Intuitionistic Propositional Logic

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Historically, *intuitionism* is a current in the philosophy of mathematics, formulated by Brouwer starting in 1905 and later. Roughly, Brouwer's idea is that mathematics is a construction of the mind, which contrasts with the view that mathematics is about deriving objective truth. The key notion here is that of *construction*, which did not have a formal meaning in Brouwer's philosophy, but lead him to reject various non-constructive reasoning principles such as the law of excluded middle $\phi \vee \neg\phi$ or *reductio ad absurdum* $\neg\neg\phi \Rightarrow \phi$.

Intuitionistic logic is the logic behind intuitionistic mathematics. It can be precisely defined in various ways. For instance, in the propositional case, it is obtained from classical propositional logic, presented through the natural deduction system NK_0, by replacing the *reductio ad absurdum* rule (Abs) by the *ex-falso quod libet* rule ($\bot_\mathrm{E}$ in fig. 1). The resulting proof system is called NJ_0, and intuitionistic logic is the set of theorems that it allows to derive.

In these lectures we will consider intuitionistic propositional logic, first given by NJ_0. We will introduce Kripke semantics and show that the proof system is sound and complete with respect to this semantics of formulas. We will also discuss how intuitionistic logic takes a particularly important role in computer science, where constructive proofs become programs.

1 The proof system NJ_0

The formulas of *intuitionistic* propositional logic are the same as those of *classical* propositional logic: we write $\mathcal{F}_0$ for the set of propositional formulas, built on top of a set $\mathcal{P}$ of propositional constants, denoted by the letters P, Q, etc.

A sequent $\Gamma \vdash \phi$ is built from a formula ϕ and a *multiset* of formulas Γ. It should be read as "the conjunction of all formulas in Γ implies ϕ". The union of multisets Γ and Δ, where multiplicities add up, is simply noted with a comma, i.e. Γ, Δ.

Definition 1.1. The rules of intuitionistic natural deduction NJ_0 are given in fig. 1. We write $\Gamma \vdash_{\mathrm{NJ}} \phi$ when the sequent $\Gamma \vdash \phi$ admits a derivation in NJ_0.

Apart from the axiom rule (ax), the rules consist of introduction and elimination principles for all logical connectives. Introduction rules, noted $\star_\mathrm{I}$ where $\star$ is a logical connective or constant, indicate how one might derive formulas of the form $\star(\phi_1, \dots, \phi_n)$. Elimination rules, noted $\star_\mathrm{E}$, indicate what one might deduce from such formulas.

Note that the rules for negation can be derived from the rules for implication if one reads $\neg\phi$ as $\phi \Rightarrow \bot$. Rule $\neg_\mathrm{E}$ is sometimes presented with conclusion $\Gamma \vdash \psi$ for an arbitrary ψ: this stronger variant can be obtained by using $\bot_\mathrm{E}$ together with the simpler variant present in fig. 1.

The weakening rule is sometimes present in natural deduction systems. Here we made the choice to not take it as primitive, but we can show that it is admissible in the following sense.

$$\frac{}{\Gamma, \phi \vdash \phi}\ \text{ax} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \bot_{\mathrm{E}} \qquad \frac{}{\Gamma \vdash \top}\ \top_{\mathrm{I}}$$

$$\frac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2}\ \wedge_{\mathrm{I}} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\ \wedge_{\mathrm{E}}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee_{\mathrm{I}} \qquad \frac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi}\ \vee_{\mathrm{E}}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_{\mathrm{I}} \qquad \frac{\Gamma \vdash \phi \Rightarrow \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}\ \Rightarrow_{\mathrm{E}}$$

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi}\ \neg_{\mathrm{I}} \qquad \frac{\Gamma \vdash \neg\phi \quad \Gamma \vdash \phi}{\Gamma \vdash \bot}\ \neg_{\mathrm{E}}$$

Figure 1: Inference rules for NJ_0

Lemma 1.2 (Weakening)**.** If $\Gamma \vdash_{\mathrm{NJ}} \phi$, then $\Gamma, \Delta \vdash_{\mathrm{NJ}} \phi$.

