_k$), and repeat $k$ more times the cycle $C$, to obtain a path $AC^{x+k}B$ of the same length as $A_1 C^x A_2$, and with at least the same sum of weights, since $C$ has the largest average value among the cycles in $A_1 A_2$. Repeat this construction until $|AB| \leq |V|^3$ to conclude the proof. ☐

The optimal sup-value for arbitrary paths and best-case distribution can be computed in polynomial time as follows. We consider two cases depending on whether an optimal distribution exists. In both cases, we show that the range of possible values for the left-maximizer, which is a priori the interval $[0, T]$ and thus contains a pseudo-polynomial number of values (namely, $O(T)$), can be restrained to a small (polynomial) number of values.

- If no optimal distribution exists, let $0 \leq t_1 \leq T$ be a left-maximizer and let $\nu$ be the slope of the optimal line. We show that either $t_1 \leq |V|$, or $t_1 \geq T - |V|$ (which gives a range of values for $t_1$ of size $O(|V|)$). Consider an optimal plan $\rho$ and its left-maximizer $t_1$. If $|V| < t_1 < T - |V|$, we construct another plan $\rho'$ with sup-value at least the sup-value of $\rho$, and with left-maximizer $t_1'$ either $t_1' \leq |V|$, or $t_1' \geq T - |V|$.
  Since $|V| < t_1$, in $\rho$ there exists a cycle $C$ before $t_1$. Let $S_C$ be the sum of weights along $C$, and let $|C|$ be the length of $C$. The mean value of $C$ is $M_C = \frac{S_C}{|C|}$.
  - If $M_C \geq \nu$, then consider the path $\rho'$ obtained from $\rho$ by repeating the cycle $C$ once more, and let $t_1' = t_1 + |C|$. Note that $t_1' \leq T$, thus the sup-value of $\rho'$ is at least (assuming $u$ and $u'$ are the sequences of utilities induced by $\rho$ and $\rho'$, respectively):
  $$\begin{aligned} & u'_{t_1'} + \nu \cdot (T - t_1') \\ = & u_{t_1} + S_C + \nu \cdot (T - (t_1 + |C|)) \\ = & u_{t_1} + \nu \cdot (T - t_1) + S_C - \nu \cdot |C| \\ = & u_{t_1} + \nu \cdot (T - t_1) + (M_C - \nu) \cdot |C| \\ \geq & u_{t_1} + \nu \cdot (T - t_1) \quad \text{since } M_C \geq \nu \end{aligned}$$
    which is the sup-value of $\rho$.
  - The case $M_C < \nu$ is impossible because removing the cycle $C$ from $\rho$ would then give a better plan than $\rho$ (which we assumed to be optimal): consider $t_1' = t_1 - |C|$ (note that $t_1' \geq 0$) and we have
  $$\begin{aligned} & u'_{t_1'} + \nu \cdot (T - t_1') \\ = & u_{t_1} - S_C + \nu \cdot (T - (t_1 - |C|)) \\ = & u_{t_1} + \nu \cdot (T - t_1) - (M_C - \nu) \cdot |C| \\ > & u_{t_1} + \nu \cdot (T - t_1) \quad \text{since } M_C < \nu \end{aligned}$$

Consider the algorithm that enumerates the possible values of $t_1$ (in $[0,|V|]\cup[T-|V|,T]$), and of the vertex $v_1$ at time $t_1$, then computes the sum $u_1$ of weights of the best path to $v_1$ of length $t_1$, and mean value $M_C$ of the best cycle reachable from $v_1$ (which can be done in polynomial time, see Section 3 and [12]). Store $t_1$ and $v_1$ that gives the largest value of $u_1+(T-t_1)\cdot M_C$. Call $u^*$ this value.

- If an optimal distribution exists, let $\{t_1,t_2\}$ be its support, and by the argument in the proof of Theorem 5 we have $0\leq t_1\leq T$ and $T\leq t_2\leq T+|V|$. By Lemma 16 the segment of the optimal plan up to time $t_1$ has shape $AC_1^xB$ and the segment from $t_1$ to $T$ has shape $DC_2^yE$ with $|AB|<|V|^3$ and $|DE|<|V|^3$. We denote by $F$ the segment from $T$ to $t_2$

