Logique

David Baelde
<baelde@lsv.ens-cachan.fr>

Exercice 1

We consider a Kripke structure built like the semantic trees previously used in this course. We assume $\mathcal{P} = \{ P_i \ : \ i \in \mathbb{N} \}$. We take as worlds the w_I where I is a partial interpretation of domain $\{ P_i \ : \ i \leq n \}$ for some $n \in \mathbb{N}$. We choose $\alpha(w_I) = \{ P \ : \ I(P) = 1 \}$. Finally, we set $w_I \leq w_J$ iff the domain of I is contained in that of J and for all P in the domain of I, $I(P) = J(P)$.

1. Give a formula that is satisfied in (all worlds of) this Kripke structure, but is not valid in intuitionistic logic.

Exercice 2

Show compacity for intuitionistic logic : if a set of formulas is unsatisfiable, it must contain a finite subset that is unsatisfiable.

Hint : this is a simple exercise, an application of known results.

Exercice 3

Let ϕ_1, ϕ_2 and ψ be formulas, and P a propositional variable. If $\phi_1 \vdash \phi_2$ is derivable in LJ_0, what can we say about $\psi[\phi_1/P] \vdash \psi[\phi_2/P]$?

Hint : It is not derivable, e.g.*, for $\phi_1 = A \wedge B$, $\phi_2 = B$ and $\psi = \neg P$. However we can prove (this is the technical content of the exercise) that it is derivable for any ψ which contains only* positive *occurrences of P — an occurrence is said to be positive when it is on the left of an even number of implications.*

Exercice 4

We consider the *multicut* rule

$$\frac{\Gamma \vdash \psi \quad \Delta, \psi^n \vdash \phi}{\Gamma, \Delta \vdash \phi}$$

where $n \in \mathbb{N}$ and Δ, ψ^n is the multiset Δ to which n occurrences of ψ are added. The multicut rule obviously generalizes the cut rule. From now on we consider the system LJ_0 as containing the multicut instead of the cut rule. We seek to show that the (multi)cut rule is admissible.

1. Given two (multi)cut-free derivations

$$\frac{\Pi}{\Gamma \vdash \psi} \qquad \frac{\Pi'}{\Delta, \psi^n \vdash \phi}$$

we consider their multicut, of conclusion $\Gamma, \Delta \vdash \phi$. Show that we can always transform such a derivation into a (multi)cut-free derivation.

Hint : Proceed by structural induction on ψ*, followed by an induction on the sum of the heights of* Π *and* Π'*.*

2. Conclude.