Proof. The proof is by induction on the derivation, essentially showing that each rule still applies when the left-hand side is enriched with Δ. □

The weakening lemma can be seen as an *admissible* rule, which we will sometimes use under the following notation:

$$\frac{\Gamma \vdash \phi}{\overline{\Gamma, \Delta \vdash \phi}}$$

2 Kripke semantics

In classical logic, interpretations are sets of propositional constants: each interpretation indicates which atomic statements are satisfied; from there a general notion of satisfaction is derived. The analogue notion in intuitionistic logic is that of Kripke structure, which was invented around 1960 by Kripke and Joyal for modal logics, i.e. logics that do not talk about permanent truth but truth relative to time, location, observer, etc. It is thus surprising that Kripke semantics is the right one for intuitionistic logic, the logic of constructive reasoning.

Kripke semantics is sometimes called "possible world semantics" as it considers a notion of truth which is relative to a world. As one moves from one world to another, more propositions may become true — for general modal logic constants may also become false, but that is excluded when considering intuitionistic logic.

Definition 2.1 (Kripke structure)**.** A *Kripke structure* is given by:

- a set $\mathcal{W}$ of worlds;
- an order $\leq$ on worlds, often called *accessibility relation*;
- a monotonic mapping $\alpha : \mathcal{W} \to 2^{\mathcal{P}}$.

The monotonicity condition means that $\alpha(w) \subseteq \alpha(w')$ whenever $w \leq w'$.

When $\mathcal{K}$ is a Kripke structure, we shall denote its set of worlds by $\mathcal{W}(\mathcal{K})$.

Definition 2.2 (Satisfaction)**.** Given a Kripke structure $\mathcal{K}$, a world $w \in \mathcal{W}(\mathcal{K})$ and a formula $\phi \in \mathcal{F}_0(\mathcal{P})$, the *satisfaction* relation is defined by induction on ϕ:

- $\mathcal{K}, w \models P$ iff $P \in \alpha(w)$, for $P \in \mathcal{P}$;
- $\mathcal{K}, w \models \top$ always holds;
- $\mathcal{K}, w \models \bot$ never holds;
- $\mathcal{K}, w \models \phi \wedge \psi$ iff $\mathcal{K}, w \models \phi$ and $\mathcal{K}, w \models \psi$;
- $\mathcal{K}, w \models \phi \vee \psi$ iff $\mathcal{K}, w \models \phi$ or $\mathcal{K}, w \models \psi$;
- $\mathcal{K}, w \models \phi \Rightarrow \psi$ iff for all $w' \geq w$, $\mathcal{K}, w' \models \phi$ implies $\mathcal{K}, w' \models \psi$;
- $\mathcal{K}, w \models \neg\psi$ iff for all $w' \geq w$, $\mathcal{K}, w' \not\models \phi$.

We say that a set of formulas E is satisfied by $w \in \mathcal{W}(\mathcal{K})$ when $\mathcal{K}, w \models \phi$ for all $\phi \in E$. When $\mathcal{K}$ is obvious, we simply omit it and write $w \models \phi$ or $w \models \phi$.

Definition 2.3 (Validity, logical consequence)**.** Let ϕ, ψ be formulas. We define *validity* ($\models \phi$) and *logical consequence* ($\phi \models \psi$) as follows:

- $\models \phi$ when for all $\mathcal{K}$ and all $w \in \mathcal{W}(\mathcal{K})$, $w \models \phi$.
- $\phi \models \psi$ when $\mathcal{K}, w \models \psi$ for all $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $\mathcal{K}, w \models \phi$.

When E is a set of formulas, $E \models \phi$ means that $\mathcal{K}, w \models \phi$ for all $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $w \models E$.