  It follows that $T=|AB|+|DE|+x\cdot|C_1|+y\cdot|C_2|$ with $x\geq 0$ and $y\geq 0$, which we can equivalently express as $x=x_0+a\cdot t$ and $y=y_0+b\cdot t$ for $B_i\leq t\leq B_s$ where $x_0,y_0,a,b,B_i,B_s$ are integer constants. The sup-value of the plan is given by

$$u=\frac{(t_2-T)\cdot u_{t_2}+(T-t_1)\cdot u_{t_1}}{t_2-t_1}$$

  where (denoting by $u_{AB}$ the sum of weights in the path $AB$, by $u_i$ the sum of weights in the cycle $C_i$, etc.):

$$\begin{aligned}t_1 =&|AB|+x\cdot|C_1|\\ t_2 =&t_1+|DE|+y\cdot|C_2|+|F|\\ u_{t_1} =&u_{AB}+x\cdot u_1\\ u_{t_2} =&u_{t_1}+u_{DE}+y\cdot u_2+u_F\end{aligned}$$

  Hence the sup-value of the plan can be expressed as the fraction of a quadratic function of $t$, and a linear function of $t$:

$$u=\frac{a_0t^2+a_1t+a_2}{b_0t+b_1}$$

  and solving $\frac{du}{dt}=0$ gives at most two values $\tilde{t}_0$ and $\tilde{t}_1$. It follows that the optimal plan is obtained for $t\in\{B_i,B_s,\lfloor\tilde{t}_0\rfloor,\lceil\tilde{t}_0\rceil,\lfloor\tilde{t}_1\rfloor,\lceil\tilde{t}_1\rceil\}$. Consider the algorithm that enumerates the possible lengths and end-points of the segments in the optimal plan, namely vertices $v_1,v_2,v_3,v_4$ and lengths $\ell_A$, $\ell_B$, $\ell_D$, $\ell_E$, $\ell_F,\ell_1,\ell_2$ such that $\ell_A+\ell_B\leq|V|^3$, $\ell_D+\ell_E\leq|V|^3$, and $\ell_F,\ell_1,\ell_2\leq n$, and computes the value of the best paths
  from $v_0$ to $v_1$ of length $\ell_A$,
  from $v_1$ to $v_2$ of length $\ell_B$,
  from $v_2$ to $v_3$ of length $\ell_D$,
  from $v_3$ to $v_4$ of length $\ell_E+\ell_F$,
  and of the best cycles of length $\ell_1$ around $v_1$, and of length $\ell_2$ around $v_3$. Using those values to compute the optimal sup-value of a path with shape $AC_1^xBDC_2^yEF$, and storing the length and vertices that give the largest value. Call $u^{**}$ this value.

The optimal sup-value is $\max(u^*,u^{**})$ and can be computed in polynomial time since $u^*$ is computed in $O(|V|^5\cdot\log(T))$ (factor $|V|^2$ for enumeration, and $|V|^3\cdot\log(T)$ for computation of best paths of fixed length less than $T$) and $u^{**}$ is computed in $O(|V|^{16}\cdot\log(V))$ (factor $|V|^{13}$ for enumeration, and $|V|^3\cdot\log(|V|)$ for computation of best paths of fixed length less than $|V|^3$).

We show that optimal plans for best-case distributions have a shape that consists of simple cycles and connecting segments of polynomial length. As we have a polynomial algorithm to compute the best path of a fixed length (Theorem 1) we obtain a polynomial algorithm for the best-case distribution problem by enumerating the possible lengths and end-points of the segments and cycles, and then computing the best utility such segments can have.

THEOREM 7. *Given a weighted graph $G$ and expected finite-horizon $T$, the optimal sup-value can be computed in time $O(|V|^{16} \cdot \log(V \cdot T))$, thus in polynomial time.*

## 6 CONCLUSION

In this work we consider the expected finite-horizon problem. Our most interesting results are for worst-case distribution of stopping times, for which we establish stationary plans are sufficient, and present polynomial-time algorithms (in contrast with the case of specified distribution and best-case distribution where memory is necessary and computing an optimal plan among stationary plans is NP-complete). In terms of algorithmic complexity, our main goal was to establish polynomial-time algorithms, and we expect that better algorithms and refined complexity analysis can be obtained.

A natural extension of this problem is to consider models of graphs with stochastic transitions, that is Markov decision processes (MDP). The problem immediately becomes much harder, as even for Markov chains, which are a special class of MDP without nondeterministic choice, there is a reduction from the Skolem problem for linear recurrence sequences (and even from the Positivity problem), whose decidability is a longstanding open question [3].

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