Remark 2.4. Note that $\neg\phi$ is logically equivalent to $\phi \Rightarrow \bot$.

Example 2.5. Consider the validity of a few interesting formulas:

- $\neg\neg\phi \Rightarrow \phi$ and $\phi \Rightarrow \neg\neg\phi$;
- de Morgan laws;
- $\big((\phi \wedge \phi') \vee \psi\big) \Rightarrow \big((\phi \vee \psi) \wedge (\phi' \vee \psi)\big)$ and the converse;
- $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$;
- $\phi \vee \neg\phi$.

Proposition 2.6 (Satisfaction is monotonic)**.** $w \models \phi$ and $w \leq w'$ implies $w' \models \phi$.

Proof. By (structural) induction on ϕ. This is obvious for logical constants (their satisfaction does not depend on the world being considered) and propositional variables (because α is assumed to be monotonic). It follows immediately from induction hypotheses for disjunction and conjunction formulas. We consider the case of implication: assuming $w \leq w'$ and $w \models \phi \Rightarrow \psi$, let us show that $w' \models \phi \Rightarrow \psi$. We have to show that $w'' \models \psi$ for all $w'' \geq w'$ such that $w'' \models \phi$. By transitivity of the accessibility relation, we have $w'' \geq w$. By $w \models \phi \Rightarrow \psi$ and $w'' \models \phi$, we conclude $w'' \models \psi$. The case of negation is similar, as expected given remark 2.4. □

Proposition 2.7. Intuitionistically valid formulas are also classically valid.

Proof. It suffices to observe that any classical interpretation $\mathcal{I} \subseteq \mathcal{P}$ can be seen as Kripke structure $\mathcal{K}_\mathcal{I}$ with a single world w_0 such that $\alpha(w_0) = \mathcal{I}$, in such a way that $\mathcal{I} \models \phi$ (in the classical sense) is equivalent to $\mathcal{K}_\mathcal{I}, w_0 \models \phi$ (in the intuitionistic sense). □

3 Soundness

A sequent $\Gamma \vdash \phi$ is said to be valid when $\Gamma \models \phi$, *i.e.*, ϕ is a logical consequence of Γ seen as a set of formulas.

Theorem 3.1 (Soundness)**.** $\Gamma \vdash_{\mathrm{NJ}} \phi$ implies $\Gamma \models \phi$.

Proof. Straightforward (structural) induction on ϕ: for each rule of NJ_0 we can show that, if the premises are valid, the conclusion is also valid. □

Corollary 3.2. The sequent $\phi \vee \neg\phi$ is not derivable in NJ_0.

4 Completeness

We shall now establish completeness: any sequent that is valid with respect to Kripke semantics can be derived in NJ_0. To do so, we introduce the universal Kripke structure in which satisfaction is closely related to derivability.

We shall work under the assumption that the set of propositional variables $\mathcal{P}$ is countably infinite. This implies that there exists a bijection $r : \mathcal{F}_0 \to \mathbb{N}$.

Definition 4.1 (Saturated)**.** Given a (possibly infinite) set E of formulas, we write $E \vdash_{\mathrm{NJ}} \phi$ when there is a finite *set* $\Gamma \subseteq E$ such that $\Gamma \vdash_{\mathrm{NJ}} \phi$. A set of formulas E is *saturated* if, for any ϕ such that $E \vdash_{\mathrm{NJ}} \phi$, we have $\phi \in E$.

Proposition 4.2. Given a set E, the set $E^* = \{\ \phi\ :\ E \vdash_{\mathrm{NJ}} \phi\ \}$ is saturated.

Proof. Assume $E^* \vdash_{\mathrm{NJ}} \phi$, i.e. $\Gamma \vdash_{\mathrm{NJ}} \phi$ for some finite $\Gamma \subseteq E^*$. We show that $\phi \in E^*$ by induction on the number of formulas of Γ that are not in E. If $\Gamma \subseteq E$ we have $\phi \in E^*$ by definition of E^*. Otherwise, we can write $\Gamma = \Gamma', \psi$ with $\psi \in E^* \setminus E$ and Γ' has one less formula in $E^* \setminus E$ than Γ. In other words we have a finite $\Delta \subseteq E$ such that $\Delta \vdash_{\mathrm{NJ}} \psi$. We can now derive $\Gamma' \cup \Delta \vdash \phi$ (where we use the usual union of the two sets Γ' and Δ) as follows, using the admissible rule of weakening as well as $\Rightarrow_{\mathrm{I}}$ and $\Rightarrow_{\mathrm{E}}$:

$$\dfrac{\dfrac{\dfrac{\Gamma', \psi \vdash \phi}{\Gamma' \cup \Delta, \psi \vdash \phi}}{\Gamma' \cup \Delta \vdash \psi \Rightarrow \phi} \quad \dfrac{\Delta \vdash \psi}{\Gamma' \cup \Delta \vdash \psi}}{\Gamma' \cup \Delta \vdash \phi}$$

We can apply our induction hypothesis on $\Gamma' \cup \Delta$ to conclude that $\phi \in E^*$. □

Definition 4.3 (World-set)**.** We say that E is *consistent* if $\bot \notin E$. We say that E has the *disjunction property* if for all $\phi_1 \vee \phi_2 \in E$, there is some $i \in \{1, 2\}$ such that $\phi_i \in E$. We say that Γ is a *world-set* when it is saturated, consistent and has the disjunction property.

Definition 4.4 (Universal Kripke structure)**.** The universal structure $\mathcal{U}$ is defined by: $\mathcal{W}(\mathcal{U}) = \{\ w_E\ :\ E \text{ is a world-set}\ \}$; $w_E \leq w_{E'}$ iff $E \subseteq E'$; $\alpha(w_E) = E \cap \mathcal{P}$.

Lemma 4.5. Let E be a set of formulas, and ϕ a formula such that $E \nvdash_{\mathrm{NJ}} \phi$. There exists a world-set E' such that $E \subseteq E'$ and $E' \nvdash_{\mathrm{NJ}} \phi$.

Proof. We define an increasing sequence $(E_i)_{i \in \mathbb{N}}$ of saturated sets such that for all i, $\phi \notin E_i$. We set $E_0 = E^*$. If E_n enjoys the disjunction property, then $E_{n+1} = E_n$. Otherwise, let $\phi_1 \vee \phi_2$ be the formula in E_n such that $\phi_1 \notin E_n$ and $\phi_2 \notin E_n$, and such that $r(\phi_1 \vee \phi_2)$ is minimal among the formulas having that property. It cannot be that both $E_n \cup \{\phi_1\} \vdash_{\mathrm{NJ}} \phi$

and $E_n \cup \{\phi_2\} \vdash_{\mathrm{NJ}} \phi$, because by rule $\vee_{\mathrm{E}}$ (and weakening) that would contradict $E_n \not\vdash_{\mathrm{NJ}} \phi$. Let i be such that $E_n \cup \{\phi_i\} \not\vdash_{\mathrm{NJ}} \phi$, and let $E_{n+1} = (E_n \cup \{\phi_i\})^*$.

Let us show that $E' = \bigcup_{i \in \mathbb{N}} E_i$ satisfies the expected conditions. The set is saturated: if for a finite subset $\Gamma \subseteq E'$, we have $\Gamma \vdash_{\mathrm{NJ}} \psi$, then because Γ is finite we have $\Gamma \subseteq E_k$ for some k, and by saturation of E_k we have $\psi \in E_k \subseteq E'$. The same argument shows that $E' \not\vdash_{\mathrm{NJ}} \phi$, and thus E' is consistent: if $\bot$ could be derived, ϕ would also be derivable by rule $\bot_{\mathrm{E}}$. It only remains to show that E' enjoys the disjunction property. Let $\phi = \phi_1 \vee \phi_2 \in E'$, there must be some k such that $\phi \in E_k$. By construction, the disjunction property will be restored for that formula in at least $r(\phi)$ steps, thus we have $\phi_1 \in E_{k+r(\phi)}$ or $\phi_2 \in E_{k+r(\phi)}$, and the disjunction property is satisfied for ϕ in E'. □

Lemma 4.6. Let E be a world-set and ϕ a formula. We have $\mathcal{U}, w_E \models \phi$ iff $\phi \in E$.

Proof. We proceed by (structural) induction on the formula.

- Case of $\top$. We always have $w_E \models \top$ and also always have $\top \in E$ by saturation and rule $\top_{\mathrm{I}}$.
- Case of $\bot$. We never have $w_E \models \bot$, and never have $\bot \in E$ for a consistent E.
- Case of P. By definition, $w_E \models P$ iff $P \in \alpha(w_E) = E \cap \mathcal{P}$ iff $P \in E$.
- Case of $\phi_1 \wedge \phi_2$.
  - ($\Rightarrow$) From $w_E \models \phi_1 \wedge \phi_2$ we obtain $w_E \models \phi_1$ and $w_E \models \phi_2$. By induction hypotheses we thus have $E \vdash_{\mathrm{NJ}} \phi_1$ and $E \vdash_{\mathrm{NJ}} \phi_2$, and we can conclude by rule $\wedge_{\mathrm{I}}$ and weakening.
  - ($\Leftarrow$) By assumption we have $E \vdash_{\mathrm{NJ}} \phi_1 \wedge \phi_2$. This allows us to conclude $E \vdash_{\mathrm{NJ}} \phi_i$ for each $i \in \{1, 2\}$, using rules $\wedge_{\mathrm{E}}$, cut and axiom. By induction hypotheses this yields $w_E \models \phi_i$ for each i, which allows us to conclude.
- Case of $\phi_1 \vee \phi_2$.
  - ($\Rightarrow$) As in the previous case, but using rule $\vee_{\mathrm{I}}$ instead of $\wedge_{\mathrm{I}}$.
  - ($\Leftarrow$) If $\phi_1 \vee \phi_2 \in E$, then by the disjunction property of world-sets we have $\phi_i \in E$ for some i. By induction hypothesis this yields $w_E \models \phi_i$ and thus $w_E \models \phi_1 \vee \phi_2$.
- Case of $\phi_1 \Rightarrow \phi_2$.
  - ($\Rightarrow$) By rule $\Rightarrow_{\mathrm{I}}$ it suffices to show $E \cup \{\phi_1\} \vdash_{\mathrm{NJ}} \phi_2$. Assume the contrary. Then by Lemma 4.5 there is some world-set E' such that $E \subseteq E'$, $\phi_1 \in E'$ and $\phi_2 \notin E'$. By induction hypothesis $w_{E'} \models \phi_1$, but then by our assumption $w_E \models \phi_1 \Rightarrow \phi_2$ we must also have $w_{E'} \models \phi_2$. We then have $\phi_2 \in E'$ by induction hypothesis, which is a contradiction.
  - ($\Leftarrow$) Assuming $E \vdash_{\mathrm{NJ}} \phi_1 \Rightarrow \phi_2$, we show $w_E \models \phi_1 \Rightarrow \phi_2$. We simply follow the definition of satisfaction for an implication. For any $E \leq E'$ such that $w_{E'} \models \phi_1$, we have to establish $w_{E'} \models \phi_2$. By induction hypothesis we have $\phi_1 \in E'$, or in other words $E' \vdash_{\mathrm{NJ}} \phi_1$. Since we also have $E' \vdash_{\mathrm{NJ}} \phi_1 \Rightarrow \phi_2$, we conclude $E' \vdash_{\mathrm{NJ}} \phi_2$ by $\Rightarrow_{\mathrm{E}}$. By induction hypothesis we can finally conclude: $w_{E'} \models \phi_2$.

□

Theorem 4.7 (Completeness)**.** $\Gamma \models \phi$ implies $\Gamma \vdash_{\mathrm{NJ}} \phi$.

Proof. Assume $\Gamma \models \phi$ and $\Gamma \not\vdash_{\mathrm{NJ}} \phi$. By Lemma 4.5 we have some world-set E such that $\Gamma \subseteq E$ and $\phi \notin E$. We obviously have $w_E \models \Gamma$, so by $\Gamma \models \phi$ we also have $w_E \models \phi$. By Lemma 4.6, this implies $\phi \in E$, which is a contradiction. □

5 Proof normalization

The basis for rejecting the law of excluded middle is that it allows to prove $\phi \vee \neg\phi$ for any ϕ, without saying explicit which of ϕ and $\neg\phi$ holds — something which is indeed not always possible. In a constructive proof, we would like that a proof of $\phi \vee \psi$ consists of either a proof of ϕ or one of ψ. We cannot ask this in general: in NJ_0, $P \vee \neg P \vdash P \vee \neg P$ can be derived (and the constructivity of its proofs is clear) but both $P \vee \neg P \vdash P$ and $P \vee \neg P \vdash \neg P$ are unprovable. We will see that we can obtain this disjunction property in NJ_0 for sequents without hypotheses, although this is still not obvious: it requires to transform proofs into a normal form where eliminations are never applied to introductions.

Definition 5.1. A *détour* in a proof is an instance of an introduction rule used to deduce the first premise of an elimination rule. A proof is said to be détour-free when it contains no détour.

Note that the shape of inference rules implies that the introduction and elimination rules involved in a détour are necessarily introduction and elimination rules for the same logical connective.

Proposition 5.2. For any ϕ, a détour-free proof of $\vdash \phi$ starts[1] with an introduction rule.

Proof. Assume the contrary: then the first rule must be an elimination since the proof is détour-free. We will show that this is absurd: intuitively, the first subderivation of the elimination cannot be an axiom (there is no hypothesis) nor an introduction (that would yield a détour) so it is another elimination, and so on, which means that we have an infinite leftmost branch in a finite proof.

We now write this argument formally. Assume that there is a détour-free derivation π of a sequent of the form $\vdash \psi$ that starts with an elimination rule. We show by induction on π that this is absurd. Consider the elimination rule at the conclusion of π. By inspection of the rules, we see that its first premise is still of the form $\vdash \psi'$. So the corresponding subderivation cannot start with an axiom. It also cannot start with an introduction since that would form a détour. Hence we have a smaller derivation of $\vdash \psi'$ which is still détour-free: by induction hypothesis, this is absurd. □

This result allows us to prove the disjunction property for détour-free proofs, and also a consistency result for détour-free proofs.

Corollary 5.3. Assume there is a détour-free proof of $\vdash \phi \vee \psi$, Then there is either a proof of $\vdash \phi$ or one of $\vdash \psi$.

Corollary 5.4. There is no détour-free proof of $\vdash \bot$.

We now turn to the task of transforming any derivation into a détour-free derivation of the same sequent. This is done by transforming derivations step by step, reducing détours at each step. Some reductions are obvious but others introduce new détours, so we will need a convenient well-founded ordering to prove that we can apply our reductions in a way that terminates.

[1]This means that the conclusion of the proof is derived by means of an introduction rule.

Definition 5.5. Given a strict order $(E, <)$ we define the strict order $(E^\sharp, <^\sharp)$ as follows:

- $E^\sharp$ is the set of finite multisets with elements in E. We write s, e for the addition of an element e to a multiset s.
- $<^\sharp$ is the least strict ordering on $E^\sharp$ such that, for all $n \in \mathbb{N}$, for all $e, e_1, \ldots, e_n \in E$ such that $e_i < e$ for all i,
$$s, e_1, \ldots, e_n <^\sharp s, e.$$

Proposition 5.6. If $(E, <)$ is a well-order, then so is $(E^\sharp, <^\sharp)$.

Proof. Assume that there exists an infinite chain $s_1 >^\sharp s_2 >^\sharp \ldots >^\sharp s_i >^\sharp s_{i+1} >^\sharp \ldots$. We can assume wlog. that each step in this chain corresponds to an elementary replacement of an element e by a finite number of smaller elements $e_1, \ldots, e_n$ since $>^\sharp$ is the transitive closure of this elementary relation by transitivity.

We show that there cannot be such an infinite chain, by induction on the number of elements of s_1 (counting duplicates, i.e. summing all multiplicities). The result is obvious if s_1 is empty. Otherwise, $s_1 = s_1', e$. Consider, if it exists, the first step $s_i >^\sharp s_{i+1}$ where e is replaced by some $e_1, \ldots, e_n$ with $e_i < e$ for all i. Consider then, for each e_i, if it exists, the first step where it is replaced by smaller elements $e^i_1, \ldots, e^i_m$. Repeating this process we extract a possibly infinite tree containing all the descendants of e in our chain. This tree cannot have an infinite branch, since this would contradict the well-foundedness of $<$ over E. Since it is finitely branching, it must be finite. Hence we can remove e and its descendants from our infinite chain, to obtain a new chain $s_1' >^\sharp \ldots >^\sharp s_i' >^\sharp s_{i+1}' >^\sharp \ldots$ that is still infinite. We conclude by induction hypothesis on s_1'. □

This results yields an induction principle over finite multisets: if we can prove $P(s)$ for any multiset s by assuming that the property also holds for all $s' <^\sharp s$, then the property must hold for all multisets. We use it in our final result.

Proposition 5.7. If $\Gamma \vdash \phi$ is derivable in NJ_0, then it also admits a détour-free derivation.

Proof sketch. Let π be a derivation of $\Gamma \vdash \phi$. We define the degree of a détour as the size of the formula which is the conclusion of the introduction rule of the detour. We show that $\Gamma \vdash \phi$ admits a détour-free derivation by induction on the multiset of the degrees of the détours of π.

The result is immediate if there is no détour. If π contains a détour involving a conjunction, we easily simplify π into a new derivation π' which has one less détour, and conclude by induction hypothesis. Consider now a détour involving an implication, which must have the following structure:

$$\dfrac{\dfrac{\dfrac{\pi'}{\Delta, \psi_1 \vdash \psi_2}}{\Delta \vdash \psi_1 \Rightarrow \psi_2} \quad \dfrac{\pi''}{\Delta \vdash \psi_1}}{\Delta \vdash \psi_2}$$

We can then replace in π' all uses of the axiom rule on (the new occurrence of) ψ_1 by π'' (possibly using weakening first) to obtain a new proof of $\Delta \vdash \psi_2$: this construction is noted $\pi'[\pi''/\psi_1]$. Since we can also assume wlog. that the considered détour is of maximal depth, this transformation does not duplicate existing détours. It can create a new détour on ψ_1, but it would be of lesser degree than the détour on $\psi_1 \Rightarrow \psi_2$ that we eliminated, which allows to conclude by induction hypothesis. The cases of détours on negation and disjunction are similarly handled. □

The successive transformations used in the previous lemma may be seen as a computation. In fact, proofs of NJ_0 correspond closely to type derivations in simply-typed λ-calculus, and the above procedure is a strategy for computing a β-normal form for the corresponding λ-term. This is an instance of the so-called Curry-Howard correspondence, where proofs are programs, formulas are types, and proof normalization is program reduction. In the context of intuitionistic logic, it gives a very satisfying meaning to the notion of constructive proof: it is simply a program transforming normal-form proofs of hypotheses to a normal-form proof of the conclusion. Surprisingly, the Curry-Howard correspondence also extends to several other logics, including classical logic: this does not mean that classical logic is constructive, the associated notion of computation is just more complex than with intuitionistic logic…but that is another story.